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SOLUTION OF CONVECTION-DIFFUSION PROBLEMS USING FOURTH ORDER ADAPTIVE CUBIC SPLINE METHOD

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Abstract: In this paper, using adaptive cubic spline, we have suggested a numerical scheme for solving a convection-diffusion problem having layer structure. The numerical scheme is derived with this spline and non-standard finite differences of the first derivative. The tridiagonal solver is used to solve the system of the numerical method. The analysis of convergence of the method is briefly discussed and the fourth order is shown. The numerical results of the examples were tabulated and compared to the existing computational results in order to support the higher accuracy of the proposed numerical scheme.

Keywords: singular perturbation; adaptive cubic spline; convection-diffusion problems; convergence.

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1. INTRODUCTION

It is well known that many physical problems with many small parameters often involve the

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solution of boundary value problems. This paper deals with convection-diffusion boundary value problems involving small parameter. These problems are characterized by the inclusion of a small perturbation parameters ε which multiply the second order derivative. In many fields of engineering and science, such types of problems exist such as chemical reactor theory, transport phenomena in chemistry, lubrication theory and biology.

A broad verity of books has been found in the literature for the convection – diffusion problems or singular perturbation problems (SPPs) [2,3,4,10,14]. One can refer a book on splines by Micula [9]. The survey papers [6, 8] provides a detailed research work on SPP problems. In [1], the authors suggested a difference schemes of second and fourth order based on cubic spline in compression for SPP. A variable-mesh second-order difference scheme via cubic splines is proposed to solve SPP in [5]. In the paper [7], authors used the artificial viscosity in B-spline collocation method to capture the layer behaviour of the problem. Phaneendra and Lalu [12] derived numerical scheme using Gaussian quadrature for the solution of SPP with one end layer, dual layer and internal layer. The authors in [13] extended the Numerov scheme to the SPP with first order derivative. Soujanya et al. [15] introduced a scheme having a fitting factor in Dahlquist scheme to get the solution of SPP having dual layers. Uniform difference schemes based on a class of splines are proposed by Stojanovic [16] for the solution of non-self-adjoint SPP.

In this paper, we present a fourth order finite difference method using adaptive cubic spline to solve singularly perturbed boundary value problems. We introduce a new parameter η in the difference scheme to achieve fourth order convergence for the proposed problem. The paper is organized as follows: In section 2, Description of the problem along with conditions for layer behavior is given. In section 3, we define the adaptive spline function. In section 4, we describe the numerical method for solving singularly perturbed boundary value problems, in Section 5, the truncation error and classification of various orders of the proposed method are given. In section 6, we discuss the convergence analysis of the method. Finally, numerical results and comparison with other methods are given in section 7.

2. DESCRIPTION OF THE METHOD

To describe the method, we considered convection-diffusion boundary value problem of the type:

$$\varepsilon w''(s) + P(s)w'(s) + Q(s)w(s) = R(s), \tag{1}$$

with boundary conditions

$$w(a) = \alpha, \quad w(b) = \beta \tag{2}$$

with perturbation parameter $0 < \varepsilon << 1$. The functions $P(s), Q(s), R(s)$ and are assumed to be sufficiently smooth functions in $[a, b]$, and α, β are finite constants. The layer exists in the vicinity of $s = a$, if $P(s) \geq L > 0$ all through the domain $[a, b]$, where L is positive constant. If $\overline{L}$ is a negative constant such that such that $P(s) \leq \overline{L} < 0$ over the domain$[a, b]$, then the layer is in the range of $s = b$.

3. ADAPTIVE SPLINE

With grid points s_i in $[a, b]$, consider the mesh such that $\Omega: a = s_0 < s_1 <\cdots< s_n = b$, where $h = s_i - s_{i-1}$ for $i = 1,2 \ldots, N$. A function $\psi(s,\tau)$ interpolate $w(s)$ at the grid points s_i, depend on a variable τ , leads to cubic spline $\psi(s)$ in $[a, b]$ as $\tau \to 0$is named as adaptive spline function. Following [4], If $\psi(s,\tau)$ is an adaptive spline function then

$$\varepsilon\, \psi''(s,\tau) - p\, \psi'(s,\tau) = \tfrac{s-s_{i-1}}{h}(\varepsilon M_i - p m_i) + \tfrac{s_i - s}{h}(\varepsilon M_{i-1} - p m_{i-1}) \tag{5}$$

where $s_{i-1} \leq s \leq s_i$,ε and $\psi'(s,\tau) = m_i$, $\psi''(s,\tau) = M_i$. Solving Eq. (5) and using the interpolatory conditions $\psi(s_{i-1},\tau) = w_{i-1}$, $\psi(s_i,\tau) = w_i$, we have

$$\psi(s,\tau) = A_i + B_i e^{\tau z} - \tfrac{h^2}{\tau^3}\left[\tfrac{1}{2}\tau^2 z^2 + \tau z + 1\right]\left(M_i - \tfrac{\tau}{h} m_i\right)$$
$$+ \tfrac{h^2}{\tau^3}\left[\tfrac{1}{2}\tau^2 (1-z)^2 + \tau\,(1\text{-}z) + 1\right]\left(M_{i-1} - \tfrac{\tau}{h} m_{i-1}\right) \tag{6}$$

where

$$A_i(e^\tau - 1) = -w_i + w_{i-1}e^\tau - \frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2} + \tau + 1\right) - \tau\, e^\tau\right]\left(M_i - \frac{\tau}{h} m_i\right)$$

$$-\frac{h^2}{\tau^3}\left[\left(\frac{\tau^2}{2}-\tau+1\right)-\tau\right]\left(M_{i-1}-\frac{\tau}{h}m_{i-1}\right)$$

$$B_i(e^\tau-1)=w_i-w_{i-1}e^\tau+\frac{h^2}{\tau^3}\left[\left(\frac{\tau}{2}+1\right)-\tau\, e^\tau\right]\left(M_i-\frac{\tau}{h}m_i\right)$$
$$+\left[\left(\frac{\tau}{2}-1\right)-\tau\right]\left(M_{i-1}-\frac{\tau}{h}m_{i-1}\right)$$

$\tau=\frac{Ph}{\varepsilon}$ and $z=\frac{s-s_{i-1}}{h}$.

