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SOME APPLICATIONS VIA SUZUKI TYPE COMMON QUADRUPLE FIXED POINT RESULTS IN G-METRIC SPACES

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Abstract. In this paper we give some applications to integral equations as well as homotopy theory via Suzuki type for Jungck type common Quadruple fixed point theorems in complete G-metric space. We also furnish an example which supports our main result.

Keywords: quadruple fixed point; Suzuki type contraction; ω-compatible and G-completeness.

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1. Introduction

Fixed point theory is a beautiful mixture of Analysis, Topology and Geometry. It has been playing a vital role because of its wide applications in Homotopy theory, integral, integro-differential and impulsive differential equations, obtaining solutions of optimization problems, Approximation theory and Non-linear Analysis.

Recently Suzuki [1] proved generalized versions of both Banach's and Edelstein's basic results and thus initiated a lot of work in this direction (See.[2]-[6]). In 2006, Mustafa and Sims

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[7] initiated the concept of G-metric spaces and gave variant related fixed point results. Afterwards, many authors have developed beautiful fixed point results on the setting of G-metric spaces ([8]-[14]).

Very recently, E.Karapinar[15] initiated the concept of quadruple fixed point and proved some quadruple fixed results in partially ordered metric spaces. Further, many investigators ([16]-[20]) established quadruple fixed theorems in various metric spaces.

The aim of this paper is to combine the ideas of quadruple fixed points and Suzuki type fixed point theorems to obtain a unique common quadruple fixed point theorem for Jungck type mappings in a G-metric space. Also, we give example, applications to Integral equations and Homotopy theory.

2. PRELIMINARIES

First, let's review the important concepts of G-metric spaces.

Definition 2.1:([7]) Let $\mathscr{P}$ be a non-empty set and let $G : \mathscr{P} \times \mathscr{P} \times \mathscr{P} \to [0,\infty)$ be a function satisfying the following properties :

(B_0) $G(p,q,r) = 0$ if $p = q = r$;

(B_1) $0 < G(p,p,q)$ for any $p,q \in \mathscr{P}$ with $p \neq q$;

(B_2) if $G(p,p,q) \leq G(p,q,r)$ for all $p,q,r \in \mathscr{P}$ with $q \neq r$;

(B_3) $G(p,q,r) = G(P[p,q,r])$, where P is a permutation of p,q,r (symmetry);

(B_4) $G(p,q,r) \leq G(p,x,x) + G(x,q,r)$ for all $p,q,r,x \in \mathscr{P}$ (rectangle inequality)

then G is said to be a G-metric on $\mathscr{P}$ and pair $(\mathscr{P},G)$ is said to be a G-metric space.

Definition 2.2:([7]) A G- metric space $(\mathscr{P},G)$ is said to be symmetric if
$G(p,q,q) = G(q,p,p)$ for all $p,q \in \mathscr{P}$.

Definition 2.3:([7]) Let $\mathscr{P}$ be a G-metric space. A sequence $\{p_n\}$ in $\mathscr{P}$ is called:

(a) G-Cauchy sequence if for every $\varepsilon > 0$, there is an integer $n_0 \in \mathbf{Z}^+$ such that for all $n,m,l \geq n_0$, $G(p_n,p_m,p_l) < \varepsilon$.

(b) G-convergent to a point $p \in \mathscr{P}$ if for each $\varepsilon > 0$, there is an integer $n_0 \in \mathbf{Z}^+$ such that for all $n,m \geq n_0$, $G(p_n,p_m,p) < \varepsilon$.

A G-metric space on $\mathscr{P}$ is said to be G-complete if every G-Cauchy sequence in $\mathscr{P}$ is G-convergent in $\mathscr{P}$.

For more properties of a G-metric we refer the reader to ([7]).

Definition 2.4:([15]) Let $\mathscr{P}$ be a nonempty set and let $F : \mathscr{P}^4 \to \mathscr{P}$ be a mapping. If $F(p,q,r,s) = p$, $F(q,r,s,p) = q$, $F(r,s,p,q) = r$ and $F(s,p,q,r) = s$ for $p,q,r,s \in \mathscr{P}$ then (p,q,r,s) is called a Quadruple fixed point of F.

