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SOME CYCLIC CODES OF LENGTH $8p^n$ OVER $GF(q)$, WHERE ORDER OF q MODULO $8p^n$ IS $\frac{\phi(p^n)}{2}$

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Abstract. Let G be a finite group and F be finite field of prime power order q (of type $8k+5$) and order of q modulo $8p^n$ is $\frac{\phi(p^n)}{2}$. If p is prime of type $4k+1$, then the semi-simple ring $R_{8p^n} \equiv \frac{GF(q)[x]}{<x^{8p^n}-1>}$ has $16n+6$ primitive idempotents and for p of type $4k+3$, then R_{8p^n} has $12n+6$ primitive idempotents. The explicit expression for these idempotents are obtained, the generating polynomials and minimum distance bounds for cyclic codes are also completely described.

Keywords: cyclotomic cosets; primitive idempotents; generating polynomials; minimum distance.

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1. Introduction

Let $G = C_{8p^n}$ be a finite cyclic group of order $8p^n$ and $F(= GF(q))$ is a finite field of order q, a prime power of the form $8k+5$ and $g.c.d.(q, 8p^n) = 1$, then the group algebra FC_{8p^n} is semi-simple having finite cardinality of collection of primitive idempotents which equals the cardinality of collection of q-cyclotomic cosets modulo $8p^n$[7]. The primitive idempotents of minimal cyclic codes of length m in case, when order of q modulo m is $\phi(m)$ for $m = 2, 4, p^n, 2p^n$ were computed in [2,8]. The primitive idempotents of length p^n with order of q modulo p^n is

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$\frac{\phi(p^n)}{2}$ were obtained in [1] and minimal quadratic residue codes of length p^n in [4]. Cyclic codes of length $2p^n$ over F, where order of q modulo $2p^n$ is $\frac{\phi(2p^n)}{2}$ were discussed in [5]. Minimal cyclic codes of length p^nq, where p and q are distinct odd primes were derived in [3,9]. Further, when order of q modulo p^n is $\phi(p^n)$, the minimal cyclic codes of length $8p^n$ were discussed in [10,11]. Irreducible cyclic codes of length $4p^n$ and $8p^n$, where $q \equiv 3(mod\ 8)$ and $p/(q-1)$ were obtained in [6].

In present paper, we obtained cyclic codes of length $8p^n$ over F where q is of the form $8k+5$ and order of q modulo p^n is $\frac{\phi(p^n)}{2}$. The q-cyclotomic cosets modulo $8p^n$ are obtained in section 2 and corresponding primitive idempotents in section 3. In section 4, we discussed generating polynomials and dimensions for the corresponding cyclic codes of length $8p^n$. The minimum distance or the bounds for minimum distance of these codes are obtained in section 5. At the end, an example is discussed to illustrate the various parameters for these codes.

2. Cyclotomic Cosets

Let $S = \{1,2,...,8p^n\}$. For $a,b \in S$, consider $a \sim b$ iff $a \equiv bq^i(mod\ 8p^n)$ for some integer $i \geq 0$. This is an equivalence relation on S. The equivalence classes due to this relation are called q-cyclotomic cosets modulo $8p^n$. The q-cyclotomic coset containing $s \in S$ is $\Omega_s = \{s, sq, sq^2, ..., sq^{t_s-1}\}$, where t_s is the smallest positive integer such that $sq^{t_s} \equiv s(mod\ 8p^n)$.

Lemma 2.1. *[5, Theorem 2.5] If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then the order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, $0 \leq i \leq n-1$.*

Lemma 2.2. *If $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, then for $0 \leq i \leq n-1$, order of q modulo $2p^{n-i}$ and $4p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$.*

Proof. Since $\frac{\phi(p^n)}{2}$ is the order of q modulo p^n, therefore by lemma 2.1, order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}, 1 \leq i \leq n-1$. Hence

$$q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ p^{n-i}) \tag{1}$$

Since q is of the form $8k+5$, therefore $q \equiv 1(mod\ 2)$. Hence $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2)$. As $gcd(2, p^{n-i}) = 1$ and order of q modulo p^{n-i} is $\frac{\phi(p^{n-i})}{2}$, so $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 2p^{n-i})$. This implies that $\frac{\phi(p^{n-i})}{2}$ is the smallest integer for which (2.1) holds. Hence order of q modulo $2p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$. Similar result hold for $4p^{n-i}$. □

Lemma 2.3. *If $p \equiv 1(mod\ 4)$, then order of q modulo $8p^{n-i}$ is $\frac{\phi(p^{n-i})}{2}$ and for $p \equiv 3(mod\ 4)$ order of q modulo $8p^{n-i}$ is $\phi(p^{n-i})$.*

Proof. Proof is on similar lines as that of lemma 2.2. □

Lemma 2.4. *For $0 \le i \le n-1$ and $0 \le k \le \frac{\phi(p^{n-i})}{2} - 1$, $T \not\equiv q^k(mod\ 8p^{n-i})$, where $T = \lambda = (1+2p^n)$ or $T = \mu = 2(1+2p^n)$ or $T = \nu = (1+4p^n)$ or $T = \chi = (1+6p^n)$.*

Proof. Proof can be obtained by using lemma 2.1 and lemma 2.2. □

Lemma 2.5. *Let p be an odd prime. Then there exists an integer g , $1 < g < 8p$ and is primitive root modulo p, further when p is of the form $4k+1$ then order of g modulo 4 and modulo 8 is 2, and when p is of the form $4k+3$ then order of g modulo 4 is 1 and modulo 8 is 2. Also, if q is any prime power and $g.c.d.(q, p) = 1$, then $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$.*

Proof. Consider the complete residue systems, $S_p = \{0, 1, 2, ..., p-1\}$ modulo p, $S_2 = \{0, 1\}$ modulo 2, and $S_{2p} = \{0, 1, 2, ..., 2p-1\}$ modulo $2p$. Since $g.c.d.(2, p) = 1$, so there exist an integer $v \in S_p$ such that $2v - p = 1$. Let a be any primitive root mod p in S_p. For $p \equiv 1(mod\ 4)$, let $g \equiv 2av + tp + 6ap(mod\ 8p)$ where t is a prime of the form $8k_1 + 3$ implies $g \equiv a(mod\ p)$. Hence g is primitive root modulo p. Now, $g \equiv 2av + tp + 6ap(mod\ 8)$ where t is a prime of the form $8k_1 + 3$, so $g \equiv 3(mod\ 4)$ as p is of the form $4k+1$. Hence order of g modulo 4 and modulo 8 is 2. Now for $p \equiv 3(mod\ 4)$, let $g \equiv 2av + tp + 4ap(mod\ 8p)$ where t is a prime of the form $8k_2 + 7$ implies g is primitive root modulo p and order of g modulo 4 is 1 and modulo 8 is 2. Let $g \in \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$, so $g = q^i$ for some $1 \le i \le \frac{\phi(p)}{2} - 1$ equivalently $o(g) = o(q^i)$. As order of q modulo $8p$ is $\frac{\phi(p)}{2}$, so $o(q^i) \le \frac{\phi(p)}{2}$ modulo $8p$. This implies $o(g) \le \frac{\phi(p)}{2}$ modulo $8p$, but order of g mod $8p$ is $\phi(p)$, hence $g \notin \{1, q, q^2, ..., q^{\frac{\phi(p)}{2}-1}\}$. □

Lemma 2.6. *If $p \equiv 1(mod\ 4)$, there exist a fixed integer g satisfying $gcd(g, 2pq) = 1, 1 < g < 8p, g \not\equiv q^k(mod p)$ where $0 \le k \le \frac{\phi(p)}{2} - 1$ such that for $0 \le j \le n-1$, the set*

$\{1,q,..,q^{\frac{\phi(p^{n-j})}{2}-1},g,gq,..,gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ *forms a reduced residue system modulo* p^{n-j} *and the set* $\{1,q,...,q^{\frac{\phi(p^{n-j})}{2}-1},g,gq,...,gq^{\frac{\phi(p^{n-j})}{2}-1},\lambda,\lambda q,\lambda q^2,$
$...,\lambda q^{\frac{\phi(p^{n-j})}{2}-1},\lambda g,\lambda gq,\lambda gq^2,...,\lambda gq^{\frac{\phi(p^{n-j})}{2}-1},\nu,\nu q,\nu q^2,...,\nu q^{\frac{\phi(p^{n-j})}{2}-1},$
$\nu g,\nu gq,....,\nu gq^{\frac{\phi(p^{n-j})}{2}-1},\chi,\chi q,...,\chi q^{\frac{\phi(p^{n-j})}{2}-1},\chi g,\chi gq,...,\chi gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ *forms a reduced residue system modulo* $8p^{n-j}$.

Proof. By lemma 2.1, order of q modulo p is $\frac{\phi(p)}{2}$, therefore $1,q,q^2,...,q^{\frac{\phi(p)}{2}-1}$ are incongruent modulo p. As there are exactly $\phi(p)$ numbers in the reduced residue system modulo p. Therefore there exist a number g satisfying $gcd(g,2pq)=1, 1<g<8p, g\not\equiv q^k(mod p)$ for $0\leq k\leq \frac{\phi(p)}{2}-1$. Then the set $\{1,q,q^2,...,q^{\frac{\phi(p)}{2}-1},g,gq,gq^2,...,gq^{\frac{\phi(p)}{2}-1}\}$ forms a reduced residue system modulo p. Since for $0\leq k\leq \frac{\phi(p)}{2}-1$, $g\not\equiv q^k(mod p)$. It follows that $g\not\equiv q^k(mod p^{n-j})$ for $0\leq k\leq \frac{\phi(p^{n-j})}{2}-1$. Hence the set $\{1,q,q^2,...,q^{\frac{\phi(p^{n-j})}{2}-1},g,gq,gq^2,...,gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo p^{n-j}.

Similar result holds to show that the set
$\{1,q,...,q^{\frac{\phi(p^{n-j})}{2}-1},g,gq,...,gq^{\frac{\phi(p^{n-j})}{2}-1},\lambda,\lambda q,...,\lambda q^{\frac{\phi(p^{n-j})}{2}-1},\lambda g,\lambda gq,...,\lambda gq^{\frac{\phi(p^{n-j})}{2}-1},$
$\nu,\nu q,...,\nu q^{\frac{\phi(p^{n-j})}{2}-1},\nu g,\nu gq,....,\nu gq^{\frac{\phi(p^{n-j})}{2}-1},\chi,\chi q,...,\chi q^{\frac{\phi(p^{n-j})}{2}-1},\chi g,\chi gq,\chi gq^2...,$
$\chi gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ forms a reduced residue system modulo $8p^{n-j}$. □

Lemma 2.7. *For* $p\equiv 3(mod\ 4)$, *there exist a fixed integer* g *satisfying* $gcd(g,2pq)=1, 1<g<8p, g\not\equiv q^k(mod\ p)$ *where* $0\leq k\leq \frac{\phi(p)}{2}-1$, *such that for* $0\leq j\leq n-1$, *the set* $\{1,q,q^2,...,q^{\frac{\phi(p^{n-j})}{2}-1},g,gq,gq^2,...,$
$gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ *forms a reduced residue system modulo* p^{n-j} *and the set*
$\{1,q,q^2,...,q^{\phi(p^{n-j})-1},g,gq,gq^2,...,gq^{\phi(p^{n-j})-1},\lambda,\lambda q,...,\lambda q^{\phi(p^{n-j})-1},\lambda g,\lambda gq,...,\lambda gq^{\phi(p^{n-j})-1}\}$
forms a reduced residue system modulo $8p^{n-j}$.

Proof. Proof is similar to that of lemma 2.6. □

Theorem 2.8. *Let p be an odd prime then*

(i) *The* $(16n+6)$ *q-cyclotomic cosets modulo* $8p^n$ *for* $p\equiv 1(mod\ 4)$ *are given by*
$\Omega_{ap^n}=\{ap^n\}$, $\Omega_{bp^n}=\{bp^n,bp^nq\}$, $a\in\mathbb{A}=\{0,2,4,6\}$ *and* $b\in\mathbb{B}=\{1,3\}$ *and*
$\Omega_{xp^i}=\{xp^i,xp^iq,xp^iq^2,...,xp^iq^{\frac{\phi(p^{n-i})}{2}-1}\}$

for $x \in \mathbb{X} = \{1,2,4,8,\lambda,\mu,\nu,\chi,g,2g,4g,8g,\lambda g,\mu g,\nu g,\chi g\}$.

