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# A FAST GRADIENT PROJECTION ALGORITHM FOR TIME FRACTIONAL OPTIMAL CONTROL PROBLEM

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**Abstract.** In this paper a fast gradient projection algorithm for optimal control problems governed by time fractional diffusion equation is developed. The state equation is discretized by piecewise linear FE for space variable and $L1$ scheme for time variable, while the control variable is approximated by variational discretization. Based on the block triangular Toeplitz structure of the coefficient matrix of the discretized state equation and adjoint state equation, a fast gradient projection algorithm is designed for the control problem. Numerical examples are carried out to illustrate the effectiveness of the algorithms.

**Keywords:** optimal control problem; time fractional diffusion equation; block triangular Toeplitz matrix; projection gradient algorithm; fast algorithm.

**Subject Classification:** 49J20, 49K20, 65N15, 65N30.

## 1. Introduction

The main objective of this paper is to develop a fast algorithm for optimal control problems governed by time fractional diffusion equation. Let $\Omega_T = \Omega \times (0,T)$, $\Gamma_T = \partial\Omega \times (0,T)$ with $\Omega$ being a bounded domain of $R^d (1 \leq d \leq 3)$ and sufficiently smooth boundary $\partial\Omega$. Consider the

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following optimal control problem:

$$\min_{u \in U_{ad}} J(u,q) := \frac{1}{2}\|u(\mathbf{x},t) - u_d(\mathbf{x},t)\|^2_{L^2(\Omega_T)} + \frac{\gamma}{2}\|q(\mathbf{x},t)\|^2_{L^2(\Omega_T)}. \tag{1.1}$$

subject to

$$\begin{cases} {}_0\partial_t^\alpha u - \nabla \cdot (K(\mathbf{x},t)\nabla u) = f + q, & (\mathbf{x},t) \in \Omega_T, \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Gamma_T, \\ u(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{1.2}$$

Here ${}_0\partial_t^\alpha u$ denotes the left Caputo fractional derivative of order $\alpha (0 < \alpha < 1)$, which is defined by

$$ {}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds. $$

The other details will be specified later.

Time fractional diffusion equation has been widely used to describe anomalous process in many fields. In recent years lots of numerical methods and algorithms are proposed to solve this kinds of problem, for example, finite difference methods[1, 2, 3, 4], Galerkin finite element methods[5, 6, 7, 8], spectral methods[9, 10] and fast algorithms[11, 12, 13, 14].

In contrast to fractional differential equations and optimal control problem governed by integer order differential equations, the research for fractional diffusion optimal control problem is still immature and only a few literatures are reported on numerical methods and algorithms. An initial value inverse problem for time fractional diffusion equation was investigated in [15] under optimal control framework by spectral method. In [16] finite element method combined with $L1$-scheme was applied to discretize time fractional optimal control problem with Caputo derivative and a priori error estimate for semidiscrete case was proved. Recently, a fully discrete error estimate for $L1$ and back Euler scheme was developed in [17]. A fast projection gradient algorithm for space fractional optimal control problem based on finite difference discretization of the state equation was proposed in [18]. In [19], Legendre pseudo-spectral method combining with $L1$ scheme was applied to approximate optimal control problem governed by a time-fractional diffusion equation.

To obtain the numerical solution for optimal control problem we usually need to solve the discrete first order optimality condition in an iterative manner, which consists of discrete state equation, adjoint state equation and variational inequality. The nonlocal property of time fractional derivative results in big computational cost, since one need to save all previous information. Therefore developing effective fast algorithm for fractional optimal control problem is necessary.

Note that the diffusion coefficient depends on time variable. The coefficient matrix for the discrete state equation and adjoint state equation is a block lower triangular Toeplitz-like with tri-diagonal block(BT3LB-like) matrix, where the main diagonal matrix is different due to time dependent diffusion coefficient. According to [14] , we develop a fast gradient projection algorithm to solve the discrete first order optimality condition. The total computational cost for solving the state and adjoint state equation is of $O(MN\log^2 N)$ operations, which is smaller than those of the traditional block forward substitution method (BFSM)(see, [20]) and the classical inverse method. Numerical experiments are carried out to show the efficiency of the fast algorithm.