Definition 2.5: ([17]) Let $F : \mathscr{P}^4 \to \mathscr{P}$ and $f : \mathscr{P} \to \mathscr{P}$ be two mappings. An element (p,q,r,s) is said to be a quadruple coincident point of F and f if
$F(p,q,r,s) = fp,\ F(q,r,s,p) = fq,\ F(r,s,p,q) = fr$ and $F(s,p,q,r) = fs$.

Definition 2.6:([17]) Let $F : \mathscr{P}^4 \to \mathscr{P}$ and $f : \mathscr{P} \to \mathscr{P}$ be two mappings. An element (p,q,r,s) is said to be a quadruple common point of F and f if
$F(p,q,r,s) = fp = p,\ F(q,r,s,p) = fq = q, F(r,s,p,q) = fr = r$ and $F(s,p,q,r) = fs = s$.

Definition 2.7:([17]) Let $(\mathscr{P},G)$ be a G-metric space. A pair (F,f) is called weakly compatible if $f(F(p,q,r,s)) = F(fp,fq,fr,fs)$ whenever for all $p,q,r,s \in \mathscr{P}$ such that
$F(p,q,r,s) = fp,\ F(q,r,s,p) = fq,\ F(r,s,p,q) = fr$ and $F(s,p,q,r) = fs$.

Theorem 2.8:([1]) Let $(\mathscr{P};d)$ be a complete metric space, let $T : \mathscr{P} \to \mathscr{P}$ be a map- ping and define a non increasing function $\theta : [0;1) \to (\frac{1}{2};1]$ by $\theta(t) = \begin{cases} 1, & 0 \le t \le \frac{\sqrt{5}-1}{2} \\ (1-t)t^{-2}, & \frac{\sqrt{5}-1}{2} \le t \le \frac{1}{\sqrt{2}} \\ (1+t)^{-1}, & \frac{1}{\sqrt{2}} < t \le 1 \end{cases}$

Assume that there exists $t \in [0;1)$ such that

$$\theta(t)d(p,Tp) \le d(p;q) \text{ implies } d(Tp;Tq) \le d(p;q)$$

for all $p,q \in \mathscr{P}$. Then there exists a unique fixed point a of T. Moreover, $\lim_{n\to\infty} T^n p = a$ for all $p \in \mathscr{P}$.

3. Main Results

Theorem 3.1: Let $(\mathscr{P},G)$ be a G-metric space. Suppose that $T : \mathscr{P}^4 \to \mathscr{P}$ and $f : \mathscr{P} \to \mathscr{P}$ be a two mappings satisfying the following:

$$\eta(\theta)G(fx,fx,T(x,y,z,w)) \leq \max\left\{\begin{array}{c} G(fx,fx,fp),G(fy,fy,fq), \\ G(fz,fz,fr),G(fw,fw,fs), \\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)) \end{array}\right\}$$

implies

$$G(T(x,y,z,w),T(x,y,z,w),T(p,q,r,s))$$

$$\leq \theta \max\left\{\begin{array}{c} G(fx,fx,fp),G(fy,fy,fq), \\ G(fz,fz,fr),G(fw,fw,fs), \\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)) \\ G(fp,fp,T(p,q,r,s)),G(fq,fq,T(q,r,s,p)), \\ G(fr,fr,T(r,s,p,q)),G(fs,fs,T(s,p,q,r)) \\ G(fp,fp,T(x,y,z,w)),G(fq,fq,T(y,z,w,x)), \\ G(fr,fr,T(z,w,x,y)),G(fs,fs,T(w,x,y,z)) \end{array}\right\}.$$

(1)

for all $x,y,z,w,p,q,r,s \in \mathscr{P}$, where $\theta \in [0,1)$ and $\eta : [0,1) \to [\frac{1}{2},1)$ defined as $\eta(\theta) = \frac{1}{1+\theta}$is a strictly decreasing function,

$a)$ $T(\mathscr{P}^4) \subseteq f(\mathscr{P})$ and $f(\mathscr{P})$ is complete,

$b)$ pair (T,f) is ω-compatible.