The spline function $\psi(t,\tau)$ on $[s_i,\, s_{i+1}]$ is acquired with replacing i by $(i+1)$ in Eq. (6) and utilizing the first or second derivative continuity condition of $\psi(s,\tau)$ at $s=s_i$, we get the following relationship:

$$\left(M_{i+1}-\frac{\tau}{h}m_{i+1}\right)\left[e^{-\tau}\left(\frac{\tau^2}{2}+\tau+1\right)-1\right]+\left(M_i-\frac{\tau}{h}m_i\right)\left[e^{-\tau}\left(\frac{\tau^2}{2}-\tau-2\right)-\left(\frac{\tau^2}{2}+\tau-2\right)\right]+$$
$$\left(M_{i-1}-\frac{\tau}{h}m_{i-1}\right)\left[e^{-\tau}-1+\tau-\frac{\tau^2}{2}\right]=-\frac{\tau^2}{h^3}\left[e^{-\tau}w_{i+1}-(1+e^{-\tau})w_i+w_{i-1}\right] \qquad (7)$$

Further relations are given below for the adaptive splines

$$(i)\qquad m_{i-1}=-h\left(\tilde{A}_1M_{i-1}+\tilde{A}_2M_i\right)+\frac{1}{h}(w_i-w_{i-1})$$

$$(ii)\qquad m_i=h\left(\tilde{A}_3M_{i-1}+\tilde{A}_4M_i\right)+\frac{1}{h}(w_i-w_{i-1})$$

$$(iii)\qquad \frac{\theta\, h}{2\tau}M_{i-1}=-\left(\tilde{A}_4m_{i-1}+\tilde{A}_2m_i\right)+\tilde{B}_1\frac{(w_i-w_{i-1})}{h}$$

$$(iv)\qquad \frac{\theta\, h}{2\tau}M_i=\left(\tilde{A}_3m_{i-1}+\tilde{A}_1m_i\right)+\tilde{B}_2\frac{(w_i-w_{i-1})}{h}$$

where

$$\tilde{A}_1=\frac{1}{4}(1+\theta)+\frac{\theta}{20},\;\; \tilde{A}_2=\frac{1}{4}(1-\theta)-\frac{\theta}{20},\;\; \tilde{A}_3=\frac{1}{4}(1+\theta)-\frac{\theta}{20},\;\; \tilde{A}_4=\frac{1}{4}(1-\theta)+\frac{\theta}{20},$$
$$\tilde{B}_1=\frac{1}{4}(1-\theta),\;\; \tilde{B}_2=-\frac{1}{2}(1+\theta),\;\; \text{and}\;\; \theta=coth\left(\frac{\tau}{2}\right)-\frac{2}{\tau}$$

We also obtain, $\qquad \tilde{A}_2M_{i+1}+\left(\tilde{A}_1+\tilde{A}_4\right)M_i+\tilde{A}_3M_{i-1}=\frac{1}{h^2}[w_{i+1}-2w_i+w_{i-1}] \qquad (8)$

Remark: In the limiting case when $\tau\to 0$, we have

$$\tilde{A}_1=\tilde{A}_4=\frac{1}{3},\;\; \tilde{A}_2=\tilde{A}_3=\frac{1}{6},\;\; \tilde{B}_1=\frac{1}{2},\;\; \tilde{B}_2=-\frac{1}{2},\;\; \theta=0,\;\; \frac{\theta}{\tau}=\frac{1}{6}$$

and the spline function (6) reduces to ordinary cubic spline.

4. Description of the Numerical Procedure

At the mesh point t_i, the suggested approach can be discretized by the convection-diffusion equation Eq. (1) as

$$\varepsilon\, M_i = p(s_i) w_i{}'(s) + q(s_i) w(s_i) - r(s_i) \tag{9}$$

The above equations shall be replaced by Eq. (8) and using the following approximations for the first order derivative of *x* at the mesh points $s_1, s_2, \dots, s_{N-1}$.

$$w'_{i-1} \approx \frac{-w_{i+1} + 4w_i - 3w_{i-1}}{2h}$$

$$w'_{i+1} \approx \frac{3w_{i+1} - 4w_i + w_{i-1}}{2h}$$

$$w'_i \approx \left(\frac{1 + 2\eta h^2 q_{i+1} + \eta h[3p_{i+1} + p_{i-1}]}{2h}\right) w_{i+1} - 2\eta\,[p_{i+1} + p_{i-1}] w_i$$
$$- \left(\frac{1+2\eta h^2 q_{i-1} - \eta h[\,p_{i+1} + 3p_{i-1}]}{2h}\right) w_{i-1} + \eta\, h[r_{i+1} - r_{i-1}].$$

We get the following tridiagonal system

$$L_i x_{i-1} + C_i\, x_i + U_i x_{i+1} = W_i \quad \text{for } i = 1,2,\dots,N-1 \tag{10}$$

Where

$$L_i = -\,\varepsilon - \frac{3}{2}\tilde{A}_3\, P_{i-1}\, h\; - \left(\tilde{A}_1 + \tilde{A}_4\right) P_i h[1 + 2\eta h^2 Q_{i-1} - \eta h(P_{i+1} + 3P_{i-1})] \;+ \frac{\tilde{A}_2}{2} P_{i+1} h$$
$$+ \tilde{A}_3\, Q_{i-1} h^2$$

$$C_i = 2\varepsilon + 2\tilde{A}_3\, P_{i-1}\, h - 4\left(\tilde{A}_1 + \tilde{A}_4\right) P_i h_i^2 \eta [P_{i+1} + P_{i-1}] \; - 2\tilde{A}_2\, P_{i+1} h + 2\left(\tilde{A}_1 + \tilde{A}_4\right) Q_i h^2$$

$$U_i = -\varepsilon - \frac{\tilde{A}_3}{2}\, P_{i-1}\, h \; + \left(\tilde{A}_1 + \tilde{A}_4\right) P_i h\left[1 + 2\eta h^2\, Q_{i-1} + \eta h(3P_{i+1} + P_{i-1})\right] + \frac{3}{2}\tilde{A}_2 \eta P_{i+1} h +$$
$$\tilde{A}_2 Q_{i+1} h^2$$

$$W_i = \; h^2\left[\left(\tilde{A}_2 - 2\eta\left(\tilde{A}_1 + \tilde{A}_4\right) P_i h\right) R_{i+1} + 2\left(\tilde{A}_1 + \tilde{A}_4\right) R_i + \left(\tilde{A}_3 + 2\eta\left(\tilde{A}_1 + \tilde{A}_4\right) P_i h\right) R_{i-1}\right]$$

The tridiagonal system Eq. (10) is solved for i = 1, 2, …, N-1 to obtain the approximations $w_{1,} w_2, \dots, w_{N-1}$ of the solution *w(s)* at $s_1, s_2, \dots \dots s_{N-1}$.