(ii) *The* $(12n+6)$ *q-cyclotomic cosets modulo* $8p^n$ *for* $p \equiv 3(mod\ 4)$ *are given by*

$\Omega_{ap^n} = \{ap^n\}$, $\Omega_{bp^n} = \{bp^n, bp^n q\}$ *and*

Ω_{yp^i}=$\{yp^i, yp^i q, yp^i q^2, ..., yp^i q^{\phi(p^{n-i})-1}\}$ *for* $y \in \mathbb{Y} = \{1,\lambda,g,\lambda g\}$.

Ω_{zp^i}=$\{zp^i, zp^i q, zp^i q^2, ..., zp^i q^{\frac{\phi(p^{n-i})}{2}-1}\}$ *for* $z \in \mathbb{Z} = \{2,4,8,\mu,2g,4g,8g,\mu g\}$.

Proof. (i) $\Omega_0 = \{0\}$ is trivial. Since q is of the form $8k+5$, so $q^2 \equiv 1(mod\ 8)$, therefore $p^n q^2 \equiv p^n(mod\ 8p^n)$ and hence $\Omega_{p^n} = \{p^n, p^n q\}$. Similarly $\Omega_{3p^n} = \{3p^n, 3p^n q\}$.
Since, $q \equiv 1(mod\ 4)$, so $2p^n q \equiv 2p^n(mod\ 8p^n)$ and hence $\Omega_{2p^n} = \{2p^n\}$.
Similarly $\Omega_{4p^n} = \{4p^n\}$ and $\Omega_{6p^n} = \{6p^n\}$.
By lemma 2.3, $q^{\frac{\phi(p^{n-i})}{2}} \equiv 1(mod\ 8p^{n-i})$. Equivalently, $p^i q^{\frac{\phi(p^{n-i})}{2}} \equiv p^i(mod\ 8p^n)$.
Therefore, Ω_{xp^i}=$\{xp^i, xp^i q, xp^i q^2, ..., xp^i q^{\frac{\phi(p^{n-i})}{2}-1}\}$.
Obviously, $|\Omega_0| = 1$. Also $|\Omega_{2p^n}| = |\Omega_{4p^n}| = |\Omega_{6p^n}| = 1$, $|\Omega_{p^n}| = |\Omega_{3p^n}| = 2$ and $|\Omega_{xp^i}| = \frac{\phi(p^{n-i})}{2}$.
Therefore, $\sum_{i=0}^{n-1} |\Omega_{p^i}| = \sum_{i=0}^{n-1} \frac{\phi(p^{n-i})}{2} = \frac{p^n-1}{2}$.
Hence, $|\Omega_0| + |\Omega_{p^n}| + |\Omega_{2p^n}| + |\Omega_{3p^n}| + |\Omega_{4p^n}| + |\Omega_{6p^n}| + \sum_{i=0}^{n-1}\sum_x |\Omega_{xp^i}| = 8 + \frac{16(p^n-1)}{2} = 8 + 8(p^n-1) = 8p^n$.
(ii) Proof is similar to that of (i). □

3. PRIMITIVE IDEMPOTENTS

Throughout this paper, we consider α as $8p^n th$ root of unity in some extension field of F. Let M_s be the minimal ideal in $R_{8p^n} = \frac{F[x]}{<x^{8p^n}-1>} \equiv FC_{8p^n}$, generated by $\frac{(x^{8p^n}-1)}{m_s(x)}$, where $m_s(x)$ is the minimal polynomial for α^s, $s \in \Omega_s$. We denote $P_s(x)$, the primitive idempotent in R_{8p^n}, corresponding to the minimal ideal M_s, given by $P_s(x) = \frac{1}{8p^n}\sum_{t=0}^{8p^n-1} \rho_i^s x^t$ where $\rho_i^s = \sum_{s\in\Omega_s} \alpha^{-is}$ and $\overline{Z}_t = \sum_{s\in\Omega_s} x^s$.

Then,

$$P_s(x) = \frac{1}{8p^n}[\sum_{a\in\mathbb{A}} \rho^s_{ap^n}\overline{Z}_{ap^n} + \sum_{b\in\mathbb{B}} \rho^s_{bp^n}\overline{Z}_{bp^n} + \sum_{i=0}^{n-1}\sum_{x\in\mathbb{X}} \rho^s_{xp^i}\overline{Z}_{xp^i}] \text{ for } p \equiv 1(mod\ 4)$$

(2)

$$\textit{and } P_s(x) = \frac{1}{8p^n}[\sum_{a\in\mathbb{A}} \rho^s_{ap^n}\overline{Z}_{ap^n} + \sum_{b\in\mathbb{B}} \rho^s_{bp^n}\overline{Z}_{bp^n} + \sum_{i=0}^{n-1}[\sum_{y\in\mathbb{Y}} \rho^s_{yp^i}\overline{Z}_{yp^i} + \sum_{z\in\mathbb{Z}} \rho^s_{zp^i}\overline{Z}_{zp^i}]] \textit{ for } p \equiv 3(mod\ 4)$$

Lemma 3.1. *For any odd prime p and a positive integer k, if β is primitive p^kth root of unity in some extension field of F, then*

$$\sum_{t=0}^{\frac{\phi(p^k)}{2}-1}(\beta^{q^t}+\beta^{gq^t})=\begin{cases}-1, & k=1\\ 0, & k\geq 2\end{cases}$$ *, when q is quadratic residue modulo p^k and*

$$\sum_{t=0}^{\phi(p^k)-1}\beta^{q^t}=\begin{cases}-1, & k=1\\ 0, & k\geq 2\end{cases}$$ *, when q is primitive root modulo p^k.*

Proof. By lemma 2.6, the set $\{1,q,q^2,...,q^{\frac{\phi(p^k)}{2}-1},g,gq,gq^2,...,gq^{\frac{\phi(p^k)}{2}-1}\}$ is a reduced residue system $(mod\ p^k)$. So, $\sum_{t=0}^{\frac{\phi(p^k)}{2}-1}(\beta^{q^t}+\beta^{gq^t})=\sum_{t=0}^{p^k-1}\beta^t-\sum_{t=1,p/t}^{p^k}\beta^t=-\sum_{t=1}^{p^{k-1}}\beta^{pt}$.

If $k=1$, then $-\beta^p=-1$. If $k\geq 2$, then $\beta^p\neq 1$, therefore, $\sum_{t=1}^{p^{k-1}}\beta^{pt}=\beta^p(1+\beta^p+...+\beta^{pk-1})=\beta^p\frac{(\beta^{pk}-1)}{\beta^p-1}=0$. For the remaining part see [3, lemma 4]. □

Lemma 3.2. *For $0\leq i\leq n-1$, $\lambda^2\Omega_{p^i}=\nu^2\Omega_{p^i}=\chi^2\Omega_{p^i}=\Omega_{p^i}=\lambda\Omega_{\lambda p^i}=\nu\Omega_{\nu p^i}=\chi\Omega_{\chi p^i}$ and $\mu^2\Omega_{p^i}=4\Omega_{p^i}=\mu\Omega_{\mu p^i}=2\Omega_{2p^i}=\Omega_{4p^i}$.*

Proof. Since $\lambda^2,\nu^2,\chi^2\equiv 1(mod\ 8p^n)$ and $\mu^2\equiv 4(mod\ 8p^n)$ so, the required result holds. □

Lemma 3.3. *For Ω_{p^n} ,Ω_{2p^n}, Ω_{3p^n}, Ω_{4p^n} and Ω_{6p^n}, $\Omega_{p^n}=-\Omega_{3p^n}$, $\Omega_{2p^n}=-\Omega_{6p^n}$ and $\Omega_{4p^n}=-\Omega_{4p^n}$.*

Proof. We have $\Omega_{p^n}=\{p^n,p^nq\}$ and $\Omega_{3p^n}=\{3p^n,3p^nq\}$ and we claim that $-p^n\equiv 3p^nq(mod\ 8p^n)$.

For this, we have, $3p^nq+p^n=(3q+1)p^n$ =$\{3(8k+5)+1\}p^n=24k+16p^n\equiv 0(mod 8\ p^n)$ so, $-\Omega_{p^n}=\Omega_{3p^n}$. Other equalities holds trivially. □

Lemma 3.4. *If $p\equiv 1(mod\ 4)$, then $\Omega_\lambda=-\Omega_1$ and for $p\equiv 3(mod\ 4)$, $\Omega_{\lambda g}=-\Omega_1$.*

Proof. If $p\equiv 1(mod\ 4)$, then clearly $\lambda=1+2p^n\equiv 3(mod\ 8)$ and $q\equiv -3(mod\ 8)$.
Equivalently, $q^{\frac{\phi(p^n)}{4}}\equiv -3(mod\ 8)$ and $\lambda q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 8)$.
As $q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ p^n)$ and $(1+2p^n)\equiv 1(mod\ p^n)$, so $\lambda q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ p^n)$.
Also, $(8,p^n)=1$, thus $\lambda q^{\frac{\phi(p^n)}{4}}\equiv -1(mod\ 8p^n)$. Hence the result holds.

Further, when $p \equiv 3(mod\ 8)$ and for even k, $q^k \equiv 1(mod\ 8)$, $gq^k \equiv 5(mod\ 8)$ and $\lambda \equiv 3(mod\ 8)$. Also $\lambda gq^k \equiv -1(mod\ 8)$ and $q^{\frac{\phi(p^n)}{2}} \equiv 1(mod\ p^n)$. Here $\frac{\phi(p^n)}{2}$ is odd so, $q^k \not\equiv -1(mod\ p^n)$ for any k.

As the set $\{1, q, q^2, ..., q^{\frac{\phi(p^{n-j})}{2}-1}, g, gq, gq^2, ..., gq^{\frac{\phi(p^{n-j})}{2}-1}\}$ form a reduced residue system $mod\ p^n$ so, $gq^k \equiv 1(mod\ p^n)$ equivalently, $\lambda gq^k \equiv 1(mod\ p^n)$.

Also $(8, p^n) = 1$, therefore, $\lambda gq^k \equiv -1(mod\ 8p^n)$. Hence the result holds. $\square$

Notation 3.5. For $0 \leq j \leq n-1$, define

$A_j = p^j \sum_{s\varepsilon\Omega_{gp^j}} \alpha^s$, $B_j = p^j \sum_{s\varepsilon\Omega_{\lambda gp^j}} \alpha^s$, $C_j = p^j \sum_{s\varepsilon\Omega_{p^j}} \alpha^s$, $D_j = p^j \sum_{s\varepsilon\Omega_{\lambda p^j}} \alpha^s$, $E_j = p^j \sum_{s\varepsilon\Omega_{2gp^j}} \alpha^s$,

$F_j = p^j \sum_{s\varepsilon\Omega_{2p^j}} \alpha^s$, $G_j = p^j \sum_{s\varepsilon\Omega_{4gp^j}} \alpha^s$, $H_j = p^j \sum_{s\varepsilon\Omega_{4p^j}} \alpha^s$, $I_j = p^j \sum_{s\varepsilon\Omega_{8gp^j}} \alpha^s$, $J_j = p^j \sum_{s\varepsilon\Omega_{8p^j}} \alpha^s$.

Here $A_j^q = A_j$, so $A_j \in F$. Similarly B_j, C_j, D_j, E_j, F_j, G_j, H_j, I_j and $J_j \in F$.

The set M is defined as follows:

$M = \{\nu, \chi\}$ for $p \equiv 1(mod\ 4)$ and $M = \emptyset$, the empty set, for $p \equiv 3(mod\ 4)$.