Then there is a unique common quadruple fixed point of T and f in $\mathscr{P}$.

Proof. Let $x_0,y_0,z_0,w_0 \in \mathscr{P}$ be arbitrary, and from (a), we construct the sequences $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ in $\mathscr{P}$ as

$$T(x_n,y_n,z_n,w_n) = fx_{n+1}, \quad T(y_n,z_n,w_n,x_n) = fy_{n+1},$$

$$T(z_n,w_n,x_n,y_n) = fz_{n+1}, \quad T(w_n,x_n,y_n,z_n,) = fw_{n+1}, \text{ where } n = 0,1,2,\ldots.$$

case(i): Assume that

$$fx_n \neq fx_{n+1} \text{ or } fy_n \neq fy_{n+1} \text{ or } fz_n \neq fz_{n+1} \text{ or } fw_n \neq fw_{n+1} \forall\, n. \tag{2}$$

Since

$$\eta(\theta)G(fx_0,fx_0,T(x_0,y_0,z_0,w_0)) = \eta(\theta)G(fx_0,fx_0,fx_1) \leq G(fx_0,fx_0,fx_1)$$
$$\leq \max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1), \\ G(fx_0,fx_0,T(x_0,y_0,z_0,w_0)),G(fy_0,fy_0,T(y_0,z_0,w_0,x_0)), \\ G(fz_0,fz_0,T(z_0,w_0,x_0,y_0)),G(fw_0,fw_0,T(w_0,x_0,y_0,z_0)) \end{array}\right\}$$

Then from (1), we can get

$$\begin{aligned} G(fx_1,fx_1,fx_2) &= G(T(x_0,y_0,z_0,w_0),T(x_0,y_0,z_0,w_0),T(x_1,y_1,z_1,w_1)) \\ &\leq \theta\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1), \\ G(fx_0,fx_0,T(x_0,y_0,z_0,w_0)),G(fy_0,fy_0,T(y_0,z_0,w_0,x_0)), \\ G(fz_0,fz_0,T(z_0,w_0,x_0,y_0)),G(fw_0,fw_0,T(w_0,x_0,y_0,z_0)) \\ G(fx_1,fx_1,T(x_1,y_1,z_1,w_1)),G(fy_1,fy_1,T(y_1,z_1,w_1,x_1)), \\ G(fz_1,fz_1,T(z_1,w_1,x_1,y_1)),G(fw_1,fw_1,T(w_1,x_1,y_1,z_1)) \\ G(fx_1,fx_1,T(x_0,y_0,z_0,w_0)),G(fy_1,fy_1,T(y_0,z_0,w_0,x_0)), \\ G(fz_0,fz_1,T(z_0,w_0,x_0,y_0)),G(fw_1,fw_1,T(w_0,x_0,y_0,z_0)) \end{array}\right\}. \\ &\leq \theta\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1),G(fz_0,fz_0,fz_1), \\ G(fw_0,fw_0,fw_1),G(fx_1,fx_1,fx_2),G(fy_1,fy_1,fy_2), \\ G(fz_1,fz_1,fz_2),G(fw_1,fw_1,fw_2) \end{array}\right\}. \end{aligned}$$

(3)

Similarly, we can prove that

$$G(fy_1,fy_1,fy_2) \leq \theta\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1),G(fz_0,fz_0,fz_1), \\ G(fw_0,fw_0,fw_1),G(fx_1,fx_1,fx_2),G(fy_1,fy_1,fy_2), \\ G(fz_1,fz_1,fz_2),G(fw_1,fw_1,fw_2) \end{array}\right\}.$$

(4)

$$G(fz_1, fz_1, fz_2) \leq \theta \max \left\{ \begin{array}{c} G(fx_0, fx_0, fx_1), G(fy_0, fy_0, fy_1), G(fz_0, fz_0, fz_1), \\ G(fw_0, fw_0, fw_1), G(fx_1, fx_1, fx_2), G(fy_1, fy_1, fy_2), \\ G(fz_1, fz_1, fz_2), G(fw_1, fw_1, fw_2) \end{array} \right\}.$$