5. TRUNCATION ERROR

Developed local truncation error associated with the scheme in Eq. (10) is

$$T_i(h) = \varepsilon\left[1 - \left(2\left(\tilde{A}_1 + \tilde{A}_4\right) + \tilde{A}_2 + \tilde{A}_3\right)\right]h_i^2 w'' + \varepsilon\left(\tilde{A}_3 - \tilde{A}_2\right)h_i^3 w'''$$

$$+\left[\frac{\tilde{A}_2+\tilde{A}_3}{2} - 4\eta\varepsilon\left(\tilde{A}_1 + \tilde{A}_4\right) - \frac{1}{6}\left[2\left(\tilde{A}_1 + \tilde{A}_4\right) + \tilde{A}_2 + \tilde{A}_3\right]\right] P_i h_i^4 w''' + \frac{\varepsilon}{12}\left[1 - 6\left(\tilde{A}_2 + \tilde{A}_3\right)\right]h_i^4 w^{iv} - \frac{1}{12}\left(\tilde{A}_3 - \tilde{A}_2\right)\left[P_i w^{iv} + 2(P_i' + Q_i)w''' + 6(P_i'' + Q_i')w'' + 2(P_i''' + 3Q_i'')w' + 2Q_i'''\, w - 2R'''\right]h_i^5 + O(h_i^6)$$

Thus, for different values of $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{A}_1 + \tilde{A}_4$ in the scheme Eq. (10), indicates different orders:

(i) If $\tilde{A}_2 = \tilde{A}_3$, for any choice of arbitrary $\tilde{A}_2, \tilde{A}_1 + \tilde{A}_4$ with $\left(\tilde{A}_1 + \tilde{A}_4\right) + \tilde{A}_2 = \frac{1}{2}$ and for any value of ψ , method is obtained for second order.

(ii) For $\tilde{A}_2 = \tilde{A}_3 = \frac{1}{12}$, $\left(\tilde{A}_1 + \tilde{A}_4\right) = \frac{5}{12}$ and $\eta = -\frac{1}{20\varepsilon}$, fourth order method is derived.

6. CONVERGENCE ANALYSIS

The convergence analysis of the method described in previous section for the problem Eq. (1) is now being considered. The system of equations Eq. (10) in the matrix form with the boundary conditions is obtained as

$$(D + F)W + G + T(h) = 0 \tag{11}$$

where
$$D = [-\varepsilon,\ 2\varepsilon,\ -\varepsilon] = \begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & \dots\dots.. & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \dots\dots.. & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \dots\dots.. & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

and

$$F = [\tilde{z}_i, \tilde{v}_i, \widetilde{w}_i] = \begin{bmatrix} \tilde{v}_1 & \widetilde{w}_1 & 0 & 0 & \dots\dots.. & 0 \\ \tilde{z}_2 & \tilde{v}_2 & \widetilde{w}_2 & 0 & \dots\dots.. & 0 \\ 0 & \tilde{z}_3 & \tilde{v}_3 & \widetilde{w}_3 & \dots\dots.. & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & \tilde{z}_{N-1} & \tilde{v}_{N-1} \end{bmatrix}$$

where

$$\tilde{z}_i = -\frac{3}{2}\tilde{A}_3\, P_{i-1}\, h_i \ - \left(\tilde{A}_1 + \tilde{A}_4\right)P_i h_i[1 + 2\eta h_i^2 Q_{i-1} - \eta h_i(P_{i+1} + 3P_{i-1})] \ + \frac{\tilde{A}_2}{2}P_{i+1}h_i + \tilde{A}_3\, Q_{i-1}h_i^2$$

$$\tilde{v}_i = 2\tilde{A}_3\, P_{i-1}\, h_i - 4\left(\tilde{A}_1 + \tilde{A}_4\right) P_i h_i^2 \eta[P_{i+1} + P_{i-1}] \ - 2\tilde{A}_2\, P_{i+1}h_i + 2\left(\tilde{A}_1 + \tilde{A}_4\right) Q_i h_i^2$$

$$\widetilde{w}_i = -\frac{\tilde{A}_3}{2}\, P_{i-1}\, h_i \ + \left(\tilde{A}_1 + \tilde{A}_4\right)P_i h_i[1 + 2\eta h_i^2 Q_{i-1} + \eta h_i(3P_{i+1} + P_{i-1})]P + \frac{3}{2}\tilde{A}_2 P_{i+1}h_i + \tilde{A}_2 Q_{i+1}h_i^2,$$

for $i = 1,2,\dots,N-1$ and $G = [q_1 - \tilde{z}_1\alpha, q_2, q_3, \dots, q_{N-1} - \widetilde{w}_{N-1}\beta]$ where

$$q_i = h_i^2\left[\left(\tilde{A}_2 - 2\eta\left(\tilde{A}_1 + \tilde{A}_4\right) p_i h_i\right)r_{i+1} + 2\left(\tilde{A}_1 + \tilde{A}_4\right) r_i + \left(\tilde{A}_3 + 2\eta\left(\tilde{A}_1 + \tilde{A}_4\right) p_i h_i\right)r_{i-1}\right],$$

for $i = 2,3,\dots,N-1, T(h) = O(h^6)$for $\tilde{A}_2 = \tilde{A}_3 = \frac{1}{12}$, $\left(\tilde{A}_1 + \tilde{A}_4\right) = \frac{5}{12}$ and $\eta = -\frac{1}{20\varepsilon}$ and $W = [W_1, W_2, W_3, \dots, W_{N-1}]^T$, $T(h) = [T_1, T_2, \dots, T_{N-1}]^T, O = [0,0,\dots,0]^T$ are associated vectors of Eq. (11).