The paper is organized as follows. In Section 2, for the optimal control problems, the first order optimality condition is given. In Section 3, a fully discrete scheme for the control problem is presented based on $L1$ discretization for time variable. The structure of the coefficient matrix for discrete state and adjoint state equation is investigated. A fast projection gradient algorithm is developed in section 4. Numerical results are given to show the effectiveness of fast algorithm in Section 5.

## 2. Optimal control problem

Consider the following time-fractional optimal control problem

$$\min_{q\in U_{ad}} J(u,q), \tag{2.1}$$

subject to

$$\begin{cases} {}_0\partial_t^{\alpha} u - \nabla\cdot(K(\mathbf{x},t)\nabla u) = f+q, & (\mathbf{x},t)\in\Omega_T, \\ u(\mathbf{x},t)=0, & (\mathbf{x},t)\in\Gamma_T, \\ u(\mathbf{x},0)=0, & \mathbf{x}\in\Omega. \end{cases} \tag{2.2}$$

Here $U_{ad}$ is the admissible set defined by

$$U_{ad} = \{u \in L^2(\Omega_T): \ u_a \le u(\mathbf{x},t) \le u_b \ \text{ a.e. in } \Omega_T \ \text{ with } \ u_a, u_b \in \mathrm{R}\}.$$

$f, u_d \in L^\infty(0,T;L^2(\Omega))$ are given functions, $K(\mathbf{x},t)$ is the diffusion coefficient. Since the state equation is linear and the objective functional is strictly convex, the existence and uniqueness of the solution of above control problem can be guaranteed by standard theory(see, [16]).

For above optimal control problem the following first order optimality conditions holds(see,[16]).

**Theorem 2.1.** *Assume that $u \in U_{ad}$ is the solution to optimal control problem (2.1)-(2.2) and $u$ is the corresponding state variable given by (2.2). Then there exists an adjoint state $z$ such that $(u,z,q)$ satisfies the following optimality conditions:*

$$\begin{cases} {}_0\partial_t^\alpha u - \nabla\cdot(K(\boldsymbol{x},t)\nabla u) = f + q, & (\mathbf{x},t) \in \Omega_T, \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Gamma_T, \\ u(\mathbf{x},0) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{2.3}$$

$$\begin{cases} {}_t\partial_T^\alpha z - \nabla\cdot(K(\boldsymbol{x},t)\nabla z) = u - u_d, & (\mathbf{x},t) \in \Omega_T, \\ z(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Gamma_T, \\ z(\mathbf{x},T) = 0, & \mathbf{x} \in \Omega. \end{cases} \tag{2.4}$$

$$\int_{\Omega_T} (\gamma q + z)(v - u) \ge 0, \forall v \in U_{ad}. \tag{2.5}$$

*Here ${}_t\partial_T^\alpha z$ denotes the right Caputo fractional derivative of order $\alpha (0 < \alpha < 1)$, which is defined by the following formula*

$${}_t\partial_T^\alpha z = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{z'(s)}{(s-t)^\alpha} ds,$$

Let

$$P_{U_{ad}}(q(\mathbf{x},t)) = \max\{q_a, \min(z(\mathbf{x},t), q_b)\}$$

denote the pointwise projection onto the admissible set $U_{ad}$. Then (2.5) is equivalent to

$$q = P_{U_{ad}}(-\frac{1}{\gamma} z).$$

# 3. Finite element approximation

## 3.1 Finite element discrete scheme for control problem

Let $V_h$ be the finite element space consisting of continuous piecewise linear functions over the triangulation $T_h$:

$$V_h = \{v_h \in H_0^1(\Omega) \cap C(\Omega);\ v_h \text{ is a linear function over } \mathscr{K}, \forall \mathscr{K} \in T_h\}.$$

To define the fully discrete scheme we introduce a time partition. Let $\Delta_\tau : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a time grid with $\tau = T/N$, $t_n = n\tau, n = 0, 1, \ldots, N$.