(5)

$$G(fw_1, fw_1, fw_2) \leq \theta \max \left\{ \begin{array}{c} G(fx_0, fx_0, fx_1), G(fy_0, fy_0, fy_1), G(fz_0, fz_0, fz_1), \\ G(fw_0, fw_0, fw_1), G(fx_1, fx_1, fx_2), G(fy_1, fy_1, fy_2), \\ G(fz_1, fz_1, fz_2), G(fw_1, fw_1, fw_2) \end{array} \right\}.$$

(6)

Due to $(3)-(6)$, we conclude that

$$\begin{aligned} & \max\{G(fx_1, fx_1, fx_2), G(fy_1, fy_1, fy_2), G(fz_1, fz_1, fz_2), G(fw_1, fw_1, fw_2)\} \\ \leq \quad & \theta \max \left\{ \begin{array}{c} G(fx_0, fx_0, fx_1), G(fy_0, fy_0, fy_1), G(fz_0, fz_0, fz_1), \\ G(fw_0, fw_0, fw_1), G(fx_1, fx_1, fx_2), G(fy_1, fy_1, fy_2), \\ G(fz_1, fz_1, fz_2), G(fw_1, fw_1, fw_2) \end{array} \right\}. \end{aligned}$$

(7)

If $\max \left\{ \begin{array}{c} G(fx_0, fx_0, fx_1), G(fy_0, fy_0, fy_1), \\ G(fz_0, fz_0, fz_1), G(fw_0, fw_0, fw_1) \end{array} \right\} \leq \max \left\{ \begin{array}{c} G(fx_1, fx_1, fx_2), G(fy_1, fy_1, fy_2), \\ G(fz_1, fz_1, fz_2), G(fw_1, fw_1, fw_2) \end{array} \right\}$.
Then from (7), we have $fx_1 = fx_2$ or $fy_1 = fy_2$ or $fz_1 = fz_2$ or $fw_1 = fw_2$. It is contradiction to (2). Hence from (7), we have
$\max \left\{ \begin{array}{c} G(fx_1, fx_1, fx_2), G(fy_1, fy_1, fy_2), \\ G(fz_1, fz_1, fz_2), G(fw_1, fw_1, fw_2) \end{array} \right\} \leq \theta \max \left\{ \begin{array}{c} G(fx_0, fx_0, fx_1), G(fy_0, fy_0, fy_1), \\ G(fz_0, fz_0, fz_1), G(fw_0, fw_0, fw_1) \end{array} \right\}$.

Continuing in this way, we get

$$\begin{aligned}
\max\left\{\begin{array}{c} G(fx_n,fx_n,fx_{n+1}), \\ G(fy_n,fy_n,fy_{n+1}), \\ G(fz_n,fz_n,fz_{n+1}), \\ G(fw_n,fw_n,fw_{n+1}) \end{array}\right\}
&\leq \theta\max\left\{\begin{array}{c} G(fx_{n-1},fx_{n-1},fx_n),G(fy_{n-1},fy_{n-1},fy_n), \\ G(fz_{n-1},fz_{n-1},fz_n),G(fw_{n-1},fw_{n-1},fw_n) \end{array}\right\} \\
&\leq \theta^2\max\left\{\begin{array}{c} G(fx_{n-2},fx_{n-2},fx_{n-1}),G(fy_{n-2},fy_{n-2},fy_{n-1}), \\ G(fz_{n-2},fz_{n-2},fz_{n-1}),G(fw_{n-2},fw_{n-2},fw_{n-1}) \end{array}\right\} \\
&\vdots \\
&\leq \theta^n\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1) \end{array}\right\}.
\end{aligned}$$

Thus $G(fx_n,fx_n,fx_{n+1}) \leq \theta^n\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1) \end{array}\right\}$,

$$G(fy_n,fy_n,fy_{n+1}) \leq \theta^n\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1) \end{array}\right\},$$

$$G(fz_n,fz_n,fz_{n+1}) \leq \theta^n\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1) \end{array}\right\}$$

and

$$G(fw_n,fw_n,fw_{n+1}) \leq \theta^n\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1), \\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1) \end{array}\right\}.$$