Theorem 3.6. *The expressions for primitive idempotents corresponding to Ω_0, Ω_{p^n}, Ω_{2p^n}, Ω_{3p^n}, Ω_{4p^n} and Ω_{6p^n} are given by:*

$P_0(x) = \frac{1}{8p^n}[\overline{Z}_0 + \overline{Z}_{p^n} + \overline{Z}_{2p^n} + \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{6p^n} + \sum_{i=0}^{n-1} \{\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} + \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} + \sum_{m\varepsilon M} (\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\}]$

$P_{p^n}(x) = \frac{1}{8p^n}[2\overline{Z}_0 - 2\alpha^{2p^{2n}}\overline{Z}_{2p^n} - 2\overline{Z}_{4p^n} + 2\alpha^{2p^{2n}}\overline{Z}_{6p^n} + \sum_{i=0}^{n-1} \{-2\alpha^{2p^{n+i}}\overline{Z}_{2p^i} - 2\overline{Z}_{4p^i} + 2\overline{Z}_{8p^i} + 2\alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} + 2\alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - 2\overline{Z}_{4gp^i} + 2\overline{Z}_{8gp^i} - 2\alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i}]$

$P_{2p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 - \alpha^{2p^{2n}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} + \alpha^{2p^{2n}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{6p^n} + \sum_{i=0}^{n-1} \{-\alpha^{2p^{n+i}}\overline{Z}_{p^i} - \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{2p^{n+i}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} + \alpha^{2p^{n+i}}\overline{Z}_{gp^i} - \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{2p^{n+i}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} + \alpha^{2p^{n+i}} \sum_{m\varepsilon M} (\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\}]$

$P_{3p^n}(x) = \frac{1}{8p^n}[2\overline{Z}_0 + 2\alpha^{2p^{2n}}\overline{Z}_{2p^n} - 2\overline{Z}_{4p^n} - 2\alpha^{2p^{2n}}\overline{Z}_{6p^n} + \sum_{i=0}^{n-1} \{2\alpha^{2p^{n+i}}\overline{Z}_{2p^i} - 2\overline{Z}_{4p^i} + 2\overline{Z}_{8p^i} - 2\alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} - 2\alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - 2\overline{Z}_{4gp^i} + 2\overline{Z}_{8gp^i} + 2\alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i}]$

$P_{4p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 - \overline{Z}_{p^n} + \overline{Z}_{2p^n} - \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{6p^n} + \sum_{i=0}^{n-1} \{-\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} - \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} - \sum_{m\varepsilon M} (\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\}]$

$P_{6p^n}(x) = \frac{1}{8p^n}[\overline{Z}_0 + \alpha^{2p^{2n}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} - \alpha^{2p^{2n}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{6p^n} + \sum_{i=0}^{n-1}\{\alpha^{2p^{n+i}}\overline{Z}_{p^i} - \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{n+i}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} - \alpha^{2p^{n+i}}\overline{Z}_{gp^i} - \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{2p^{n+i}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} - \alpha^{2p^{n+i}}\sum_{m\varepsilon M}(\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\}]$.

Proof. To evaluate $P_0(x)$, take $s=0$ in (3.1), then $\rho_k^0 = \sum_{s\varepsilon\Omega_0}\alpha^0 = 1$ for all $0 \le k \le 8p^n-1$.

Therefore, $P_0(x) = \frac{1}{8p^n}[\overline{Z}_0 + \overline{Z}_{p^n} + \overline{Z}_{2p^n} + \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{6p^n} + \sum_{i=0}^{n-1}\{\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} + \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} + \sum_{m\varepsilon M}(\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\}]$

For the evaluation of $P_{p^n}(x)$, take $s = p^n$. so we have to compute $\rho_k^{p^n}$ for $k = 0, p^n, 2p^n, 3p^n, 4p^n, 6p^n, p^i, 2p^i,$
$4p^i, 8p^i, \lambda p^i, \mu p^i, \nu p^i, \chi p^i, gp^i, 2gp^i, 4gp^i, 8gp^i, \lambda gp^i, \mu gp^i, \nu gp^i, \chi gp^i$.

Here $\rho_k^{p^n} = \sum_{s\varepsilon\Omega_{p^n}}\alpha^{-ks} = \sum_{s\varepsilon\Omega_{3p^n}}\alpha^{ks} = \alpha^{3p^nk} + \alpha^{3p^nkq} = \alpha^{3p^nk} + \alpha^{3p^nk(8k+5)} = \alpha^{3p^nk} + \alpha^{7p^nk}$.

Therefore, $\rho_0^{p^n} = -\rho_{4p^n}^{p^n} = -\rho_{4p^i}^{p^n} = \rho_{8p^i}^{p^n} = -\rho_{4gp^i}^{p^n} = \rho_{8gp^i}^{p^n} = 2$.

$\rho_{p^n}^{p^n} = \rho_{3p^n}^{p^n} = \rho_{p^i}^{p^n} = \rho_{\lambda p^i}^{p^n} = \rho_{\nu p^i}^{p^n} = \rho_{\chi p^i}^{p^n} = \rho_{gp^i}^{p^n} = \rho_{\lambda gp^i}^{p^n} = \rho_{\nu gp^i}^{p^n} = \rho_{\chi gp^i}^{p^n} = 0$,

$\rho_{2p^n}^{p^n} = -\rho_{6p^n}^{p^n} = -2\alpha^{2p^{2n}}$, $\rho_{2p^i}^{p^n} = -\rho_{\mu p^i}^{p^n} = -\rho_{2gp^i}^{p^n} = -\rho_{\mu gp^i}^{p^n} = -2\alpha^{2p^{n+i}}$.

Thus $P_{p^n}(x) = \frac{1}{8p^n}[2\overline{Z}_0 - 2\alpha^{2p^{2n}}\overline{Z}_{2p^n} - 2\overline{Z}_{4p^n} + 2\alpha^{2p^{2n}}\overline{Z}_{6p^n} + \sum_{i=0}^{n-1}\{-2\alpha^{2p^{n+i}}\overline{Z}_{2p^i} - 2\overline{Z}_{4p^i} + 2\overline{Z}_{8p^i} + 2\alpha^{2p^{n+i}}\overline{Z}_{\mu p^i} + 2\alpha^{2p^{n+i}}\overline{Z}_{2gp^i} - 2\overline{Z}_{4gp^i} + 2\overline{Z}_{8gp^i} - 2\alpha^{2p^{n+i}}\overline{Z}_{\mu gp^i}]$

Similarly $P_{2p^n}(x)$,$P_{3p^n}(x)$,$P_{4p^n}(x)$ and $P_{6p^n}(x)$ can be obtained using lemma 3.3. □

Lemma 3.7. *For $0 \le i \le n$, $0 \le j \le n-1$ and $p \equiv 1(mod\ 4)$,*

$$\sum_{s\varepsilon\Omega_{4gp^j}}\alpha^{p^is} = \sum_{s\varepsilon\Omega_{4gp^j}}\alpha^{\lambda p^is} = \sum_{s\varepsilon\Omega_{4gp^j}}\alpha^{\nu p^is} = \sum_{s\varepsilon\Omega_{4gp^j}}\alpha^{\chi p^is} = \begin{cases} -\frac{\phi(p^{n-j})}{2}, & if\ i+j \ge n, \\ \frac{1}{p^j}G_{i+j}, & if\ i+j \le n-1,\ g \ne 1, \\ \frac{1}{p^j}H_{i+j}, & if\ i+j \le n-1,\ g = 1. \end{cases}$$

$$\sum_{s\varepsilon\Omega_{8gp^j}}\alpha^{p^is} = \sum_{s\varepsilon\Omega_{8gp^j}}\alpha^{\lambda p^is} = \sum_{s\varepsilon\Omega_{8gp^j}}\alpha^{\mu p^is} = \sum_{s\varepsilon\Omega_{8gp^j}}\alpha^{\chi p^is} = \begin{cases} \frac{\phi(p^{n-j})}{2}, & if\ i+j \ge n, \\ \frac{1}{p^j}I_{i+j}, & if\ i+j \le n-1,\ g \ne 1, \\ \frac{1}{p^j}J_{i+j}, & if\ i+j \le n-1,\ g = 1. \end{cases}$$

For $0 \le i \le n$, $0 \le j \le n-1$ and $p \equiv 3(mod\ 4)$.

$$\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{s\varepsilon\Omega_{\lambda p^j}} \alpha^{4gp^is} = \begin{cases} -\phi(p^{n-j}),\ if\ i+j \geq n, \\ \frac{2}{p^j}G_{i+j},\ if\ i+j \leq n-1,\ g \neq 1, \\ \frac{2}{p^j}H_{i+j},\ if\ i+j \leq n-1,\ g = 1. \end{cases}$$

$$\sum_{s\varepsilon\Omega_{p^j}} \alpha^{8gp^is} = \sum_{s\varepsilon\Omega_{\lambda p^j}} \alpha^{8gp^is} = \begin{cases} \phi(p^{n-j}),\ if\ i+j \geq n, \\ \frac{2}{p^j}I_{i+j},\ if\ i+j \leq n-1,\ g \neq 1, \\ \frac{2}{p^j}J_{i+j},\ if\ i+j \leq n-1,\ g = 1. \end{cases}$$

Proof. Here $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{4(1+2p^n)gp^{i+j}q^t} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{4gp^{i+j}q^t} = \sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is}$

Let $\beta = \alpha^{4p^{i+j}}$. Then, $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \beta^{gq^t}$.

If $i+j \geq n$, then β is $2nd$ root of unity so, $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \alpha^{4gp^{i+j}q^t} = -\frac{\phi(p^{n-j})}{2}$.

If $i+j \leq n-1$, β is $2p^{n-i-j}th$ root of unity.

Then $\beta^{gq^l} \equiv \beta^{gq^r}$ if and only if $gq^l \equiv gq^r (mod\ 2p^{n-i-j})$ if and only if $l \equiv r(mod\ \frac{\phi(p^{n-i-j})}{2})$.

So $\sum_{s\varepsilon\Omega_{p^j}} \alpha^{4gp^is} = \sum_{t=0}^{\frac{\phi(p^{n-j})}{2}-1} \beta^{gq^t} = \frac{p^{i+j}}{p^j} \sum_{t=0}^{\frac{\phi(p^{n-i-j})}{2}-1} \beta^{gq^t} = \frac{1}{p^j}G_{i+j}$.

Similar result holds for other expressions using lemma 3.1. □

Theorem 3.8. *The expressions for primitive idempotents corresponding to Ω_{4p^j} and Ω_{8p^j} are given by*

$$P_{4p^j}(x) = \frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \overline{Z}_{p^n} + \overline{Z}_{2p^n} - \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} - \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} - \sum_{m\varepsilon M}(\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{H_{i+j}\overline{Z}_{p^i} + J_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + H_{i+j}\overline{Z}_{\lambda p^i} + J_{i+j}\overline{Z}_{\mu p^i} + G_{i+j}\overline{Z}_{gp^i} + I_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + G_{i+j}\overline{Z}_{\lambda gp^i} + I_{i+j}\overline{Z}_{\mu gp^i} + H_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mp^i} + G_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mgp^i}\}\Big]$$

$$P_{8p^j}(x) = \frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \overline{Z}_{p^n} + \overline{Z}_{2p^n} + \overline{Z}_{3p^n} + \overline{Z}_{4p^n} + \overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\overline{Z}_{p^i} + \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \overline{Z}_{\lambda p^i} + \overline{Z}_{\mu p^i} + \overline{Z}_{gp^i} + \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \overline{Z}_{\lambda gp^i} + \overline{Z}_{\mu gp^i} + \sum_{m\varepsilon M}(\overline{Z}_{mp^i} + \overline{Z}_{mgp^i})\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{J_{i+j}\overline{Z}_{p^i} + J_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + J_{i+j}\overline{Z}_{\lambda p^i} + J_{i+j}\overline{Z}_{\mu p^i} + I_{i+j}\overline{Z}_{gp^i} + I_{i+j}\overline{Z}_{2gp^i} +$$

$I_{i+j}\overline{Z}_{4gp^i}+I_{i+j}\overline{Z}_{8gp^i}+I_{i+j}\overline{Z}_{\lambda gp^i}+I_{i+j}\overline{Z}_{\mu gp^i}+J_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mp^i}+I_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mgp^i}\}]$ *where* $G_{n-1}=\frac{1}{2}(\sqrt{p^{2n-1}}+p^{n-1})$, $H_{n-1}=-\frac{1}{2}(\sqrt{p^{2n-1}}-p^{n-1})$, $I_{n-1}=\frac{1}{2}(\sqrt{p^{2n-1}}-p^{n-1})$ *and* $J_{n-1}=-\frac{1}{2}(\sqrt{p^{2n-1}}+p^{n-1})$ *for* $p\equiv(1(mod\ 4)$.
and $G_{n-1}=\frac{1}{2}(\sqrt{-p^{2n-1}}+p^{n-1})$, $H_{n-1}=-\frac{1}{2}(\sqrt{-p^{2n-1}}-p^{n-1})$, $I_{n-1}=\frac{1}{2}(\sqrt{-p^{2n-1}}-p^{n-1})$ *and* $J_{n-1}=-\frac{1}{2}(\sqrt{-p^{2n-1}}+p^{n-1})$ *for* $p\equiv(3(mod\ 4)$ *and for all* $j\leq n-2$, $G_j=H_j=I_j=J_j=0$.