Let $w = [w_1, w_2, \dots, w_{N-1}]^T \cong W$ which satisfies the equation

$$(D + F)w + G = 0 \tag{12}$$

Let the discretization error be $e_i = w_i - W_i$, $i = 1,2,\dots,N-1$ so that, $E = [e_1, e_2, \dots, e_{N-1}]^T = w - W$. Subtracting Eq. (11) from Eq. (12), we obtain the error equation

$$(D + F)E = T(h) \tag{13}$$

Let $|P(s)| \le \xi_1$ and $|Q(s)| \le \xi_2$where ξ_1, ξ_2 are positive constants. If $F_{i,j}$ be the $(i,j)^{th}$element of F,

Then $\left|P_{i,i+1}\right| = |\widetilde{w}_i| \le \varepsilon + \left(h\left(\tilde{A}_2 + \left(\tilde{A}_1 + \tilde{A}_4\right)\right)\xi_1 + h^2\tilde{A}_2\xi_2 + 4\left(\tilde{A}_1 + \tilde{A}_4\right)\eta h^2\xi_1^2 + 2h^3\left(\tilde{A}_1 + \tilde{A}_4\right)\eta\xi_1\xi_2\right)$, for $i = 1,2,\dots,N-2$

$$\left|P_{i,i-1}\right| = |\tilde{z}_i| \le +\left(h\left(\tilde{A}_2 + \left(\tilde{A}_1 + \tilde{A}_4\right)\right)\eta_1 + h^2\tilde{A}_2\eta_2 + 4\left(\tilde{A}_1 + \tilde{A}_4\right)\eta h^2\xi_1^2 + 2h^3\left(\tilde{A}_1 + \tilde{A}_4\right)\eta\xi_1\xi_2\right) \text{ for } i = 2,3,\dots,N-1$$

Thus, for sufficiently small h_i, we have

$$\left|P_{i,i+1}\right| < \varepsilon, \qquad i = 1,2,\dots,N-2 \tag{14a}$$

$$\left|P_{i,i-1}\right| < \varepsilon, \qquad i = 2,3,\dots,N-1 \tag{14b}$$

Hence $(D+F)$ is irreducible.

Let $\bar{S}_i$ be the sum of the elements of the i^{th} row of the matrix $(D+F)$, then we have

$$\bar{S}_i = \varepsilon - \frac{\tilde{A}_2 h}{2}(P_{i+1} - 3P_{i-1}) + h(\tilde{A}_1 + \tilde{A}_4)P_i + h^2(\tilde{A}_2 Q_{i-1} + 2(\tilde{A}_1 + \tilde{A}_4)Q_i)$$
$$+ h^2(\tilde{A}_1 + \tilde{A}_4)\eta P_i(3P_{i-1} + P_{i+1}) - 2h^3(\tilde{A}_1 + \tilde{A}_4)\eta P_i Q_{i-1} \quad \text{for } i = 1$$

$$\bar{S}_i = h^2\left(Q_{i-1} + 2\left(\tilde{A}_1 + \tilde{A}_4\right)Q_i + \tilde{A}_2 Q_{i+1}\right) + 2h^3\left(\tilde{A}_1 + \tilde{A}_4\right)P_i\eta\left(Q_{i+1} - Q_{i-1}\right) \text{ for } i = 2,3,\dots,N-2$$

$$\bar{S}_i = \varepsilon + \frac{\tilde{A}_2 h}{2}(P_{i-1} - 3P_{i+1}) - h(\tilde{A}_1 + \tilde{A}_4)P_i + h^2(\tilde{A}_2 Q_{i-1} + 2(\tilde{A}_1 + \tilde{A}_4)Q_i)$$
$$- h^2(\tilde{A}_1 + \tilde{A}_4)\eta P_i(3P_{i+1} + P_{i-1}) - 2h^3(\tilde{A}_1 + \tilde{A}_4)\eta P_i Q_{i-1} \quad \text{for } i = 2,3 \dots N-1$$

Let $\xi_{1^*} = \min_{1 \le i \le N} |P(t_i)|$ and $\xi_1^* = \max_{1 \le i \le N} |P(t_i)|$, $\xi_{2^*} = \min_{1 \le i \le N} |Q(t_i)|$ and $\xi_2^* = \max_{1 \le i \le N} |Q(t_i)|$.

Since $0 < \varepsilon << 1$ and $\varepsilon \propto O(h)$, it is verified that for sufficiently small h, $(D+F)$is monotone [16, 17]. Hence $(D+F)^{-1}$ exists and $(D+F)^{-1} \ge 0$.

Thus, using Eq. (13), we get

$$\|E\| \le \|(D+F)^{-1}\| \ \ \|T\| \tag{15}$$

Let $(D+F)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+F)^{-1}$ and we define

$$\|(D+F)^{-1}\| = \max_{1 \le i \le N-1} \sum_{k=1}^{N-1} (D+F)^{-1}_{i,k} \quad \text{and} \quad \|T(h)\| = \max_{1 \le i \le N-1} |T(h)| \tag{16a}$$

Since $(D+F)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{N-1} (D+F)^{-1}_{i,k} \bar{S}_k = 1 \quad for \ i = 1,2,\dots,N-1$.

Hence,

$$(D+F)^{-1}_{i,1} \le \frac{1}{\bar{S}_1} < \frac{1}{h^2\left[(\tilde{A}_2 + 2(\tilde{A}_1 + \tilde{A}_4))\xi_{2^*} - 4(\tilde{A}_1 + \tilde{A}_4)\psi \xi_1^2\right]} \tag{16b}$$

$$(D+F)^{-1}_{i,N-1} \le \frac{1}{\bar{S}_{N-1}} < \frac{1}{h^2\left[(\tilde{A}_2 + 2(\tilde{A}_1 + \tilde{A}_4))\xi_{2^*} - 4(\tilde{A}_1 + \tilde{A}_4)\psi \xi_1^2\right]} \tag{16c}$$

Furthermore,

$$\sum_{k=2}^{N-2} (D+F)^{-1}_{i,k} \le \frac{1}{\min_{2 \le k \le N-2} \bar{S}_k} < \frac{1}{h^2[2(\tilde{A}_2 + (\tilde{A}_1 + \tilde{A}_4))\xi_{2^*}]}, \ i = 2,3,\dots,N-2 \tag{16d}$$

By the help of Eqs. (16a) - (16d), using Eq. (15), we get

$$\|E\| \le O(h^4). \tag{17}$$

Hence, the method given in Eq. (10) is fourth order convergent for

$$\tilde{A}_2 = \tilde{A}_3 = \frac{1}{12}, (\tilde{A}_1 + \tilde{A}_4) = \frac{5}{12} \text{ and } \quad \eta = -\frac{1}{20\varepsilon}$$

7. NUMERICAL EXAMPLES

We consider convection diffusion problems to demonstrate the vitality of our proposed method computationally based on adaptive spline. These problems were chosen because they were widely covered in the literature and the exact solutions for the comparison are available. Maximum absolute solutions errors in the solution are tabulated and compared with the existing method results which have demonstrated improvement.