Then the Galerkin finite element discrete scheme for the control problem (2.1)-(2.2) is to find $(U^{n+1}, Q^{n+1}) \in V_h \times U_{ad}$ such that

$$\min_{Q^{n+1} \in U_{ad}} J_{h,\tau}(U^{n+1}, Q^{n+1}) \tag{3.1}$$

subject to

$$\begin{cases} ({}_0L_t^\alpha U^{n+1}, \chi_h) + (K^{n+1}\nabla U^{n+1}, \nabla \chi_h) = (f^{n+1} + Q^{n+1}, \chi_h), \chi_h \in V_h, \\ U^0 = 0. \end{cases} \tag{3.2}$$

Here $K^{n+1} = K(\mathbf{x}, t_{n+1})$, and the cost functional is discretized by the right rectangular rule:

$$J_{h,\tau}(U^{n+1}, Q^{n+1}) := \frac{1}{2}\sum_{n=0}^{N-1} \tau\Big( \| U^{n+1} - u_d^{n+1} \|_{L^2(\Omega)}^2 + \gamma \| Q^{n+1} \|_{L^2(\Omega)}^2 \Big). \tag{3.3}$$

The time fractional derivative is discretized by $L1$-scheme(see, [9]),

$${}_0\partial_t^\alpha u^{n+1} \approx {}_0L_t^\alpha U^{n+1} = \frac{1}{\Gamma(2-\alpha)} \sum_{i=0}^{n} B_i \frac{U^{n+1-i} - U^{n-i}}{\tau^\alpha},$$

where $B_i = (i+1)^{1-\alpha} - i^{1-\alpha}$.

According [16], the following discrete optimality conditions can be derived based on 'first discretize, then optimize' approach:

$$\begin{cases} (a)({}_0L_t^\alpha U^{n+1}, w_h) + (K^{n+1}\nabla U^{n+1}, \nabla w_h) = (f^{n+1} + Q^{n+1}, w_h), & \forall w_h \in V^h, \\ (b)({}_tL_T^\alpha Z^n, w_h) + (K^{n+1}\nabla Z^n, \nabla w_h) = (U^{n+1} - u_d^{n+1}, w_h), & \forall w_h \in V^h, \\ (c)(\gamma Q^{n+1} + Z^n, v_h - Q^{n+1}) \geq 0, & \forall v_h \in U_{ad}, \\ (d)U^0 = 0, Z^N = 0, n = 0, 1, \ldots, N-1. \end{cases} \tag{3.4}$$

Here

$$ {}_tL_T^{\alpha}Z^n = \frac{1}{\Gamma(2-\alpha)}\sum_{i=n}^{N-1} B_{i-n}\frac{Z^i - Z^{i+1}}{\tau^{\alpha}}. $$

By the projection operator $P_{U_{ad}}$ the discrete control variable $Q^n$ can be expressed as follows

$$ Q^n = P_{U_{ad}}(-\frac{1}{\gamma}Z^{n-1}). $$

### 3.2 The coefficient matrix of discrete state and adjoint state equation

In this part we are going to investigate the structure of the coefficient matrix of discrete state and adjoint state equation. The finite element space $V_h$ takes the following form

$$ V_h = span\{\varphi_1, \varphi_2, \cdots, \varphi_M\}. $$

We denote the mass matrix and stiff matrix by $\mathscr{M}$ and $\mathscr{S}^{n+1}$, respectively, whose entries are calculated by $(\varphi_i, \varphi_j)$ and $(K^{n+1}\nabla\varphi_i, \nabla\varphi_j), i, j = 1, 2, \ldots, M$.

For convenience, we set $g^{(\alpha)} = \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}$. By rearranging the discrete state equation, we have the following formula