Proof. To evaluate $P_{4p^j}(x)$, take $s=4p^j$ in $P_s(x)$ so we have to compute $\rho_k^{4p^j}$ for $k=0,p^n,2p^n,3p^n,4p^n,$
$6p^n,p^i,2p^i,4p^i,8p^i,\lambda p^i,\mu p^i,\nu p^i,\chi p^i,gp^i,2gp^i,4gp^i,8gp^i,\lambda gp^i,\mu gp^i,\nu gp^i,\chi gp^i$.
Since in this case $\Omega_{4p^j}=-\Omega_{4p^j}$, using lemma 3.4. So $\rho_k^{4p^j}=\sum_{s\varepsilon\Omega_{4p^j}}\alpha^{-sk}=\sum_{s\varepsilon\Omega_{p^j}}\alpha^{4ks}$.
Therefore, using lemma 3.7, we have

$$\rho_0^{4p^j}=-\rho_{p^n}^{4p^j}=\rho_{2p^n}^{4p^j}=-\rho_{3p^n}^{4p^j}=\rho_{4p^n}^{4p^j}=-\rho_{5p^n}^{4p^j}=\rho_{6p^n}^{4p^j}=-\rho_{7p^n}^{4p^j}=\frac{\phi(p^{n-j})}{2},$$

$$\rho_{p^i}^{4p^j}=\rho_{\lambda p^i}^{4p^j}=\rho_{\nu p^i}^{4p^j}=\rho_{\chi p^i}^{4p^j}=\begin{cases}-\frac{\phi(p^{n-j})}{2},\ if\ i+j\geq n,\\ \frac{1}{p^j}H_{i+j},\ if\ i+j\leq n-1.\end{cases}$$

$$\rho_{2p^i}^{4p^j}=\rho_{4p^i}^{4p^j}=\rho_{8p^i}^{4p^j}=\rho_{\mu p^i}^{4p^j}=\begin{cases}\frac{\phi(p^{n-j})}{2},\ if\ i+j\geq n,\\ \frac{1}{p^j}J_{i+j},\ if\ i+j\leq n-1.\end{cases}$$

$$\rho_{gp^i}^{4p^j}=\rho_{\lambda gp^i}^{4p^j}=\rho_{\nu gp^i}^{4p^j}=\rho_{\chi gp^i}^{4p^j}=\begin{cases}-\frac{\phi(p^{n-j})}{2},\ if\ i+j\geq n,\\ \frac{1}{p^j}G_{i+j},\ if\ i+j\leq n-1.\end{cases}$$

$$\rho_{2gp^i}^{4p^j}=\rho_{4gp^i}^{4p^j}=\rho_{8gp^i}^{4p^j}=\rho_{\mu gp^i}^{4p^j}=\begin{cases}\frac{\phi(p^{n-j})}{2},\ if\ i+j\geq n,\\ \frac{1}{p^j}I_{i+j},\ if\ i+j\leq n-1.\end{cases}$$

So, $P_{4p^j}(x)=\frac{1}{8p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0-\overline{Z}_{p^n}+\overline{Z}_{2p^n}-\overline{Z}_{3p^n}+\overline{Z}_{4p^n}+\overline{Z}_{6p^n}\}+\frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\overline{Z}_{p^i}+\overline{Z}_{2p^i}+\overline{Z}_{4p^i}+\overline{Z}_{8p^i}-\overline{Z}_{\lambda p^i}+\overline{Z}_{\mu p^i}-\overline{Z}_{gp^i}+\overline{Z}_{2gp^i}+\overline{Z}_{4gp^i}+\overline{Z}_{8gp^i}-\overline{Z}_{\lambda gp^i}+\overline{Z}_{\mu gp^i}-\sum_{m\varepsilon M}(\overline{Z}_{mp^i}+\overline{Z}_{mgp^i})\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{H_{i+j}\overline{Z}_{p^i}+J_{i+j}\overline{Z}_{2p^i}+J_{i+j}\overline{Z}_{4p^i}+J_{i+j}\overline{Z}_{8p^i}+H_{i+j}\overline{Z}_{\lambda p^i}+J_{i+j}\overline{Z}_{\mu p^i}+G_{i+j}\overline{Z}_{gp^i}+I_{i+j}\overline{Z}_{2gp^i}+I_{i+j}\overline{Z}_{4gp^i}+I_{i+j}\overline{Z}_{8gp^i}+G_{i+j}\overline{Z}_{\lambda gp^i}+I_{i+j}\overline{Z}_{\mu gp^i}+H_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mp^i}+G_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mgp^i}\}]$
Similarly using lemma 3.4 and lemma 3.7, we can obtain the expression for $P_{8p^j}(x)$. □

We can obtain the expressions for $P_{4gp^j}(x),P_{8gp^j}(x)$ by interchanging G and I by H and J respectively in the expressions of $P_{4p^j}(x),P_{8p^j}(x)$ for $p\equiv 1(mod\ 4)$. The expressions for $P_{4p^j}(x)$,$P_{8p^j}(x)$, $P_{4gp^j}(x)$ and $P_{8gp^j}(x)$ above also represents $P_{4gp^j}(x)$,$P_{8gp^j}(x)$, $P_{4p^j}(x)$ and $P_{8p^j}(x)$ respectively in case when $p \equiv 3(mod\ 4)$.

Lemma 3.9. *For* $0 \leq i \leq n$ *and* $0 \leq j \leq n-1$,

$$\sum_{s\varepsilon\Omega_{2gp^j}} \alpha^{p^i s} = \sum_{s\varepsilon\Omega_{2gp^j}} \alpha^{\nu p^i s} = \sum_{s\varepsilon\Omega_{\mu gp^j}} \alpha^{\lambda p^i s} = \begin{cases} -\frac{\phi(p^{n-j})}{2}\alpha^{2p^{i+j}},\ if\ i+j \geq n,\ p \equiv 1(mod\ 4), \\ \frac{\phi(p^{n-j})}{2}\alpha^{2p^{i+j}},\ if\ i+j \geq n,\ p \equiv 3(mod\ 4), \\ \frac{1}{p^j}E_{i+j},\ if\ i+j \leq n-1. \end{cases}$$

$$\sum_{s\varepsilon\Omega_{2p^j}} \alpha^{p^i s} = \sum_{s\varepsilon\Omega_{2p^j}} \alpha^{\nu p^i s} = \sum_{s\varepsilon\Omega_{\mu p^j}} \alpha^{\lambda p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{2}\alpha^{2p^{i+j}},\ if\ i+j \geq n, \\ \frac{1}{p^j}F_{i+j},\ if\ i+j \leq n-1. \end{cases}$$

and for $p \equiv 3(mod\ 4)$,

$$\sum_{s\varepsilon\Omega_{p^j}} \alpha^{2gp^i s} = \sum_{s\varepsilon\Omega_{\lambda p^j}} \alpha^{\mu gp^i s} = -\sum_{s\varepsilon\Omega_{p^j}} \alpha^{\mu gp^i s} = \begin{cases} \phi(p^{n-j})\alpha^{2p^{i+j}},\ if\ i+j \geq n, \\ \frac{2}{p^j}E_{i+j},\ if\ i+j \leq n-1,\ g \neq 1, \\ \frac{2}{p^j}F_{i+j},\ if\ i+j \leq n-1,\ g = 1. \end{cases}$$

Proof. Proof can be obtained on similar lines as that of lemma 3.7 and using lemmas 3.2 and 3.4. □

Theorem 3.10. *The expressions for primitive idempotents corresponding to* Ω_{2p^j} *and* $\Omega_{\mu p^j}$ *are given by*

$$P_{2p^j}(x) = \frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{2p^{n+j}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} + \alpha^{2p^{n+j}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{2p^{i+j}}\overline{Z}_{p^i} - \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} + \alpha^{2p^{i+j}}\overline{Z}_{gp^i} - \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} - \alpha^{2p^{i+j}}\sum_{m\varepsilon M}(\overline{Z}_{mp^i} - \overline{Z}_{mgp^i})\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-F_{i+j}\overline{Z}_{p^i} + H_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + F_{i+j}\overline{Z}_{\lambda p^i} + H_{i+j}\overline{Z}_{\mu p^i} - E_{i+j}\overline{Z}_{gp^i} + G_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + E_{i+j}\overline{Z}_{\lambda gp^i} + G_{i+j}\overline{Z}_{\mu gp^i} - F_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mp^i} - E_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mgp^i}\}\Big]$$

$$P_{\mu p^j}(x) = \frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{2p^{n+j}}\overline{Z}_{p^n} - \overline{Z}_{2p^n} - \alpha^{2p^{n+j}}\overline{Z}_{3p^n} + \overline{Z}_{4p^n} - \overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{2p^{i+j}}\overline{Z}_{p^i} - \overline{Z}_{2p^i} + \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\lambda p^i} - \overline{Z}_{\mu p^i} - \alpha^{2p^{i+j}}\overline{Z}_{gp^i} - \overline{Z}_{2gp^i} + \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\lambda gp^i} - \overline{Z}_{\mu gp^i} - \alpha^{2p^{i+j}}\sum_{m\varepsilon M}(\overline{Z}_{mp^i} - \overline{Z}_{mgp^i})\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{F_{i+j}\overline{Z}_{p^i} + H_{i+j}\overline{Z}_{2p^i} + J_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} - F_{i+j}\overline{Z}_{\lambda p^i} + H_{i+j}\overline{Z}_{\mu p^i} + E_{i+j}\overline{Z}_{gp^i} + G_{i+j}\overline{Z}_{2gp^i} + I_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} - E_{i+j}\overline{Z}_{\lambda gp^i} + G_{i+j}\overline{Z}_{\mu gp^i} + F_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mp^i} + E_{i+j}\sum_{m\varepsilon M}\overline{Z}_{mgp^i}\}\Big]$$

where $E_{n-1} =$

$\frac{1}{2}(\sqrt{-p^{2n-1}-2p^{2(n-1)}}+p^{n-1})$, $F_{n-1}=-\frac{1}{2}(\sqrt{-p^{2n-1}-2p^{2(n-1)}}-p^{n-1})$ *for* $p\equiv 1(mod\ 4)$. *and* $E_{n-1}=\frac{1}{2}(\sqrt{2p^{2(n-1)}+p^{2n-1}}+p^{n-1})$, $F_{n-1}=-\frac{1}{2}(\sqrt{2p^{2(n-1)}+p^{2n-1}}-p^{n-1})$ *for* $p\equiv 3(mod\ 4)$ *and for all* $j\leq n-2$, $E_j=F_j=0$.

Proof. Proof can be obtained on similar lines as that of theorem 3.8 using lemmas 3.4, 3.7 and 3.9. □

We can obtain the expressions for $P_{2gp^j}(x), P_{\mu gp^j}(x)$ by interchanging E,G and I by $-F$,H and J respectively in the expression of $P_{\mu p^j}(x), P_{2p^j}(x)$ for $p\equiv 1(mod\ 4)$.The expressions for $P_{2p^j}(x)$,$P_{\mu p^j}(x)$, $P_{2gp^j}(x)$ and $P_{\mu gp^j}(x)$ above also represents $P_{2gp^j}(x)$,$P_{\mu gp^j}(x)$, $P_{2p^j}(x)$ and $P_{\mu p^j}(x)$ respectively in case when $p\equiv 3(mod\ 4)$.