EXAMPLE 1. $-\varepsilon w''(s) + \frac{1}{s+1} w'(s) + \frac{1}{s+2} w(s) = f(s), \quad w(0) = 1 + 2^{\frac{-1}{\varepsilon}}, \quad w(1) = e + 2.$

where $f(s) = (-\varepsilon + \frac{1}{s+1} + \frac{1}{s+2}) e^s + \frac{1}{s+2} 2^{\frac{-1}{\varepsilon}} (s+1)^{1+\frac{1}{\varepsilon}}.$

The exact solution of this problem is $w(s) = e^s + 2^{\frac{-1}{\varepsilon}} (s+1)^{1+\frac{1}{\varepsilon}}$. The results obtained by our method are shown in Table 1 and layer structure is pictured in Figure 1.

EXAMPLE 2. $-\varepsilon w''(s) - w'(s) = 0, \quad w(0) = 1, \quad w(1) = exp\left(\frac{-1}{\varepsilon}\right).$

The exact solution is given by $w(s) = exp\left(\frac{-s}{\varepsilon}\right)$. The maximum absolute errors in solution of example 3 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$. The results are shown in Table 2 and layer behaviour is pictured in Figure 2.

EXAMPLE 3. $\varepsilon w''(s) + (1+s)^2 w'(s) + 2(1+s) w(s) = \frac{exp\left(\frac{-s}{2}\right)}{2}\left[(1+s)(3-s) + \frac{\varepsilon}{2}\right]$,

with $w(0) = 0, \ w(1) = exp\left(\frac{-1}{2}\right) - exp\left(\frac{-7}{3\varepsilon}\right)$.

Its exact solution is given by $w(s) = exp\left(\frac{-s}{2}\right) - exp\left[\frac{-s(s^2+3s+3)}{\varepsilon}\right]$

The numerical results are tabulated in Table 3 and layer profile is displayed in Figure 3.

EXAMPLE 4. $-\varepsilon w'' + w' = 2$ with $w(0) = 0, w(1) = 1.$

The exact solution of this problem is $w(s) = 2s + \frac{1 - e^{-\left(\frac{s}{\varepsilon}\right)}}{e^{-\left(\frac{1}{\varepsilon}\right)} - 1}.$

The maximum errors in solution of Example 4 are pictured in Table 4 and layer profile is displayed in Figure 4.

EXAMPLE 5. $\varepsilon w'' + w' + w = 0\,;\ \ w(0) = 1\,, w(1) = 2.$

The exact solution of this problem is

$$w(s) = \frac{[(2 - e^{r_2})e^{r_1 s} + (e^{r_1} - 2)e^{r_2 s}]}{(e^{r_1} - e^{r_2})}$$

where $r_1 = \frac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}$, $r_2 = \frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}$.

EXAMPLE 6. $\varepsilon w''(s) + 2(2s-1)w'(s) - 4w(s) = 0$ with $w(0) = 1,\ \ w(1) = 1.$

The exact solution of this problem is

$$w(s) = \frac{-e^{\frac{1}{2\varepsilon}-\frac{(1-2s)^2}{2\varepsilon}}\left(2e^{\frac{(1-2s)^2}{2\varepsilon}}\sqrt{2\pi}w\ erf\left(\frac{1-2s}{\sqrt{2\varepsilon}}\right) - e^{\frac{(1-2s)^2}{2\varepsilon}}\sqrt{2\pi}\ erf\left(\frac{1-2s}{\sqrt{2\varepsilon}}\right) - 2\varepsilon\right)}{e^{\frac{1}{2\varepsilon}}\sqrt{2\pi}\ erf\left(\frac{1}{\sqrt{2\varepsilon}}\right) + 2\sqrt{\varepsilon}}.$$

This problem has an internal layer at $s = \frac{1}{2}$. The numerical results are shown in the Table 6 and layer structure is pictured in Figure 6.

EXAMPLE 7. $\varepsilon w''(t) - 2(2s-1)w'(t) - 4w(t) = 0$ with $x(0) = 1, x(1) = 1.$

This problem exhibits dual layers at $t = 0$and $t = 1.$

The exact solution is given by $w(t) = exp(-2s(1-s)/\varepsilon)$. Table 7 shows the maximum errors in the solution. The maximum absolute errors are posed in Table 8 for different values of ε and h.

8. CONCLUSION

In this paper, we demonstrated a numerical scheme to solve a convection-diffusion problem using adaptive cubic spline. We introduce a new parameter η in the difference scheme to achieve fourth order convergence for the suggested problem. We have obtained a three-term relation with the help of difference scheme which involves a parameter η. The tridiagonal scheme obtained by the method is solved using the discrete invariant imbedding algorithm.

The convergence of the method has been discussed. For the standard test examples for the left-end, right-end, dual and internal layer chosen from the literature, maximum absolute errors are provided in Tables 1-7 to illustrate the efficiency of the method and to support the method. The layer profile is pictured in Figures 1-7. We noticed that the proposed fourth order scheme has been found to produce better results.

Table 1: The maximum errors in Example 1 for $A_1 = A_4 = \frac{1}{3}$ and $A_2 = A_3 = \frac{1}{6}$

ε	h =1/64	h =1/128	h =1/256	h =1/512	h =1/1024
Proposed Method					
2^{-4}	3.35(-4)	8.36(-5)	2.09(-5)	5.22(-6)	1.31(-6)
2^{-5}	7.82(-4)	1.94(-4)	4.84(-5)	1.21(-5)	3.03(-6)
2^{-6}	1.76(-3)	4.24(-4)	1.05(-4)	2.63(-5)	6.56(-6)
2^{-7}	4.53(-3)	9.43(-4)	2.23(-4)	5.51(-5)	1.37(-5)
2^{-8}	2.39(-2)	2.83(-3)	5.09(-4)	1.16(-4)	2.82(-5)
2^{-9}	1.36(-1)	1.95(-2)	1.96(-3)	2.86(-4)	6.01(-5)
Results in [11]					
2^{-4}	8.12(-4)	2.03(-4)	5.07(-5)	1.26(-5)	3.17(-6)
2^{-5}	3.53(-3)	8.79(-4)	2.19(-4)	5.48(-5)	1.37(-5)
2^{-6}	1.50(-2)	3.68(-3)	9.17(-4)	2.29(-4)	5.72(-5)
2^{-7}	6.75(-2)	1.54(-2)	3.77(-3)	9.37(-4)	2.34(-4)
2^{-8}	2.66(-1)	6.83(-2)	1.55(-2)	3.81(-3)	9.48(-4)
2^{-9}	6.92(-1)	2.68(-1)	6.87(-2)	1.56(-2)	3.83(-3)