$$ \begin{aligned} &(g^{(\alpha)}B_0U^1, w_h) + (K^1\nabla U^1, \nabla w_h) = (f^1 + Q^1, w_h) + (g^{(\alpha)}B_0U^0, w_h), \\ &(g^{(\alpha)}(B_1 - B_0)U^1, w_h) + (g^{(\alpha)}B_0U^2, w_h) + (K^2\nabla U^2, \nabla w_h) \\ &\qquad\qquad\qquad\qquad = (f^2 + Q^2, w_h) + (g^{(\alpha)}B_1U^0, w_h), \\ &(g^{(\alpha)}(B_2 - B_1)U^1, w_h) + (g^{(\alpha)}(B_1 - B_0)U^2, w_h) + (g^{(\alpha)}B_0U^3, w_h) + (K^3\nabla U^3, \nabla w_h) \\ &\qquad\qquad\qquad\qquad = (f^3 + Q^3, w_h) + (g^{(\alpha)}B_2U^0, w_h), \\ &\quad\vdots \\ &(g^{(\alpha)}(B_{N-1} - B_{N-2})U^1, w_h) + \cdots + (g^{(\alpha)}B_0U^N, w_h) + (K^N\nabla U^N, \nabla w_h) \\ &\qquad\qquad\qquad\qquad = (f^N + Q^N, w_h) + (g^{(\alpha)}B_{N-1}U^0, w_h). \end{aligned} $$

Then above discrete state equation can be rewritten as follows

$$ A\mathbf{U} = \mathbf{b}. \tag{3.5} $$

Here

$$ \mathbf{U} = (\mathbf{U}^1, \mathbf{U}^2, \ldots, \mathbf{U}^N)^T, $$

$$ \mathbf{b} = (\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^N)^T $$

with $\mathbf{b}^n = ((f^n + Q^n, \varphi_j))_{M\times 1}, n = 1,\ldots,N, j = 1,2,\ldots,M$, because the initial value $\mathbf{U}^0 = 0$. The coefficient matrix $A$ takes the following form

$$A = \begin{pmatrix} A_0^0 & & & & \\ A_1 & A_0^1 & & & \\ \vdots & \vdots & \ddots & & \\ A_{N-2} & A_{N-3} & \cdots & A_0^{N-2} & \\ A_{N-1} & A_{N-2} & \cdots & A_1 & A_0^{N-1} \end{pmatrix}, \tag{3.6}$$

where

$$\begin{cases} A_0^n = g^{(\alpha)} B_0 \mathscr{M} + \mathscr{S}^{n+1}, & n = 0,\ldots,N-1, \\ A_n = g^{(\alpha)}(B_n - B_{n-1})\mathscr{M}, & n = 1,\ldots,N-1. \end{cases}$$

In an analogous way we can rearrange the adjoint state equation as follows

$$\begin{aligned}
&(g^{(\alpha)}B_0 Z^{N-1}, w_h) + (K^N \nabla Z^{N-1}, \nabla w_h) = (U^N - u_d^N, w_h) + (g^{(\alpha)} B_0 Z^N, w_h), \\
&(g^{(\alpha)}(B_1 - B_0) Z^{N-1}, w_h) + (g^{(\alpha)} B_0 Z^{N-2}, w_h) + (K^{N-1}\nabla Z^{N-2}, \nabla w_h) \\
&\qquad\qquad = (U^{N-1} - u_d^{N-1}, w_h) + (g^{(\alpha)} B_1 Z^N, w_h), \\
&(g^{(\alpha)}(B_2 - B_1) Z^{N-1}, w_h) + (g^{(\alpha)}(B_1 - B_0) Z^{N-1}, w_h) + (g^{(\alpha)} B_0 Z^{N-3}, w_h) + (K^{N-2}\nabla Z^{N-3}, \nabla w_h) \\
&\qquad\qquad = (U^{N-2} - u_d^{N-2}, w_h) + (g^{(\alpha)} B_2 Z^N, w_h), \\
&\quad\vdots \\
&(g^{(\alpha)}(B_{N-1} - B_{N-2}) Z^{N-1}, w_h) + \cdots + (g^{(\alpha)} B_0 Z^0, w_h) + (K^1 \nabla Z^0, \nabla w_h) \\
&\qquad\qquad = (U^1 - u_d^1, w_h) + (g^{(\alpha)} B_{N-1} Z^N, w_h),
\end{aligned}$$