Lemma 3.11. *For* $0\leq i\leq n$ *and* $0\leq j\leq n-1$,

$$\sum_{s\varepsilon\Omega_{p^j}}\alpha^{gp^is}=\sum_{s\varepsilon\Omega_{\lambda p^j}}\alpha^{\lambda gp^is}=\sum_{s\varepsilon\Omega_{\nu p^j}}\alpha^{\nu gp^is}=\begin{cases}0\ ,\ if\ i+j\geq n,\\ \frac{1}{p^j}A_{i+j}\ ,if\ i+j\leq n-1,\ g\neq 1,\\ \frac{1}{p^j}C_{i+j}\ ,if\ i+j\leq n-1,\ g=1.\end{cases}$$

$$\sum_{s\varepsilon\Omega_{p^j}}\alpha^{\lambda gp^is}=\sum_{s\varepsilon\Omega_{\chi p^j}}\alpha^{\nu gp^is}=-\sum_{s\varepsilon\Omega_{\chi p^j}}\alpha^{gp^is}=\begin{cases}0\ ,\ if\ i+j\geq n,\\ \frac{1}{p^j}B_{i+j}\ ,if\ i+j\leq n-1,\ g\neq 1,\\ \frac{1}{p^j}D_{i+j}\ ,if\ i+j\leq n-1,\ g=1.\end{cases}$$

Proof. Proof can be obtained on similar lines as that of lemma 3.7 and using lemmas 3.2 and 3.4. □

Theorem 3.12. *The expressions for primitive idempotents corresponding to* Ω_{p^j} *and* $\Omega_{\lambda p^j}$ *are given by*

$$P_{p^j}(x)=\frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0-\alpha^{2p^{n+j}}\overline{Z}_{2p^n}-\overline{Z}_{4p^n}+\alpha^{2p^{n+j}}\overline{Z}_{6p^n}\}+\frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{2p^{i+j}}\overline{Z}_{2p^i}-\overline{Z}_{4p^i}+\overline{Z}_{8p^i}+\alpha^{2p^{i+j}}\overline{Z}_{\mu p^i}+\alpha^{2p^{i+j}}\overline{Z}_{2gp^i}-\overline{Z}_{4gp^i}+\overline{Z}_{8gp^i}-\alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i}\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\{D_{i+j}\overline{Z}_{p^i}-F_{i+j}\overline{Z}_{2p^i}+H_{i+j}\overline{Z}_{4p^i}+J_{i+j}\overline{Z}_{8p^i}+C_{i+j}\overline{Z}_{\lambda p^i}+F_{i+j}\overline{Z}_{\mu p^i}-D_{i+j}\overline{Z}_{\nu p^i}-C_{i+j}\overline{Z}_{\chi p^i}+B_{i+j}\overline{Z}_{gp^i}-E_{i+j}\overline{Z}_{2gp^i}+G_{i+j}\overline{Z}_{4gp^i}+I_{i+j}\overline{Z}_{8gp^i}+A_{i+j}\overline{Z}_{\lambda gp^i}+E_{i+j}\overline{Z}_{\mu gp^i}-B_{i+j}\overline{Z}_{\nu gp^i}-A_{i+j}\overline{Z}_{\chi gp^i}\}\Big]$$

$P_{\lambda p^j}(x) = \frac{1}{8p^n}[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \overline{Z}_{4p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{C_{i+j}\overline{Z}_{p^i} + F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} + D_{i+j}\overline{Z}_{\lambda p^i} - F_{i+j}\overline{Z}_{\mu p^i} - C_{i+j}\overline{Z}_{\nu p^i} - D_{i+j}\overline{Z}_{\chi p^i} + A_{i+j}\overline{Z}_{gp^i} + E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} + B_{i+j}\overline{Z}_{\lambda gp^i} - E_{i+j}\overline{Z}_{\mu gp^i} - A_{i+j}\overline{Z}_{\nu gp^i} - B_{i+j}\overline{Z}_{\chi gp^i}\}]$

We can obtain the expressions for $P_{\lambda gp^j}(x), P_{gp^j}(x)$ by interchanging A,B,E,G, and I by D,C,$-F$,H, and J respectively in the expression of $P_{p^j}(x), P_{\lambda p^j}(x)$ for $p \equiv 1(mod\ 4)$.

If $p \equiv 3(mod\ 4)$,

$P_{p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{Z}_0 - \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \overline{Z}_{4p^n} + \alpha^{2p^{n+j}}\overline{Z}_{6p^n}\} + \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{B_{i+j}\overline{Z}_{p^i} - 2E_{i+j}\overline{Z}_{2p^i} + 2G_{i+j}\overline{Z}_{4p^i} + 2I_{i+j}\overline{Z}_{8p^i} + A_{i+j}\overline{Z}_{\lambda p^i} + 2E_{i+j}\overline{Z}_{\mu p^i} + D_{i+j}\overline{Z}_{gp^i} - 2F_{i+j}\overline{Z}_{2gp^i} + 2H_{i+j}\overline{Z}_{4gp^i} + 2J_{i+j}\overline{Z}_{8gp^i} + C_{i+j}\overline{Z}_{\lambda gp^i} + 2F_{i+j}\overline{Z}_{\mu gp^i}\}]$

$P_{\lambda p^j}(x) = \frac{1}{8p^n}[\phi(p^{n-j})\{\overline{Z}_0 + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \overline{Z}_{4p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}\} + \phi(p^{n-j})\sum_{i=n-j}^{n-1}\{\alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} + \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{A_{i+j}\overline{Z}_{p^i} + 2E_{i+j}\overline{Z}_{2p^i} + 2G_{i+j}\overline{Z}_{4p^i} + 2I_{i+j}\overline{Z}_{8p^i} + B_{i+j}\overline{Z}_{\lambda p^i} - 2E_{i+j}\overline{Z}_{\mu p^i} + C_{i+j}\overline{Z}_{gp^i} + 2F_{i+j}\overline{Z}_{2gp^i} + 2H_{i+j}\overline{Z}_{4gp^i} + 2J_{i+j}\overline{Z}_{8gp^i} + D_{i+j}\overline{Z}_{\lambda gp^i} - 2F_{i+j}\overline{Z}_{\mu gp^i}\}]$

We can obtain the expressions for $P_{gp^j}(x), P_{\lambda gp^j}(x)$ by interchanging A,B,E,G, and I by C,D,F,H, and J respectively in the expression of $P_{p^j}(x), P_{\lambda p^j}(x)$ where $A_{n-1}, B_{n-1}C_{n-1}$ and D_{n-1} are obtained by following relations

$A_{n-1}B_{n-1} + C_{n-1}D_{n-1} = p^{2n-1}$, $A_{n-1}D_{n-1} + B_{n-1}C_{n-1} = \frac{1}{2}p^{(2n-1)} + \frac{1}{2}p^{2(n-1)}$

$A_{n-1}^2 + B_{n-1}^2 + C_{n-1}^2 + D_{n-1}^2 = 0$, $A_{n-1}C_{n-1} + B_{n-1}D_{n-1} = -\frac{1}{2}p^{(2n-1)} - \frac{1}{2}p^{2(n-1)}$, *when* $p \equiv 1(mod\ 4)$,

$A_{n-1}B_{n-1} + C_{n-1}D_{n-1} = 3p^{2(n-1)} + p^{(2n-1)} - p^n + p^{n-1}$

$A_{n-1}D_{n-1} + B_{n-1}C_{n-1} = p^{(2n-1)} - p^{2(n-1)} - p^n + p^{n-1}$

$A_{n-1}^2 + B_{n-1}^2 + C_{n-1}^2 + D_{n-1}^2 = -2p^{(2n-1)} - 6p^{2(n-1)} - 2p^n + 2p^{n-1}$

$A_{n-1}C_{n-1} + B_{n-1}D_{n-1} = -p^{2n-1} + p^{2(n-1)} + p^n - p^{n-1}$, *when* $p \equiv 3(mod\ 4)$,

and for all $j \leq n-2$, $A_j = B_j = C_j = D_j = 0$.

Proof. Proof can be obtained on similar lines as that of theorem 3.8 using lemmas 3.2, 3.4 and 3.11. $\square$

Theorem 3.13. *The expressions for primitive idempotents corresponding to $\Omega_{\nu p^j}$ and $\Omega_{\chi p^j}$ are given by*

$$P_{\nu p^j}(x) = \frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 - \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \overline{Z}_{4p^n} + \alpha^{2p^{n+j}}\overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{-\alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} + \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-D_{i+j}\overline{Z}_{p^i} - F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} - C_{i+j}\overline{Z}_{\lambda p^i} + F_{i+j}\overline{Z}_{\mu p^i} + D_{i+j}\overline{Z}_{\nu p^i} + C_{i+j}\overline{Z}_{\chi p^i} - B_{i+j}\overline{Z}_{gp^i} - E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} - A_{i+j}\overline{Z}_{\lambda gp^i} + E_{i+j}\overline{Z}_{\mu gp^i} + B_{i+j}\overline{Z}_{\nu gp^i} + A_{i+j}\overline{Z}_{\chi gp^i}\}\Big]$$

$$P_{\chi p^j}(x) = \frac{1}{8p^n}\Big[\frac{\phi(p^{n-j})}{2}\{\overline{Z}_0 + \alpha^{2p^{n+j}}\overline{Z}_{2p^n} - \overline{Z}_{4p^n} - \alpha^{2p^{n+j}}\overline{Z}_{6p^n}\} + \frac{\phi(p^{n-j})}{2}\sum_{i=n-j}^{n-1}\{\alpha^{2p^{i+j}}\overline{Z}_{2p^i} - \overline{Z}_{4p^i} + \overline{Z}_{8p^i} - \alpha^{2p^{i+j}}\overline{Z}_{\mu p^i} - \alpha^{2p^{i+j}}\overline{Z}_{2gp^i} - \overline{Z}_{4gp^i} + \overline{Z}_{8gp^i} + \alpha^{2p^{i+j}}\overline{Z}_{\mu gp^i}\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\{-C_{i+j}\overline{Z}_{p^i} + F_{i+j}\overline{Z}_{2p^i} + H_{i+j}\overline{Z}_{4p^i} + J_{i+j}\overline{Z}_{8p^i} - D_{i+j}\overline{Z}_{\lambda p^i} - F_{i+j}\overline{Z}_{\mu p^i} + C_{i+j}\overline{Z}_{\nu p^i} + D_{i+j}\overline{Z}_{\chi p^i} - A_{i+j}\overline{Z}_{gp^i} + E_{i+j}\overline{Z}_{2gp^i} + G_{i+j}\overline{Z}_{4gp^i} + I_{i+j}\overline{Z}_{8gp^i} - B_{i+j}\overline{Z}_{\lambda gp^i} - E_{i+j}\overline{Z}_{\mu gp^i} + A_{i+j}\overline{Z}_{\nu gp^i} + B_{i+j}\overline{Z}_{\chi gp^i}\}\Big]$$

Proof. Proof can be obtained on similar lines as that of theorem 3.8 using lemmas 3.2, 3.4 and 3.11. $\square$

We can obtain the expressions for $P_{\chi gp^j}(x), P_{\nu gp^j}(x)$ by interchanging A,B,E,G, and I by D,C,$-F$,H, and J respectively in the expression of $P_{\nu p^j}(x), P_{\chi p^j}(x)$ for $p \equiv 1 (mod\ 4)$.

4. Dimension and Generating Polynomials

If α is primitive $8p^n th$ root of unity in some extension field of F, then $m_s(x) = \prod_{s\in\Omega_s}(x-\alpha^s)$ denote the minimal polynomial for α^s and the generating polynomial for cyclic code M_s of length $8p^n$ corresponding to the cyclotomic coset Ω_s is $\frac{x^{8p^n}-1}{m_s(x)}$. The dimension of minimal cyclic code M_s is equal to the cardinality of the class Ω_s[11]. Thus the dimensions of the codes M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n} and M_{6p^n} are 1,2,1,2,1,1 respectively. For $p \equiv 1 (mod\ 4)$, the dimension of each M_{xp^i} is $\frac{\phi(p^{n-i})}{2}$ and for $p \equiv 3 (mod\ 4)$, dimension of each M_{yp^i} is $\phi(p^{n-i})$ and of M_{zp^i} is $\frac{\phi(p^{n-i})}{2}$.

Theorem 4.1. (i) *The generating polynomial for the codes* $M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}$ *and* M_{6p^n} *are* $(1+x+x^2+...+x^{8p^n-1})$, $(x^4-1)(x^2+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^6-x^4+x^2-1)(x+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^7-x^6+x^5-x^4+x^3-x^2+x-1)(1+x^8+...+x^{8(p^n-1)})$ *and* $(x^4-1)(x^2+\beta)(x-\beta)(1+x^8+...+x^{8(p^n-1)})$ *respectively.*
(ii) *The generating polynomial for* $M_{4p^i} \oplus M_{4gp^i}$ *and* $M_{8p^i} \oplus M_{8gp^i}$ *are* $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$ *and* $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$ *respectively.*
(iii) *For* $p \equiv 1(mod\ 4)$, *the generating polynomial for* $M_{p^i} \oplus M_{2p^i} \oplus M_{\lambda p^i} \oplus M_{\mu p^i} \oplus M_{\nu p^i} \oplus M_{\chi p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{\lambda gp^i} \oplus M_{\mu gp^i} \oplus M_{\nu gp^i} \oplus M_{\chi gp^i}$ *is* $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$.
(iv) *For* $p \equiv 3(mod\ 4)$, *the generating polynomial for* $M_{p^i} \oplus M_{2p^i} \oplus M_{\lambda p^i} \oplus M_{\mu p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{\lambda gp^i} \oplus M_{\mu gp^i}$ *is* $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$.