Table 2: The maximum errors in solution of example 2 for $A_1 = A_4 = \frac{1}{3}$ an $A_2 = A_3 = \frac{1}{6}$

ε	h =1/64	h =1/128	h =1/256	h =1/512	h =1/1024
Proposed Method					
2^{-3}	1.24(-07)	7.76(-09)	4.85(-10)	3.05(-11)	2.60(-12)
2^{-4}	2.00(-06)	1.25(-07)	7.80(-09)	4.87(-10)	3.06(-11)
2^{-5}	3.24(-05)	2.00(-06)	1.25(-07)	7.80(-09)	4.87(-10)
2^{-6}	5.42(-04)	3.24(-05)	2.00(-06)	1.25(-07)	7.80(-09)
2^{-7}	0.0075	5.42(-04)	3.24(-05)	2.00(-06)	1.25(-07)
2^{-8}	0.0586	0.0075	5.42(-04)	3.24(-05)	2.00(-06)
2^{-9}	0.2255	0.0586	0.0075	5.42 (-04)	3.24(-05)
Results in [11]					
2^{-3}	4.77(-4)	1.19(-4)	2.98(-5)	7.45(-6)	1.86(-6)
2^{-4}	1.92(-3)	4.79(-4)	1.19(-4)	2.99(-5)	7.48(-6)
2^{-5}	7.87(-3)	1.92(-3)	4.79(-4)	1.19(-4)	2.99(-5)
2^{-6}	3.45(-2)	7.87(-3)	1.92(-3)	4.79(-4)	1.19(-4)
2^{-7}	6.75(-2)	1.54(-2)	3.77(-3)	9.37(-4)	2.34(-4)
2^{-8}	3.51(-1)	1.35(-1)	3.45(-2)	7.87(-3)	1.92(-3)
2^{-9}	6.00(-1)	3.51(-1)	1.35(-1)	3.45(-2)	7.87(-3)

Table 3. The maximum errors in solution of Example 3 for $A_1 = A_4 = \frac{1}{3}$ an $A_2 = A_3 = \frac{1}{6}$

ε	h =1/64	h =1/128	h =1/256	h =1/512	h =1/1024
Proposed Method					
2^{-1}	2.31(-05)	5.79(-06)	1.45(-06)	3.62(-07)	9.04(-08)
2^{-4}	4.33(-04)	1.08(-04)	2.70(-05)	6.75(-06)	1.68(-06)
2^{-8}	6.87(-02)	9.64(-03)	9.08(-04)	1.48(-04)	3.46(-05)
10^{-4}	9.24(-01)	8.67(-01)	7.42(-01)	5.44(-01)	2.95(-01)
10^{-5}	9.55(-01)	9.73(-01)	9.72(-01)	9.43(-01)	8.86(-01)
10^{-6}	9.56(-01)	9.78(-01)	9.88(-01)	9.92(-01)	9.89(-01)
10^{-7}	9.56(-01)	9.78(-01)	9.89(-01)	9.94(-01)	9.97(-01)
Results in [11]					
2^{-1}	5.13(-5)	1.28(-5)	3.20(-7)	8.02(-7)	2.00(-7)
2^{-4}	1.45(-3)	3.64(-4)	9.09(-5)	2.27(-5)	5.68(-6)
2^{-8}	3.43(-1)	1.33(-1)	3.39(-2)	7.75(-3)	1.89(-3)
10^{-4}	2.45(00)	1.01(00)	8.95(-1)	8.11(-1)	6.59(-1)
10^{-5}	2.36(+1)	6.04(00)	1.71(00)	9.86(-1)	9.58(-1)
10^{-6}	2.36(+2)	6.00(+2)	1.51(+1)	3.88(00)	1.27(00)
10^{-7}	2.36(+3)	6.00(+2)	1.51(+2)	3.88(+1)	1.27(+1)

Table 4. The maximum errors in solution of Example 4.

ε/ h	1/32	1/64	1/128	1/256	1/512	1/1024	1/2048	1/4096
Proposed Method								
2^{-3}	1.99(-6)	1.24(-7)	7.76(-9)	4.85(-10)	3.04(-11)	2.29(-12)	8.58(-12)	3.40(-11)
2^{-4}	3.24(-5)	2.00(-6)	1.25(-7)	7.80(-09)	4.87(-10)	3.05(-11)	5.41(-12)	2.24(-11)
2^{-5}	5.42(-4)	3.24(-5)	2.00(-6)	1.25(-07)	7.80(-09)	4.87(-10)	2.90(-11)	9.65(-12)
2^{-6}	7.52(-3)	5.42(-4)	3.24(-5)	2.00(-06)	1.25(-07)	7.80(-09)	4.85(-10)	2.26(-11)
2^{-7}	5.86(-2)	7.52(-3)	5.42(-4)	3.24(-05)	2.00(-06)	1.25(-0)	7.80(-09)	4.84(-10)
2^{-8}	2.25(-1)	5.86(-2)	7.52(-3)	5.42(-04)	3.24(-05)	2.00(-06)	1.25(-07)	7.80(-09)
2^{-9}	4.73(-1)	2.25(-1)	5.87(-2)	7.52(-03)	5.41(-04)	3.24(-05)	2.00(-06)	1.25(-07)
Results in [7]								
2^{-3}	0.19(-2)	0.48(-3)	0.12(-3)	0.30(-4)	0.75(-5)	0.19(-5)	0.47(-6)	0.12(-6)
2^{-4}	0.79(-2)	0.19(-2)	0.48(-3)	0.12(-3)	0.30(-4)	0.75(-5)	0.19(-5)	0.47(-6)
2^{-5}	0.35(-1)	0.79(-2)	0.19(-2)	0.48(-3)	0.12(-3)	0.30(-4)	0.75(-5)	0.19(-5)
2^{-6}	------	0.35(-1)	0.79(-2)	0.19(-2)	0.48(-3)	0.12(-3)	0.30(-4)	0.75(-5)
2^{-7}	--------	--------	0.35(-1)	0.79(-2)	0.19(-2)	0.48(-3)	0.12(-3)	0.30(-4)
2^{-8}	--------	--------	--------	0.35(-1)	0.79(-2)	0.19(-2)	0.48(-3)	0.12(-3)
2^{-9}	------	------	------	------	0.35(-1)	0.79(-2)	0.19(-2)	0.48(-2)