Then above equations implies

$$D\mathbf{Z} = \mathbf{c}, \tag{3.7}$$

among

$$\begin{aligned}
\mathbf{Z} &= (\mathbf{Z}^{N-1}, \mathbf{Z}^{N-2}, \ldots, \mathbf{Z}^0)^T, \\
\mathbf{c} &= (\mathbf{c}^1, \mathbf{c}^2, \ldots, \mathbf{c}^N)^T. \\
\mathbf{c}^n &= ((U^{N-n+1} - u_d^{N-n+1}, \varphi_j))_{M\times 1}, n = 1,\ldots,N, j = 1,\ldots,M
\end{aligned}$$

with the initial value $\mathbf{Z}^N = 0$. The coefficient matrix takes the following form:

$$D = \begin{pmatrix} D_0^0 & & & & \\ D_1 & D_0^1 & & & \\ \vdots & \vdots & \ddots & & \\ D_{N-2} & D_{N-3} & \cdots & D_0^{N-2} & \\ D_{N-1} & D_{N-2} & \cdots & D_1 & D_0^{N-1} \end{pmatrix} \tag{3.8}$$

with

$$\begin{cases} D_0^n = g^{(\alpha)} B_0 \mathscr{M} + \mathscr{S}^{N-n}, & n = 0, \ldots, N-1, \\ D_n = g^{(\alpha)} (B_n - B_{n-1}) \mathscr{M}, & n = 1, 2, \ldots, N-1. \end{cases}$$

**Remark**: By the (3.6) and (3.8), we can observe that the coefficient matrix for state equation and adjoint state equation are both block lower triangular Toeplitz-like with tri-diagonal block matrix. For $K = K(\mathbf{x}, t)$, we can observe that $A$ and $D$ only have different main diagonal elements, i.e,

$$A_0^n = D_0^{N-1-n}, \quad n = 0, 1, \ldots, N-1,$$

$$A_n = D_n, \qquad n = 1, 2, \ldots, N-1.$$

At this point, the matrix $A$ and $D$ have the following form

$$A = \begin{bmatrix} A_0^0 & & & \\ A_1 & A_0^1 & & \\ \vdots & \ddots & \ddots & \\ A_{N-1} & \ldots & A_1 & A_0^{N-1} \end{bmatrix}, \quad D = \begin{bmatrix} A_0^{N-1} & & & \\ A_1 & A_0^{N-2} & & \\ \vdots & \ddots & \ddots & \\ A_{N-1} & \ldots & A_1 & A_0^0 \end{bmatrix}$$

in which all $A_0^n$, $n = 0, 1, \ldots, N-1, A_j, j = 1, \ldots, N-1$, are $M \times M$ tri-diagonal matrices. This matrix is called $BL3TB - like$ matrix.

## 4. Fast algorithm

In this section, a fast projection gradient algorithm is designed to solve the optimal control problem. In the following $F_N$ and $F_N^*$ denote the Fourier matrix and the inverse fourier matrix. $\otimes$ denotes the Kronecker tensor product, Note that for $K = K(\mathbf{x},t)$ the coefficient matrix of the state equation and adjoint equation are not exactly the same. Therefore, in this section, we focus on the following linear system with $BT3TB-like$ matrix

$$A\mathscr{X} = \mathbf{y}. \tag{4.9}$$

According [14], for $N \geq 1$ satisfying $N = 2^p$ with $p$ being a nonnegative integer, the matrix A can be divided into the following four parts with $k = N/2$,

$$\mathbf{A} = \begin{bmatrix} A_0^0 & & & & & \\ \vdots & \ddots & & & & \\ A_{k-1} & \cdots & A_0^{k-1} & & & \\ A_k & \cdots & A_1 & A_1^k & & \\ \vdots & & \vdots & \vdots & \ddots & \\ A_{N-1} & \cdots & A_k & A_{k-1} & \cdots & A_0^{N-1} \end{bmatrix} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{G} & \mathbf{Q} \end{bmatrix}.$$

Set

$$\mathscr{X} = \begin{bmatrix} \mathbf{v}^{(1)} \\ \mathbf{v}^{(2)} \end{bmatrix} \quad and \quad \mathbf{y} = \begin{bmatrix} \mathbf{p}^{(1)} \\ \mathbf{p}^{(2)} \end{bmatrix}.$$