Proof. (i) The minimal polynomial for α^0, α^{p^n}, α^{2p^n}, α^{3p^n}, α^{4p^n} and α^{6p^n} are $(x-1)$, $(x^2-\beta)$, $(x-\beta)$, $(x^2+\beta)$, $(x+1)$ and $(x+\beta)$ respectively. The corresponding generating polynomials are $(1+x+x^2+...+x^{8p^n-1})$, $(x^4-1)(x^2+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^6-x^4+x^2-1)(x+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^4-1)(x^2+\beta)(1+x^8+...+x^{8(p^n-1)})$, $(x^7-x^6+x^5-x^4+x^3-x^2+x-1)(1+x^8+...+x^{8(p^n-1)})$ and $(x^4-1)(x^2+\beta)(x-\beta)(1+x^8+...+x^{8(p^n-1)})$.
(ii) The product of minimal polynomial satisfied by α^{4p^i} and α^{4gp^i} is $\frac{x^{p^{n-i}}+1}{x^{p^{n-i-1}}+1}$. Therefore, the generating polynomial for $M_{4p^i} \oplus M_{4gp^i}$ is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$. The product of minimal polynomial satisfied by α^{8p^i} and α^{8gp^i} is $\frac{x^{p^{n-i}}-1}{x^{p^{n-i-1}}-1}$. Therefore, the generating polynomial for $M_{8p^i} \oplus M_{8gp^i}$ is $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$.
(iii) Also the product of minimal polynomial satisfied by α^{p^i}, α^{2p^i}, α^{gp^i}, α^{2gp^i}, $\alpha^{\lambda p^i}$, $\alpha^{\lambda gp^i}$, $\alpha^{\mu p^i}$, $\alpha^{\mu gp^i}$, $\alpha^{\nu p^i}$, $\alpha^{\nu gp^i}$, $\alpha^{\chi p^i}$ and $\alpha^{\chi gp^i}$ is $\frac{x^{2p^{n-i}}+1}{x^{2p^{n-i-1}}+1}$. Therefore, the generating polynomial for $M_{p^i} \oplus M_{2p^i} \oplus M_{\lambda p^i} \oplus M_{\mu p^i} \oplus M_{\nu p^i} \oplus M_{\chi p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{\lambda gp^i} \oplus M_{\mu gp^i} \oplus M_{\nu gp^i} \oplus M_{\chi gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(x^{4p^{n-i-1}}+1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$.
(iv) Similarly as above the generating polynomial for $M_{p^i} \oplus M_{2p^i} \oplus M_{\lambda p^i} \oplus M_{\mu p^i} \oplus M_{gp^i} \oplus M_{2gp^i} \oplus M_{\lambda gp^i} \oplus M_{\mu gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+...x^{8p^{n-i(p^i-1)}})$. $\square$

5. Minimum Distance

Here, we find the minimum distance of the minimal cyclic code M_s of length $8p^n$, generated by the primitive idempotent $P_s(x)$. If l is a cyclic code of length m generated by $g(x)$ and its minimum distance is d, then the code $\bar{l}$ of length mk generated by $g(x)(1+x^m+x^{2m}+\ldots+x^{(k-1)m})$ is a repetition code of l repeated k times and its minimum distance is dk[3].

Theorem 5.1. *The codes M_0, M_{2p^n}, M_{4p^n} and M_{6p^n} are of minimum distance $8p^n$ and M_{p^n} and M_{3p^n} are of minimum distance $4p^n$.*

Proof. Since generating polynomial for the code M_0 is $(1+x+x^2+\ldots+x^{8p^n-1})$, which is itself a polynomial of length $8p^n$, hence its minimum distance is $8p^n$. Similarly, the minimum distance of each of the cyclic codes M_{2p^n}, M_{4p^n} and M_{6p^n} is $8p^n$. Also, the generating polynomial for the cyclic code M_{p^n} is $(x^4-1)(x^2+\beta)(1+x^8+\ldots+x^{8(p^n-1)})$. If we take a cyclic code of length 4 generated by the polynomial $(x^4-1)(x^2+\beta)$, then the minimal distance of this code is 4. Since the cyclic code of length $8p^n$ with generating polynomial $(x^4-1)(x^2+\beta)(1+x^8+\ldots+x^{8(p^n-1)})$ is a repetition code of the cyclic code of length 4 with generating polynomial $(x^4-1)(x^2+\beta)$, repeated p^n times. Therefore its minimum distance is $4p^n$. Similarly, the minimum distance of the cyclic code M_{3p^n} is also $4p^n$. □

Theorem 5.2. *For $0 \leq i \leq n-1$, the minimum distance of the cyclic codes $M_{4p^i}, M_{4gp^i}, M_{8p^i}$ and M_{8gp^i} are greater than or equal $16p^i$ and for the codes $M_{p^i}, M_{gp^i}, M_{2p^i}, M_{2gp^i}, M_{\lambda p^i}, M_{\lambda gp^i}, M_{\mu p^i}, M_{\mu gp^i}, M_{\nu p^i}$, $M_{\nu gp^i}, M_{\chi p^i}$ and $M_{\chi gp^i}$ are greater than or equal to $8p^i$.*

Proof. Consider the cyclic codes M_{4p^i} and M_{4gp^i}, since the generating polynomial of the cyclic code of length $8p^n$ is $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+\ldots+x^{8p^{n-i(p^i-1)}})$, therefore, if we take a cyclic code C of length p^{n-i} generated by the polynomial $(x^{p^{n-i-1}}+1)$, then the minimum distance of this code is 2. Now consider the cyclic code C_1 of length $2p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)$ and so minimum distance of this code is 4, as it is 2 time repetition of the code C. Further, the minimum distance of the code C_2 of length $4p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)$ is

8, as it is 2 time repetition of the code C_1. Hence the minimum distance of the code C_3 of length $8p^{n-i}$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)$ and so minimum distance of this code is 16. Since the cyclic code of length $8p^n$ generated by the polynomial $(x^{p^{n-i-1}}+1)(x^{p^{n-i}}-1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$ is a repetition code of the code C_3, repeated p^i times. Hence its minimum distance is $16p^i$. The codes corresponding to M_{4p^i} and M_{4gp^i} are the sub codes of the above codes, so their minimum distances are greater than or equal to $16p^i$. Similarly, the minimum distance of the cyclic code M_{8p^i} and M_{8gp^i} of length $8p^n$ with generating polynomial $(x^{p^{n-i-1}}-1)(x^{p^{n-i}}+1)(x^{2p^{n-i}}+1)(x^{4p^{n-i}}+1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^i-1)}})$ is also greater than or equal to $16p^i$.

Now, the product of generating polynomial for the cyclic codes $M_{p^i}, M_{gp^i}, M_{2p^i}, M_{2gp^i}, M_{\lambda p^i}, M_{\lambda gp^i}, M_{\mu p^i}$, $M_{\mu gp^i}, M_{\nu p^i}, M_{\nu gp^i}, M_{\chi p^i}$ and $M_{\chi gp^i}$ is $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+ \dots x^{8p^{n-i(p^i-1)}})$,

therefore, if we take a code C of length $8p^{n-i}$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)$, then the minimum distance of this code is 8. Since the cyclic code C_1 of length $8p^n$ generated by the polynomial $(x^{2p^{n-i-1}}+1)(x^{4p^{n-i-1}}+1)(x^{2p^{n-i}}-1)(1+x^{8p^{n-i}}+...+x^{8p^{n-i(p^j-1)}})$ is a repetition code of the code C, repeated p^i times. Hence its minimum distance is $8p^i$.

The codes corresponding to Ω_{p^i}, Ω_{gp^i},Ω_{2p^i}, Ω_{2gp^i}, $\Omega_{\lambda p^i}$, $\Omega_{\lambda gp^i}$, $\Omega_{\mu p^i}$, $\Omega_{\mu gp^i}$, $\Omega_{\nu p^i}$, $\Omega_{\nu gp^i}$, $\Omega_{\chi p^i}$ and $\Omega_{\chi gp^i}$ are the subcodes of above codes so, their minimum distances are greater than or equal to $8p^i$. □

6. Example

Example 6.1. Cyclic Codes of length 40.

Take $p=5$, $n=1$, $q=29$. The q-cyclotomic cosets are

$\Omega_0=\{0\}$, $\Omega_1=\{1,29\}$, $\Omega_2=\{2,18\}$, $\Omega_3=\{3,7\}$, $\Omega_4=\{4,36\}$, $\Omega_5=\{5,25\}$, $\Omega_6=\{6,14\}$, $\Omega_8=\{8,32\}$, $\Omega_9=\{9,21\}$, $\Omega_{10}=\{10\}$, $\Omega_{11}=\{11,39\}$, $\Omega_{12}=\{12,28\}$, $\Omega_{13}=\{13,17\}$, $\Omega_{15}=\{15,35\}$, $\Omega_{16}=\{16,24\}$, $\Omega_{19}=\{19,31\}$, $\Omega_{20}=\{20\}$, $\Omega_{22}=\{22,38\}$, $\Omega_{23}=\{23,27\}$,

$\Omega_{26} = \{26,34\}$, $\Omega_{30} = \{30\}$, $\Omega_{33} = \{33,37\}$,

and the corresponding primitive idempotents in $\frac{GF(29)[x]}{<x^{40}-1>}$ are

$P_0(x) = \frac{1}{40}[\overline{Z}_0+\overline{Z}_5+\overline{Z}_{10}+\overline{Z}_{15}+\overline{Z}_{20}+\overline{Z}_{30}+\overline{Z}_1+\overline{Z}_2+\overline{Z}_4+\overline{Z}_8+\overline{Z}_{11}+\overline{Z}_{22}+\overline{Z}_{21}+\overline{Z}_{31}+\overline{Z}_3+\overline{Z}_6$
$+\overline{Z}_{12}+\overline{Z}_{24}+\overline{Z}_{33}+\overline{Z}_{26}+\overline{Z}_{23}+\overline{Z}_{13}]$

$P_1(x) = \frac{1}{40}[2\overline{Z}_0+5\overline{Z}_{10}-2\overline{Z}_{20}-5\overline{Z}_{30}-2\overline{Z}_1-8\overline{Z}_2-5\overline{Z}_4-6\overline{Z}_8-5\overline{Z}_{11}+8\overline{Z}_{22}+2\overline{Z}_{21}+5\overline{Z}_{31}+2\overline{Z}_3$
$+7\overline{Z}_6+6\overline{Z}_{12}+5\overline{Z}_{24}+5\overline{Z}_{33}-7\overline{Z}_{26}-2\overline{Z}_{23}-5\overline{Z}_{13}]$

$P_2(x) = \frac{1}{40}[2\overline{Z}_0+5\overline{Z}_5-2\overline{Z}_{10}-5\overline{Z}_{15}+2\overline{Z}_{20}-2\overline{Z}_{30}-8\overline{Z}_1-5\overline{Z}_2-6\overline{Z}_4-6\overline{Z}_8+8\overline{Z}_{11}-5\overline{Z}_{22}-8\overline{Z}_{21}$
$+8\overline{Z}_{31}+7\overline{Z}_3+6\overline{Z}_6+5\overline{Z}_{12}+5\overline{Z}_{24}-7\overline{Z}_{33}+6\overline{Z}_{26}+7\overline{Z}_{23}-7\overline{Z}_{13}]$

$P_3(x) = \frac{1}{40}[2\overline{Z}_0-5\overline{Z}_{10}-2\overline{Z}_{20}+5\overline{Z}_{30}+2\overline{Z}_1+7\overline{Z}_2+6\overline{Z}_4+5\overline{Z}_8+5\overline{Z}_{11}-7\overline{Z}_{22}-2\overline{Z}_{21}-5\overline{Z}_{31}-2\overline{Z}_3$
$-8\overline{Z}_6-5\overline{Z}_{12}-6\overline{Z}_{24}-5\overline{Z}_{33}+8\overline{Z}_{26}+2\overline{Z}_{23}+5\overline{Z}_{13}]$