Table 5. Maximum absolute errors in Example 5

ε / h	1/64	1/128	1/256	1/512	1/1024	1/2048	1/4096	1/8192
Proposed Method								
2^{-3}	3.04(-7)	1.90(-8)	1.18(-9)	8.45(-11)	4.59(-11)	1.84(-10)	7.35(-10)	2.91(-09)
2^{-4}	7.15(-6)	4.46(-7)	2.79(-8)	1.75(-09)	1.58(-10)	1.18(-10)	9.53(-10)	1.87(-09)
2^{-5}	1.30(-4)	8.07(-6)	5.03(-7)	3.14(-8)	1.95(-9)	1.79(-10)	2.26(-10)	1.20(-09)
2^{-6}	2.29(-3)	1.37(-4)	8.49(-6)	5.29(-7)	3.31(-8)	2.11(-9)	4.42(-10)	1.29(-09)
2^{-7}	3.28(-2)	2.35(-3)	1.41(-4)	8.69(-6)	5.41(-7)	3.39(-8)	2.15(-09)	8.11(-10)
2^{-8}	2.59(-1)	3.31(-2)	2.38(-3)	1.42(-4)	8.79(-6)	5.48(-7)	3.43(-08)	2.33(-09)
2^{-9}	9.10(-1)	2.59(-1)	3.32(-2)	2.39(-3)	1.43(-4)	8.84(-6)	5.51(-07)	3.45(-08)
2^{-10}	2.1(+0)	1.0(+0)	2.60(-1)	3.33(-2)	2.40(-3)	1.43(-4)	8.86(-06)	5.53(-07)
Results in [7]								
2^{-3}	0.25(-2)	0.64(-3)	0.16(-3)	0.40(-4)	0.99(-5)	0.25(-5)	0.62(-6)	0.16(-6)
2^{-4}	0.15(-1)	0.37(-2)	0.91(-3)	0.23(-3)	0.57(-4)	0.14(-4)	0.36(-5)	0.89(-6)
2^{-5}	0.71(-1)	0.17(-1)	0.44(-2)	0.11(-2)	0.27(-3)	0.68(-4)	0.17(-4)	0.42(-5)
2^{-6}	-----	0.78(-1)	0.19(-1)	0.48(-2)	0.12(-2)	0.30(-3)	0.75(-4)	0.19(-4)
2^{-7}	-----	----	0.82(-1)	0.20(-1)	0.50(-2)	0.13(-2)	0.31(-3)	0.78(-4)
2^{-8}	-----	----	----	0.84(-1)	0.21(-1)	0.52(-2)	0.13(-2)	0.32(-3)
2^{-9}	-----	----	----	----	0.86(-1)	0.21(-1)	0.52(-2)	0.13(-2)
2^{-10}	----	----	----	----	----	0.86(-1)	0.21(-2)	0.53(-2)

Table 6. Maximum absolute errors in Example 6

ε / h	1/32	1/64	1/128	1/ 256	1/ 512	1/1024
Proposed method						
2^{-5}	8.6799(-6)	5.4123(-7)	3.3837(-8)	2.1151(-9)	1.3218(-10)	8.1866(-12)
2^{-6}	2.4578(-5)	1.5331(-6)	9.5748(-8)	5.9831(-9)	3.7392(-10)	2.3330(-11)
2^{-7}	6.7327(-5)	4.3399(-6)	2.7061(-7)	1.6919(-8)	1.0575(-09)	6.6083(-11)
2^{-8}	1.9559(-4)	1.2289(-5)	7.6654(-7)	4.7874(-8)	2.9915(-09)	1.8695(-10)
2^{-9}	5.6901(-4)	3.3663(-5)	2.1700(-6)	1.3531(-7)	8.4593(-09)	5.2877(-10)
2^{-10}	1.5105(-3)	9.7794(-5)	6.1444(-6)	3.8327(-7)	2.3937(-08)	1.4958(-09)
Results in [12]						
2^{-5}	5.9701(-3)	3.3654(-3)	1.7391(-3)	8.7449(-4)	4.3697(-4)	2.1822(-4)
2^{-6}	5.3525(-3)	3.2322(-3)	1.7219(-3)	8.7336(-4)	4.3719(-4)	2.1834(-4)
2^{-7}	1.1177(-2)	2.9851(-3)	1.6827(-3)	8.6953(-4)	4.3725(-4)	2.1848(-4)
2^{-8}	2.5867(-2)	2.6763(-3)	1.6161(-3)	8.6093(-4)	4.3668(-4)	2.1860(-4)
2^{-9}	4.7842(-2)	5.5886(-3)	1.4925(-3)	8.4134(-4)	4.3477(-4)	2.1862(-4)
2^{-10}	7.5829(-2)	1.2934(-2)	1.3381(-3)	8.0805(-4)	4.3046(-4)	2.1834(-4)

Table 7. Maximum absolute errors in Example 7

ε / h	1/32	1/64	1/128	1/ 256	1/ 512	1/1024
Proposed method						
2^{-5}	6.6213(-03)	4.9949(-04)	2.9956(-05)	1.8527(-06)	1.1549(-07)	7.2202(-09)
2^{-6}	5.3873(-02)	7.0956(-03)	5.2237(-04)	3.1287(-05)	1.9343(-06)	1.2056(-07)
2^{-7}	2.1476(-01)	5.6298(-02)	7.3146(-03)	5.3245(-04)	3.1875(-05)	1.9704(-06)
2^{-8}	4.6087(-01)	2.2024(-01)	5.7467(-02)	7.4197(-03)	5.3714(-04)	3.2149(-05)
2^{-9}	6.7876(-01)	4.6685(-01)	2.2288(-01)	5.8040(-02)	7.4712(-03)	5.3941(-04)
2^{-10}	8.2386(-01)	6.8315(-01)	4.6973(-01)	2.2419(-01)	5.8325(-02)	7.4966(-03)