Then the linear system (4.9) reduces to

$$\begin{cases} \mathbf{H}\mathbf{v}^{(1)} = \mathbf{p}^{(1)}, \\ \mathbf{Q}\mathbf{v}^{(2)} = \mathbf{p}^{(2)} - \mathbf{G}\mathbf{v}^{(1)}. \end{cases} \tag{4.10}$$

Since $\mathbf{H}$ and $\mathbf{Q}$ are also BLT3B-like matrix, we can use an iterative algorithm to solve this problem. So the remaining problem is how to calculate $\mathbf{G}\mathbf{v}^{(1)}$. Note that $\mathbf{G}$ is a block toeplitz matrix. Then we can extend $\mathbf{G}$ into a $MN \times MN$ circular matrix $\widetilde{\mathbf{G}}$, whose first block column is:

$$\mathbf{R} = \begin{bmatrix} A_k \\ \vdots \\ A_{N-1} \\ 0 \\ A_1 \\ \vdots \\ A_{k-1} \end{bmatrix}_{(NM \times M)},$$

Thus the term $\mathbf{G}\mathbf{v}^{(1)}$ can be calculated by using of FFT in the following manner

$$\widetilde{\mathbf{G}} \begin{bmatrix} \mathbf{v}^{(1)} \\ 0 \end{bmatrix} = (F^* \otimes I_M) diag(\Lambda_0, \Lambda_1, \ldots, \Lambda_{N-1})(F \otimes I_M) \begin{bmatrix} \mathbf{v}^{(1)} \\ 0 \end{bmatrix}, \tag{4.11}$$

where

$$\begin{bmatrix} \Lambda_0 \\ \Lambda_1 \\ \vdots \\ \Lambda_{N-1} \end{bmatrix} = \sqrt{N}(F \otimes I_M)\mathbf{R}.$$

For the equation (4.9), the calculation process is given by the following algorithm. According [14], we called this algorithm DC-BFS, since this method combined the divided and conquer method and the block forward substitution method.

**Algorithm 1** DC-BFS for solving the $BL3TB-like$ system

1 Input $\{A_0^j\}_{j=0}^{N-1}$ , $\{A_j\}_{j=1}^{N-1}$ and $\mathbf{y}$, $d=1$, $n=N$.

2 **If** $N=1$, then solve $A_0^0\mathscr{X}=\mathbf{y}$.

3 **else**

4 Carry out function DAC(d,n)

(a) **If** $n=1$, then solve $A_0^d\mathscr{X}_d=\mathbf{y}_d$;

(b) **else**

(i) DAC(d,n/2);

(ii) according (4.11), correction right hand member $\mathbf{y}[d+n/2,d+n-1]$;

(iii) DAC(d+n/2,n/2);

(c) **Endif**

5 **Endif**

According the [14], we know the computational operator for above algorithm is $O(MN\log^2 N)$. Combined the DC-BFS method with the classical gradient projection we present the fast projection gradient algorithm.

**Algorithm 2** Fast gradient projection algorithm

1 Given the time step $\tau$, space step $h$ and tolerance $\eta$.

2 Given the initial value $\mathbf{Q}$, and set $error=1$.

3 **If** $error>\eta$

4 Solving the state equation (a) in the discrete optimal system (3.4) to get state variable $\mathbf{U}$ by using AIM or DC-BFS method;

5 Solving the adjoint state equation (b) in the discrete optimal system (3.4) to get adjoint state variable $\mathbf{Z}$ by using AIM or DC-BFS method;

6 Using the pointwise projection $P_{U_{ad}}$ onto the admissible $U_{ad}$ to compute the control variable:

$$\mathbf{Q}_{new}=P_{U_{ad}}(-\frac{1}{\gamma}\mathbf{Z}).$$

7 Calculate the error

$$error=norm(\mathbf{Q}-\mathbf{Q}_{new},inf)/norm(\mathbf{Q},inf).$$

8 Update the control variable $\mathbf{Q}=\mathbf{Q}_{new}$,

9 **Endif**

If the diffusion coefficient is independent of time, the DC-BFS algorithm is still utilized. So here we only discuss the time-dependent diffusion coefficient.