$P_4(x) = \frac{1}{40}[2\overline{Z}_0-2\overline{Z}_5+2\overline{Z}_{10}-2\overline{Z}_{15}+2\overline{Z}_{20}+2\overline{Z}_{30}-5\overline{Z}_1-6\overline{Z}_2-6\overline{Z}_4-6\overline{Z}_8-5\overline{C}_{11}-6\overline{Z}_{22}-5\overline{Z}_{21}$
$-5\overline{Z}_{31}+6\overline{Z}_3+5\overline{Z}_6+5\overline{Z}_{12}+5\overline{Z}_{24}+6\overline{Z}_{33}+5\overline{Z}_{26}+6\overline{Z}_{23}+6\overline{Z}_{13}]$

$P_5(x) = \frac{1}{40}[2\overline{Z}_0+5\overline{Z}_{10}-2\overline{Z}_{20}-5\overline{Z}_{30}+5\overline{Z}_2-2\overline{Z}_4+2\overline{Z}_8-5\overline{Z}_{22}-5\overline{Z}_6-2\overline{Z}_{12}+2\overline{Z}_{24}+5\overline{Z}_{26}]$

$P_6(x) = \frac{1}{40}[2\overline{Z}_0-5\overline{Z}_5-2\overline{Z}_{10}+5\overline{Z}_{15}+2\overline{Z}_{20}-2\overline{Z}_{30}+7\overline{Z}_1+6\overline{Z}_2+5\overline{Z}_4+5\overline{Z}_8-7\overline{Z}_{11}+6\overline{Z}_{22}+7\overline{Z}_{21}$
$-7\overline{Z}_{31}-8\overline{Z}_3-5\overline{Z}_6+5\overline{Z}_{12}+5\overline{Z}_{24}+8\overline{Z}_{33}-5\overline{Z}_{26}-8\overline{Z}_{23}+8\overline{Z}_{13}]$

$P_8(x) = \frac{1}{40}[2\overline{Z}_0+2\overline{Z}_5+2\overline{Z}_{10}+2\overline{Z}_{15}+2\overline{Z}_{20}+2\overline{Z}_{30}-6\overline{Z}_1-6\overline{Z}_2-6\overline{Z}_4-6\overline{Z}_8-6\overline{Z}_{11}-6\overline{Z}_{22}-6\overline{Z}_{21}$
$-6\overline{Z}_{31}+5\overline{Z}_3+5\overline{Z}_6+5\overline{Z}_{12}+5\overline{Z}_{24}+5\overline{Z}_{33}+5\overline{Z}_{26}+5\overline{Z}_{23}+5\overline{Z}_{13}]$

$P_{10}(x) = \frac{1}{40}[\overline{Z}_0-12\overline{Z}_5-\overline{Z}_{10}+12\overline{Z}_{15}+\overline{Z}_{20}-\overline{Z}_{30}-12\overline{Z}_1-\overline{Z}_2+\overline{Z}_4+\overline{Z}_8+12\overline{Z}_{11}-\overline{Z}_{22}-12\overline{Z}_{21}$
$-12\overline{Z}_{31}+12\overline{Z}_3-\overline{Z}_6+\overline{Z}_{12}+\overline{Z}_{24}-12\overline{Z}_{33}-\overline{Z}_{26}+12\overline{Z}_{23}+12\overline{Z}_{13}]$

$P_{11}(x) = \frac{1}{40}[2\overline{Z}_0-5\overline{Z}_{10}-2\overline{Z}_{20}+5\overline{Z}_{30}-5\overline{Z}_1+8\overline{Z}_2-5\overline{Z}_4-6\overline{Z}_8-2\overline{Z}_{11}-8\overline{Z}_{22}+5\overline{Z}_{21}+2\overline{Z}_{31}+5\overline{Z}_3$
$-7\overline{Z}_6+6\overline{Z}_{12}+5\overline{Z}_{24}+2\overline{Z}_{33}+7\overline{Z}_{26}-5\overline{Z}_{23}-2\overline{Z}_{13}]$

$P_{12}(x) = \frac{1}{40}[2\overline{Z}_0-2\overline{Z}_5+2\overline{Z}_{10}-2\overline{Z}_{15}+2\overline{Z}_{20}+2\overline{Z}_{30}+6\overline{Z}_1+5\overline{Z}_2+5\overline{Z}_4+5\overline{Z}_8+6\overline{Z}_{11}+5\overline{Z}_{22}+6\overline{Z}_{21}$
$+6\overline{Z}_{31}-5\overline{Z}_3-6\overline{Z}_6-6\overline{Z}_{12}-6\overline{Z}_{24}-5\overline{Z}_{33}-6\overline{Z}_{26}-5\overline{Z}_{23}-5\overline{Z}_{13}]$

$P_{13}(x) = \frac{1}{40}[2\overline{Z}_0+5\overline{Z}_{10}-2\overline{Z}_{20}-5\overline{Z}_{30}-5\overline{Z}_1-7\overline{Z}_2+6\overline{Z}_4+5\overline{Z}_8-2\overline{Z}_{11}+7\overline{Z}_{22}+5\overline{Z}_{21}+2\overline{Z}_{31}+5\overline{Z}_3$
$+8\overline{Z}_6-5\overline{Z}_{12}-6\overline{Z}_{24}+2\overline{Z}_{33}-8\overline{Z}_{26}-5\overline{Z}_{23}-2\overline{Z}_{13}]$

$P_{15}(x) = \frac{1}{40}[2\overline{Z}_0-5\overline{Z}_{10}-2\overline{Z}_{20}+5\overline{Z}_{30}-5\overline{Z}_2-2\overline{Z}_4+2\overline{Z}_8+5\overline{Z}_{22}+5\overline{Z}_6-2\overline{Z}_{12}+2\overline{Z}_{24}-5\overline{Z}_{26}]$

$P_{20}(x) = \frac{1}{40}[\overline{Z}_0-\overline{Z}_5+\overline{Z}_{10}-\overline{Z}_{15}+\overline{Z}_{20}+\overline{Z}_{30}-\overline{Z}_1+\overline{Z}_2+\overline{Z}_4+\overline{Z}_8-\overline{Z}_{11}+\overline{Z}_{22}-\overline{Z}_{21}-\overline{Z}_{31}-\overline{Z}_3+$
$\overline{Z}_6+\overline{Z}_{12}+\overline{Z}_{24}-\overline{Z}_{33}+\overline{Z}_{26}-\overline{Z}_{23}-\overline{Z}_{13}]$

$P_{21}(x) = \frac{1}{40}[2\overline{Z}_0+5\overline{Z}_{10}+2\overline{Z}_{20}-5\overline{Z}_{30}+2\overline{Z}_1-8\overline{Z}_2-5\overline{Z}_4-6\overline{Z}_8+5\overline{Z}_{11}+8\overline{Z}_{22}-2\overline{Z}_{21}-5\overline{Z}_{31}-2\overline{Z}_3$
$+7\overline{Z}_6+6\overline{Z}_{12}+5\overline{Z}_{24}-5\overline{Z}_{33}-7\overline{Z}_{26}+2\overline{Z}_{23}+5\overline{Z}_{13}]$

$P_{22}(x) = \frac{1}{40}[2\overline{Z}_0 - 5\overline{Z}_5 - 2\overline{Z}_{10} + 5\overline{Z}_{15} + 2\overline{Z}_{20} - 2\overline{Z}_{30} + 8\overline{Z}_1 - 5\overline{Z}_2 - 6\overline{Z}_4 - 6\overline{Z}_8 - 8\overline{Z}_{11} - 5\overline{Z}_{22} + 8\overline{Z}_{21}$
$-8\overline{Z}_{31} - 7\overline{Z}_3 + 6\overline{Z}_6 + 5\overline{Z}_{12} + 5\overline{Z}_{24} + 7\overline{Z}_{33} + 6\overline{Z}_{26} - 7\overline{Z}_{23} + 7\overline{Z}_{13}]$

$P_{23}(x) = \frac{1}{40}[2\overline{Z}_0 - 5\overline{Z}_{10} - 2\overline{Z}_{20} + 5\overline{Z}_{30} - 2\overline{Z}_1 + 7\overline{Z}_2 + 6\overline{Z}_4 + 5\overline{Z}_8 - 5\overline{Z}_{11} - 7\overline{Z}_{22} + 2\overline{Z}_{21} + 5\overline{Z}_{31} + 2\overline{Z}_3$
$-8\overline{Z}_6 - 5\overline{Z}_{12} - 6\overline{Z}_{24} + 5\overline{Z}_{33} + 8\overline{Z}_{26} - 2\overline{Z}_{23} - 5\overline{Z}_{13}]$

$P_{24}(x) = \frac{1}{40}[2\overline{Z}_0 + 2\overline{Z}_5 + 2\overline{Z}_{10} + 2\overline{Z}_{15} + 2\overline{Z}_{20} + 2\overline{Z}_{30} + 5\overline{Z}_1 + 5\overline{Z}_2 + 5\overline{Z}_4 + 5\overline{Z}_8 + 5\overline{Z}_{11} + 5\overline{Z}_{22} + 5\overline{Z}_{21}$
$+5\overline{Z}_{31} - 6\overline{Z}_3 - 6\overline{Z}_6 - 6\overline{Z}_{12} - 6\overline{Z}_{24} - 6\overline{Z}_{33} - 6\overline{Z}_{26} - 6\overline{Z}_{23} - 6\overline{Z}_{13}]$

$P_{26}(x) = \frac{1}{40}[2\overline{Z}_0 + 5\overline{Z}_5 - 2\overline{Z}_{10} - 5\overline{Z}_{15} + 2\overline{Z}_{20} - 2\overline{Z}_{30} - 7\overline{Z}_1 + 6\overline{Z}_2 + 5\overline{Z}_4 + 5\overline{Z}_8 + 7\overline{Z}_{11} + 6\overline{Z}_{22} - 7\overline{Z}_{21}$
$+7\overline{Z}_{31} + 8\overline{Z}_3 - 5\overline{Z}_6 - 6\overline{Z}_{12} - 6\overline{Z}_{24} - 8\overline{Z}_{33} - 5\overline{Z}_{26} + 8\overline{Z}_{23} - 8\overline{Z}_{13}]$

$P_{30}(x) = \frac{1}{40}[\overline{Z}_0 + 12\overline{Z}_5 - \overline{Z}_{10} - 12\overline{Z}_{15} + \overline{Z}_{20} - \overline{Z}_{30} + 12\overline{Z}_1 + \overline{Z}_2 + \overline{Z}_4 + \overline{Z}_8 - 12\overline{Z}_{11} - \overline{Z}_{22} + 12\overline{Z}_{21}$
$-12\overline{Z}_{31} - 12\overline{Z}_3 - \overline{Z}_6 + \overline{Z}_{12} + \overline{Z}_{24} + 12\overline{Z}_{33} - \overline{Z}_{26} - 12\overline{Z}_{23} + 12\overline{Z}_{13}]$

$P_{31}(x) = \frac{1}{40}[2\overline{Z}_0 - 5\overline{Z}_{10} - 2\overline{Z}_{20} + 5\overline{Z}_{30} + 5\overline{Z}_1 + 8\overline{Z}_2 - 5\overline{Z}_4 - 6\overline{Z}_8 + 2\overline{Z}_{11} - 8\overline{Z}_{22} - 5\overline{Z}_{21} - 2\overline{Z}_{31} - 5\overline{Z}_3$
$-7\overline{Z}_6 + 6\overline{Z}_{12} + 5\overline{Z}_{24} - 2\overline{Z}_{33} + 7\overline{Z}_{26} + 5\overline{Z}_{23} + 2\overline{Z}_{13}]$

$P_{33}(x) = \frac{1}{40}[2\overline{Z}_0 + 5\overline{Z}_{10} - 2\overline{Z}_{20} - 5\overline{Z}_{30} + 5\overline{Z}_1 - 7\overline{Z}_2 + 6\overline{Z}_4 + 5\overline{Z}_8 + 2\overline{Z}_{11} + 7\overline{Z}_{22} - 5\overline{Z}_{21} - 2\overline{Z}_{31} - 5\overline{Z}_3$
$+8\overline{Z}_6 - 5\overline{Z}_{12} - 6\overline{Z}_{24} - 2\overline{Z}_{33} - 8\overline{Z}_{26} + 5\overline{Z}_{23} + 2\overline{Z}_{13}]$

Minimal polynomials for $\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^8, \alpha^{10}, \alpha^{11}, \alpha^{12}, \alpha^{13}, \alpha^{15}, \alpha^{20}, \alpha^{21}, \alpha^{22}, \alpha^{23}, \alpha^{24}, \alpha^{26}, \alpha^{30}, \alpha^{31}$ and α^{33} are $x-1$, x^2-12, x^2+5x-1, x^2+12, x^2+2x+1, x^2-12, x^2-5x-1, x^2-2x+1, $x-12$, $x-12$, x^2+12, x^2+2x+1, x^2+12, $x+1$, x^2-2x+1, x^2+12, $x+1$, x^2-5x-1, x^2+12, $x+12$, $x-17$ and x^2-12 respectively.