For $m>n$, by use of the rectangle inequality, we have

$$\begin{aligned}
& G(fx_m,fx_m,fx_n)\\
\leq\ & G(fx_n,fx_{n+1},fx_{n+1})+G(fx_{n+1},fx_{n+2},fx_{n+2})+\ldots+G(fx_{m-1},fx_m,fx_m),\\
\leq\ & \left(\theta^n+\theta^{n+1}+\ldots+\theta^{m-1}\right)\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1),\\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1)\end{array}\right\}\\
\leq\ & \frac{\theta^n}{1-\theta}\max\left\{\begin{array}{c} G(fx_0,fx_0,fx_1),G(fy_0,fy_0,fy_1),\\ G(fz_0,fz_0,fz_1),G(fw_0,fw_0,fw_1)\end{array}\right\}\to 0 \text{ as } n,m\to\infty.
\end{aligned}$$

Hence $\{fx_n\}$ is a Cauchy sequence in $f(\mathscr{P})$. Similarly we can show that $\{fy_n\}$, $\{fz_n\}$ and $\{fw_n\}$ are Cauchy sequence in $f(\mathscr{P})$. Since $f(\mathscr{P})$ is complete, there exist $\alpha,\beta,\gamma,\omega$ and a,b,c,d in $\mathscr{P}$ such that

$$\begin{aligned}
&\lim_{n\to\infty} fx_n=\alpha=fa \quad \lim_{n\to\infty} fy_n=\beta=fb\\
&\lim_{n\to\infty} fz_n=\gamma=fc \quad \lim_{n\to\infty} fw_n=\omega=fd.
\end{aligned}$$

Since $fx_n\to\alpha$, $fy_n\to\beta$, $fz_n\to\gamma$ and $fw_n\to\omega$, we may assume that $fx_n\neq\alpha$, $fy_n\neq\beta$, $fz_n\neq\gamma$ and $fw_n\neq\omega$ for infinitely many n. We claim that

$$\max\left\{\begin{array}{c} G(fa,fa,T(x,y,z,w)),\\ G(fb,fb,T(y,z,w,x)),\\ G(fc,fc,T(z,w,x,y)),\\ G(fd,fd,T(w,x,y,z))\end{array}\right\}\leq\theta\max\left\{\begin{array}{c} G(fa,fa,fx),G(fb,fb,fy),\\ G(fc,fc,fz),G(fd,fd,fw)\\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)),\\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z))\end{array}\right\}$$

for all $x,y,z,w\in\mathscr{P}$ with $fa\neq fx, fb\neq fy, fc\neq fz$ and $fd\neq fw$.

Let $x,y,z,w\in\mathscr{P}$ with $fa\neq fx, fb\neq fy, fc\neq fz$ and $fd\neq fw$. Then there exists a positive integer n_0 such that for $n\geq n_0$, we have $G(fa,fa,fx_n)\leq\frac{1}{4}G(fa,fa,fx)$, $G(fb,fb,fy_n)\leq\frac{1}{4}G(fa,fa,fy)$, $G(fc,fc,fz_n)\leq\frac{1}{4}G(fc,fc,fz)$,