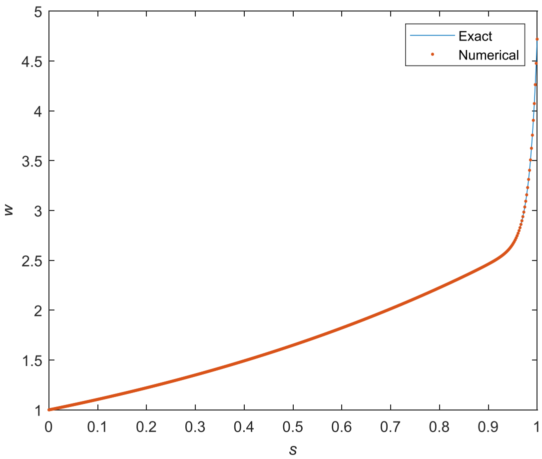

Figure 1: Solution profile for $\varepsilon = 2^{-7}$with $h = 2^{-9}$

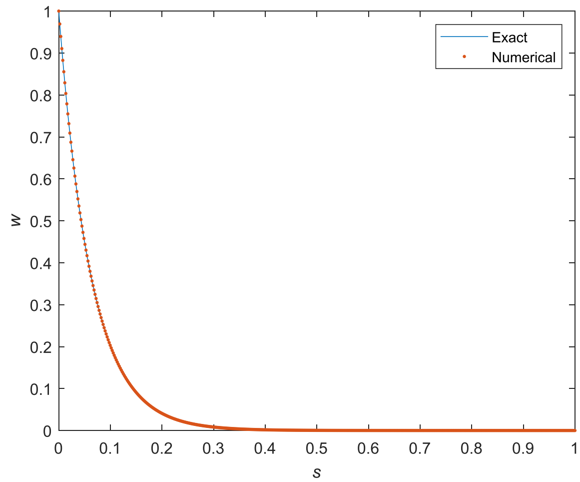

Figure 2: Solution profile for $\varepsilon = 2^{-4}$with $h = 2^{-9}$

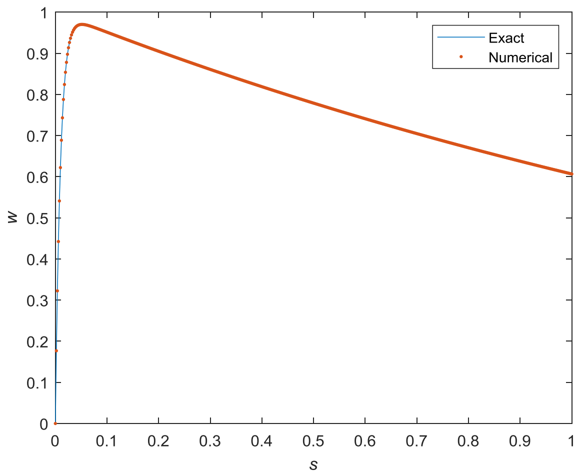

Figure 3: Solution profile for $\varepsilon = 10^{-2}$with $h = 2^{-9}$

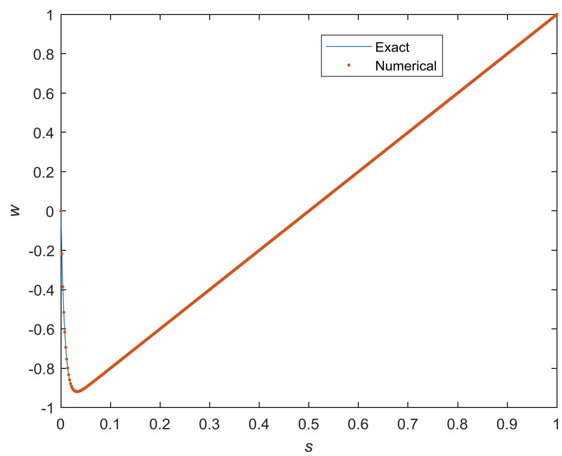

Figure 4: Solution profile for $\varepsilon = 2^{-7}$with $h = 2^{-5}$

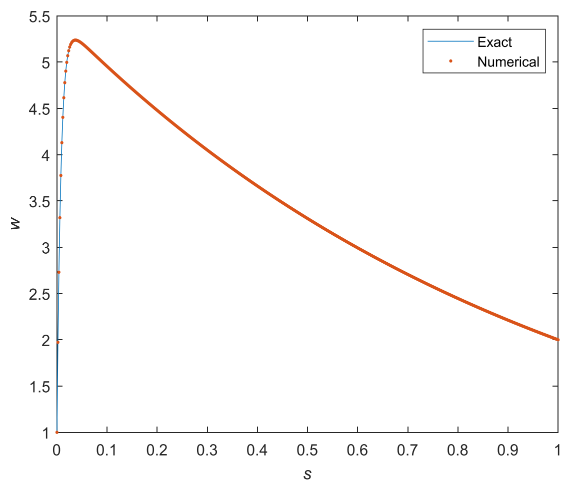

Figure 5: Solution profile for $\varepsilon = 2^{-7}$with $h = 2^{-9}$

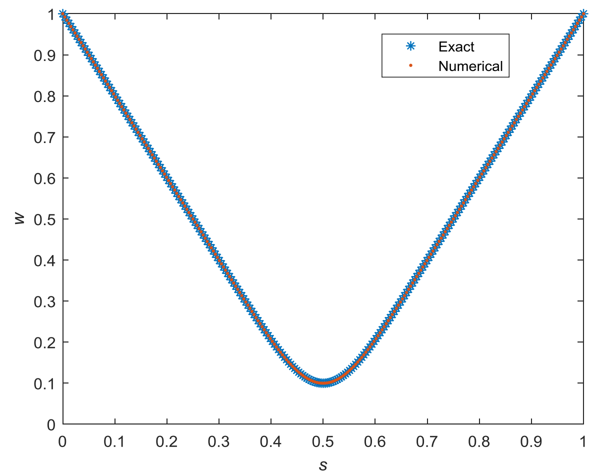

Figure 6: Solution profile for $\varepsilon = 2^{-6}$with $h = 2^{-8}$

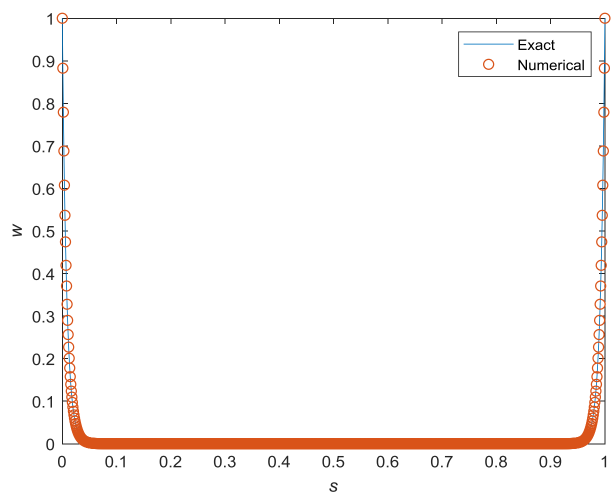

Figure 7: Solution profile for $\varepsilon = 2^{-6}$with $h = 2^{-10}$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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