The minimal codes M_0, M_1, M_2, M_3, M_4, M_5, M_6, M_8, M_{10}, M_{11}, M_{12}, M_{13}, M_{15}, M_{20}, M_{21}, M_{22}, M_{23}, M_{24}, M_{26}, M_{30}, M_{31} and M_{33} of length 40 are as follows

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_0	1	40	$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}+x^{18}+x^{19}+x^{20}+x^{21}+x^{22}+x^{23}+x^{24}+x^{25}+x^{26}+x^{27}+x^{28}+x^{29}+x^{30}+x^{31}+x^{32}+x^{33}+x^{34}+x^{35}+x^{36}+x^{37}+x^{38}+x^{39}$
M_1	2	$8 \leq d \leq 40$	$17+28x^2+12x^4+x^6+17x^8+28x^{10}+12x^{12}+x^{14}+17x^{16}+28x^{18}+12x^{20}+x^{22}+17x^{24}+28x^{26}+12x^{28}+x^{30}+17x^{32}+28x^{34}+12x^{36}+x^{38}$
M_2	2	$8 \leq d \leq 40$	$19+21x+8x^2+3x^3+23x^4+2x^5+4x^6+22x^7+27x^8+12x^9+12x^{11}+2x^{12}+22x^{13}+25x^{14}+2x^{15}+6x^{16}+3x^{17}+21x^{18}+21x^{19}+10x^{20}+13x^{21}+17x^{22}+11x^{23}+14x^{24}+23x^{25}+13x^{26}+x^{27}+18x^{28}+4x^{29}+9x^{30}+20x^{31}+22x^{32}+14x^{33}+5x^{34}+10x^{35}+26x^{36}+24x^{37}+x^{38}$
M_3	2	$8 \leq d \leq 40$	$12+28x^2+17x^4+x^6+12x^8+28x^{10}+17x^{12}+x^{14}+12x^{16}+28x^{18}+17x^{20}+x^{22}+12x^{24}+28x^{26}+17x^{28}+x^{30}+12x^{32}+28x^{34}+17x^{36}+x^{38}$
M_4	2	$16 \leq d \leq 40$	$10+20x+8x^2+22x^3+6x^4+24x^5+4x^6+26x^7+2x^8+28x^9+x^{11}+27x^{12}+3x^{13}+25x^{14}+5x^{15}+23x^{16}+7x^{17}+21x^{18}+9x^{19}+19x^{20}+11x^{21}+17x^{22}+13x^{23}+15x^{24}+15x^{25}+13x^{26}+17x^{27}+11x^{28}+19x^{29}+9x^{30}+21x^{31}+7x^{32}+23x^{33}+5x^{34}+25x^{35}+3x^{36}+27x^{37}+x^{38}$
M_5	2	20	$17+28x^2+12x^4+x^6+17x^8+28x^{10}+12x^{12}+x^{14}+17x^{16}+28x^{18}+12x^{20}+x^{22}+17x^{24}+28x^{26}+12x^{28}+x^{30}+17x^{32}+28x^{34}+12x^{36}+x^{38}$

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_6	2	$8 \leq d \leq 40$	$19+8x+8x^2+26x^3+23x^4+27x^5+4x^6+7x^7+27x^8+17x^9+17x^{11}+2x^{12}+7x^{13}+25x^{14}+27x^{15}+6x^{16}+26x^{17}+21x^{18}+8x^{19}+10x^{20}+16x^{21}+17x^{22}+18x^{23}+14x^{24}+6x^{25}+13x^{26}+28x^{27}+18x^{28}+25x^{29}+9x^{30}+9x^{31}+22x^{32}+15x^{33}+5x^{34}+19x^{35}+26x^{36}+5x^{37}+x^{38}$
M_8	2	$16 \leq d \leq 40$	$10+9x+8x^2+7x^3+6x^4+5x^5+4x^6+3x^7+2x^8+x^9+28x^{11}+27x^{12}+26x^{13}+25x^{14}+24x^{15}+23x^{16}+22x^{17}+21x^{18}+20x^{19}+19x^{20}+18x^{21}+17x^{22}+16x^{23}+15x^{24}+14x^{25}+13x^{26}+12x^{27}+11x^{28}+10x^{29}+9x^{30}+8x^{31}+7x^{32}+6x^{33}+5x^{34}+4x^{35}+3x^{36}+2x^{37}+x^{38}$
M_{10}	1	40	$17+28x+12x^2+x^3+17x^4+28x^5+12x^6+x^7+17x^8+28x^9+12x^{10}+x^{11}+17x^{12}+28x^{13}+12x^{14}+x^{15}+17x^{16}+28x^{17}+12x^{18}+x^{19}+17x^{20}+28x^{21}+12x^{22}+x^{23}+17x^{24}+28x^{25}+12x^{26}+x^{27}+17x^{28}+28x^{29}+12x^{30}+x^{31}+17x^{32}+28x^{33}+12x^{34}+x^{35}+17x^{36}+28x^{37}+12x^{38}+x^{39}$
M_{11}	2	$8 \leq d \leq 40$	$17+28x+12x^2+x^3+17x^4+28x^5+12x^6+x^7+17x^8+28x^9+12x^{10}+x^{11}+17x^{12}+28x^{13}+12x^{14}+x^{15}+17x^{16}+28x^{30}+x^{31}+17x^{32}+28x^{33}+12x^{34}+x^{35}+17x^{36}+28x^{37}+12x^{38}+x^{39}$
M_{12}	2	$16 \leq d \leq 40$	$12+28x^2+17x^4+x^6+12x^8+28x^{10}+17x^{12}+x^{14}+12x^{16}+28x^{18}+17x^{20}+x^{22}+12x^{24}+28x^{26}+17x^{28}+x^{30}+12x^{32}+28x^{34}+17x^{36}+x^{38}$

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_{13}	2	$8 \leq d \leq 40$	$10+20x+8x^2+22x^3+6x^4+24x^5+4x^6+26x^7+2x^8+28x^9+x^{11}+27x^{12}+3x^{13}+25x^{14}+5x^{15}+23x^{16}+7x^{17}+21x^{18}+9x^{19}+19x^{20}+11x^{21}+17x^{22}+13x^{23}+15x^{24}+15x^{25}+13x^{26}+17x^{27}+11x^{28}+19x^{29}+9x^{30}+21x^{31}+7x^{32}+23x^{33}+5x^{34}+25x^{35}+3x^{36}+27x^{37}+x^{38}$
M_{15}	2	20	$12+28x^2+17x^4+x^6+12x^8+28x^{10}+17x^{12}+x^{14}+12x^{16}+28x^{18}+17x^{20}+x^{22}+12x^{24}+28x^{26}+17x^{28}+x^{30}+12x^{32}+28x^{34}+17x^{36}+x^{38}$
M_{20}	1	40	$-1+x-x^2+x^3-x^4+x^5-x^6+x^7-x^8+x^9-x^{10}+x^{11}-x^{12}+x^{13}-x^{14}+x^{15}-x^{16}+x^{17}-x^{18}+x^{19}-x^{20}+x^{21}-x^{22}+x^{23}-x^{24}+x^{25}-x^{26}+x^{27}-x^{28}+x^{29}-x^{30}+x^{31}-x^{32}+x^{33}-x^{34}+x^{35}-x^{36}+x^{37}-x^{38}+x^{39}$
M_{21}	2	$8 \leq d \leq 40$	$10+9x+8x^2+7x^3+6x^4+5x^5+4x^6+3x^7+2x^8+x^9+28x^{11}+27x^{12}+26x^{13}+25x^{14}+24x^{15}+23x^{16}+22x^{17}+21x^{18}+20x^{19}+19x^{20}+18x^{21}+17x^{22}+16x^{23}+15x^{24}+14x^{25}+13x^{26}+12x^{27}+11x^{28}+10x^{29}+9x^{30}+8x^{31}+7x^{32}+6x^{33}+5x^{34}+4x^{35}+3x^{36}+2x^{37}+x^{38}$
M_{22}	2	$8 \leq d \leq 40$	$12+28x^2+17x^4+x^6+12x^8+28x^{10}+17x^{12}+x^{14}+12x^{16}+28x^{18}+17x^{20}+x^{22}+12x^{24}+28x^{26}+17x^{28}+x^{30}+12x^{32}+28x^{34}+17x^{36}+x^{38}$
M_{23}	2	$8 \leq d \leq 40$	$-1+x-x^2+x^3-x^4+x^5-x^6+x^7-x^8+x^9-x^{10}+x^{11}-x^{12}+x^{13}-x^{14}+x^{15}-x^{16}+x^{17}-x^{18}+x^{19}-x^{20}+x^{21}-x^{22}+x^{23}-x^{24}+x^{25}-x^{26}+x^{27}-x^{28}+x^{29}-x^{30}+x^{31}-x^{32}+x^{33}-x^{34}+x^{35}-x^{36}+x^{37}-x^{38}+x^{39}$

Code	Dim.	Min. Distance Bound	Generating Polynomial
M_{24}	2	$16 \leq d \leq 40$	$19+8x+8x^2+26x^3+23x^4+27x^5+4x^6+7x^7+27x^8+17x^9+17x^{11}+2x^{12}+7x^{13}+25x^{14}+27x^{15}+6x^{16}+26x^{17}+21x^{18}+8x^{19}+10x^{20}+16x^{21}+17x^{22}+18x^{23}+14x^{24}+6x^{25}+13x^{26}+28x^{27}+18x^{28}+25x^{29}+9x^{30}+9x^{31}+22x^{32}+15x^{33}+5x^{34}+19x^{35}+26x^{36}+5x^{37}+x^{38}$
M_{26}	2	$8 \leq d \leq 40$	$12+28x^2+17x^4+x^6+12x^8+28x^{10}+17x^{12}+x^{14}+12x^{16}+28x^{18}+17x^{20}+x^{22}+12x^{24}+28x^{26}+17x^{28}+x^{30}+12x^{32}+28x^{34}+17x^{36}+x^{38}$
M_{30}	1	40	$12+28x+17x^2+x^3+12x^4+28x^5+17x^6+x^7+12x^8+28x^9+17x^{10}+x^{11}+12x^{12}+28x^{13}+17x^{14}+x^{15}+12x^{16}+28x^{17}+17x^{18}+x^{19}+12x^{20}+28x^{21}+17x^{22}+x^{23}+12x^{24}+28x^{25}+17x^{26}+x^{27}+12x^{28}+28x^{29}+17x^{30}+x^{31}+12x^{32}+28x^{33}+17x^{34}+x^{35}+12x^{36}+28x^{37}+17x^{38}+x^{39}$
M_{31}	2	$8 \leq d \leq 40$	$12-x+17x^2+x^3+12x^4-x^5+17x^6+x^7+12x^8+28x^9+17x^{10}+x^{11}+12x^{12}-x^{13}+17x^{14}+x^{15}+12x^{16}-x^{17}+17x^{18}+x^{19}+12x^{20}-x^{21}+17x^{22}+x^{23}+12x^{24}-x^{25}+17x^{26}+x^{27}+12x^{28}-x^{29}+17x^{30}+x^{31}+12x^{32}-x^{33}+17x^{34}+x^{35}+12x^{36}-x^{37}+17x^{38}+x^{39}$
M_{33}	2	$8 \leq d \leq 40$	$17+28x^2+12x^4+x^6+17x^8+28x^{10}+12x^{12}+x^{14}+17x^{16}+28x^{18}+12x^{20}+x^{22}+17x^{24}+28x^{26}+12x^{28}+x^{30}+17x^{32}+28x^{34}+12x^{36}+x^{38}$

Conflict of Interests

The authors declare that there is no conflict of interests.

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