$G(fd,fd,fw_n) \leq \frac{1}{4}G(fd,fd,fw)$. Now for $n \geq n_0$

$$\begin{aligned}
\eta(\theta)G(fx_n,fx_n,T(x_n,y_n,z_n,w_n)) &\leq G(fx_n,fx_n,T(x_n,y_n,z_n,w_n)) \\
&= G(fx_n,fx_n,fx_{n+1}) \\
&\leq G(fx_n,fx_n,fa)+G(fa,fa,fx_{n+1}) \\
&\leq 2G(fx_n,fa,fa)+G(fa,fa,fx_{n+1}) \\
&\leq \frac{2}{4}G(fx,fa,fa)+\frac{1}{4}G(fa,fa,fx) \\
&\leq G(fx,fa,fa)-\frac{1}{4}G(fa,fa,fx) \\
&\leq G(fx,fa,fa)-G(fa,fa,fx_n) \\
&\leq G(fx,fx,fx_n) \\
&\leq \max\left\{\begin{array}{c}
G(fx,fx,fx_n),G(fy,fy,fy_n), \\
G(fz,fz,fz_n),G(fw,fw,fw_n), \\
G(fx_n,fx_n,T(x_n,y_n,z_n,w_n)), \\
G(fy_n,fy_n,T(y_n,z_n,w_n,x_n)), \\
G(fz_n,fz_n,T(z_n,w_n,x_n,y_n)), \\
G(fw_n,fw_n,T(w_n,x_n,y_n,z_n))
\end{array}\right\}.
\end{aligned}$$

From (1), we have

$$\begin{aligned}
&G(T(x_n,y_n,z_n,w_n),T(x_n,y_n,z_n,w_n),T(x,y,z,w)) \\
\leq \ &\theta\max\left\{\begin{array}{c}
G(fx_n,fx_n,fx),G(fy_n,fy_n,fy), \\
G(fz_n,fz_n,fz),G(fw_n,fw_n,fw), \\
G(fx_n,fx_n,fx_{n+1}),G(fy_n,fy_n,fy_{n+1}), \\
G(fz_n,fz_n,fz_{n+1}),G(fw_n,fw_n,fw_{n+1}) \\
G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\
G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)), \\
G(fx,fx,fx_{n+1}),G(fy,fy,fy_{n+1}), \\
G(fz,fz,fz_{n+1}),G(fw,fw,fw_{n+1})
\end{array}\right\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$G(fa,fa,T(x,y,z,w)) \leq \theta \max \left\{ \begin{array}{c} G(fa,fa,fx), G(fb,fb,fy), \\ G(fc,fc,fz), G(fd,fd,fw), \\ G(fx,fx,T(x,y,z,w)), G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)), G(fw,fw,T(w,x,y,z)), \end{array} \right\}.$$

Similarly we can show that

$$G(fb,fb,T(y,z,w,x)) \leq \theta \max \left\{ \begin{array}{c} G(fb,fb,fy), G(fc,fc,fz), \\ G(fd,fd,fw), G(fa,fa,fx), \\ G(fx,fx,T(x,y,z,w)), G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)), G(fw,fw,T(w,x,y,z)), \end{array} \right\},$$

$$G(fc,fc,T(z,w,x,y)) \leq \theta \max \left\{ \begin{array}{c} G(fc,fc,fz), G(fd,fd,fw), \\ G(fa,fa,fx), G(fb,fb,fy), \\ G(fx,fx,T(x,y,z,w)), G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)), G(fw,fw,T(w,x,y,z)), \end{array} \right\}$$

and

$$G(fd,fd,T(w,x,y,z)) \leq \theta \max \left\{ \begin{array}{c} G(fd,fd,fw), G(fa,fa,fx), \\ G(fb,fb,fy), G(fc,fc,fz), \\ G(fx,fx,T(x,y,z,w)), G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)), G(fw,fw,T(w,x,y,z)), \end{array} \right\}.$$

Thus

$$\max \left\{ \begin{array}{c} G(fa,fa,T(x,y,z,w)), \\ G(fb,fb,T(y,z,w,x)), \\ G(fc,fc,T(z,w,x,y)), \\ G(fd,fd,T(w,x,y,z)) \end{array} \right\} \quad \leq \quad \theta \max \left\{ \begin{array}{c} G(fa,fa,fx), G(fb,fb,fy), \\ G(fc,fc,fz), G(fd,fd,fw) \\ G(fx,fx,T(x,y,z,w)), G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)), G(fw,fw,T(w,x,y,z)) \end{array} \right\}$$

(8)