## 5. Numerical experiments

In this example, we consider the problem with $\Omega = [0,1]$, $\gamma = 1$, $K = 1+t^2$ and $T = 1$ . The exact solutions are given by

$$u = t^2 \sin(\pi x),$$

$$z = (1-t)^2 \sin(2\pi x),$$

$$q = \max(-0.8, \min(-z, -0.1)).$$

The right hand term $f$ and the desired state $u_d$ can be calculated by the exact solutions and governing equation. In this example the diffusion coefficient depends on time variable. Therefore, the fast algorithm is equipped with DC-BFS.

The space-time surfaces of discrete state variable, adjoint state variable and the control variable are displayed in Fig. 1.

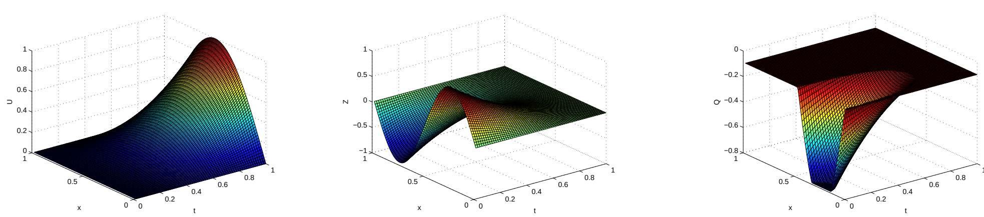

FIGURE 1. discrete state U, adjoint state Z and control Q.

In table 1, we give a comparison of the computing time for the gradient projection algorithm equipped with DC-BFS and BFSM. We can observe from Table 3 that fast algorithm can effectively reduce computation time. In Fig. 2 and 3, the relationship between time growth and time division for the gradient projection algorithm equipped with DC-BFS and BFSM are given. We can see that in the fast algorithm equipped with DC-BSF the computation time is almost linear with $N$, which is in agreement with the theoretical result.

| M | N | $2^5$ | $2^6$ | $2^7$ | $2^8$ | $2^9$ | $2^{10}$ |
|---|---|---|---|---|---|---|---|
| $2^5$ | DC-BFS | 0.0453 | 0.0871 | 0.191 | 0.403 | 0.864 | 1.843 |
| | BFSM | 0.069 | 0.211 | 0.694 | 2.435 | 9.295 | 46.017 |
| $2^6$ | DC-BFS | 0.0682 | 0.148 | 0.320 | 0.691 | 1.531 | 3.201 |
| | BFSM | 0.151 | 0.457 | 1.693 | 11.628 | 47.972 | 218.080 |
| $2^7$ | DC-BFS | 0.135 | 0.297 | 0.650 | 1.418 | 3.080 | 6.618 |
| | BFSM | 0.493 | 2.931 | 11.918 | 50.269 | 207.310 | 1024.181 |

TABLE 1. computation time

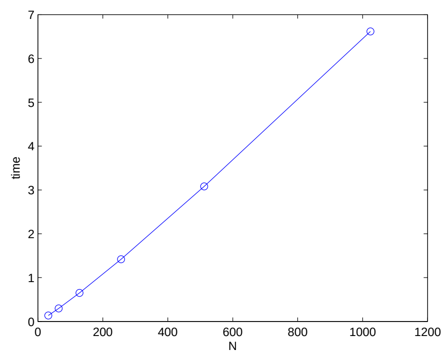


FIGURE 2. the time of DC-BFS

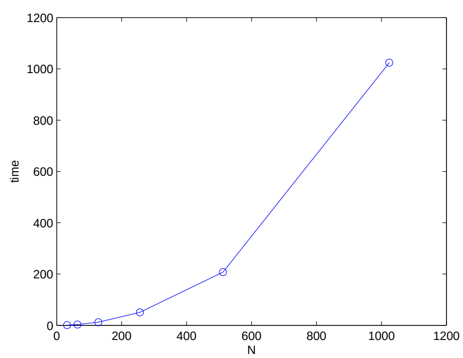


FIGURE 3. the time of BFSM

## Conflict of Interests

The authors declare that there is no conflict of interests.

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