Hence the claim. Now consider

$$\begin{aligned} G(fx,fx,T(x,y,z,w)) &\leq G(fx,fx,fa)+G(fa,fa,T(x,y,z,w)) \\ &\leq G(fx,fx,fa)+\theta \max\left\{\begin{array}{c} G(fa,fa,fx),G(fb,fb,fy), \\ G(fc,fc,fz),G(fd,fd,fw), \\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)), \end{array}\right\}. \\ &\leq (1+\theta)\max\left\{\begin{array}{c} G(fa,fa,fx),G(fb,fb,fy), \\ G(fc,fc,fz),G(fd,fd,fw), \\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)), \end{array}\right\}. \end{aligned}$$

Thus

$$\eta(\theta)G(fx,fx,T(x,y,z,w)) \leq \max\left\{\begin{array}{c} G(fa,fa,fx),G(fb,fb,fy), \\ G(fc,fc,fz),G(fd,fd,fw), \\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)), \end{array}\right\}.$$

Hence from (1), we have

$$\begin{aligned} &G(T(a,b,c,d),T(a,b,c,d),T(x,y,z,w)) \\ &\leq \quad \theta \max\left\{\begin{array}{c} G(fa,fa,fx),G(fb,fb,fy), \\ G(fc,fc,fz),G(fd,fd,fw), \\ G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\ G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)), \\ G(fa,fa,T(a,b,c,d)),G(fb,fb,T(b,c,d,a)), \\ G(fc,fc,T(c,d,a,b)),G(fd,fd,T(d,a,b,c)) \\ G(fx,fx,T(a,b,c,d)),G(fy,fy,T(b,c,d,a)), \\ G(fz,fz,T(c,d,a,b)),G(fw,fw,T(d,a,b,c)) \end{array}\right\}. \end{aligned}$$

(9)

Now

$$
\begin{aligned}
& G(fa,fa,T(a,b,c,d)) = \lim_{n\to\infty} G(fx_{n+1},fx_{n+1},T(a,b,c,d)) \\
= \;& \lim_{n\to\infty} G(T(x_n,y_n,z_n,w_n),T(x_n,y_n,z_n,w_n),T(a,b,c,d)) \\
\leq \;& \lim_{n\to\infty} \theta \max \left\{ \begin{array}{c}
G(fx_n,fx_n,fa), G(fy_n,fy_n,fb), \\
G(fz_n,fz_n,fc), G(fw_n,fw_n,fd), \\
G(fa,fa,T(a,b,c,d)), G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), G(fd,fd,T(d,a,b,c)), \\
G(fx_n,fx_n,T(x_n,y_n,z_n,w_n)), G(fy_n,fy_n,T(y_n,z_n,w_n,x_n)), \\
G(fz_n,fz_n,T(z_n,w_n,x_n,y_n)), G(fw_n,fw_n,T(w_n,x_n,y_n,z_n)) \\
G(fa,fa,T(x_n,y_n,z_n,w_n)), G(fb,fb,T(y_n,z_n,w_n,x_n)), \\
G(fc,fc,T(z_n,w_n,x_n,y_n)), G(fd,fd,T(w_n,x_n,y_n,z_n))
\end{array} \right\} \\
\leq \;& \theta \max \left\{ \begin{array}{c}
G(fa,fa,T(a,b,c,d)), G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), G(fd,fd,T(d,a,b,c))
\end{array} \right\}.
\end{aligned}
$$

Similarly, we can have

$$
G(fb,fb,T(b,c,d,a)) \leq \theta \max \left\{ \begin{array}{c}
G(fa,fa,T(a,b,c,d)), G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), G(fd,fd,T(d,a,b,c))
\end{array} \right\}
$$

and

$$
G(fc,fc,T(c,d,a,b)) \leq \theta \max \left\{ \begin{array}{c}
G(fa,fa,T(a,b,c,d)), G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), G(fd,fd,T(d,a,b,c))
\end{array} \right\}
$$

also

$$
G(fd,fd,T(d,a,b,c)) \leq \theta \max \left\{ \begin{array}{c}
G(fa,fa,T(a,b,c,d)), G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), G(fd,fd,T(d,a,b,c))
\end{array} \right\}.
$$

Thus

$$
\max \left\{ \begin{array}{c}
G(fa,fa,T(a,b,c,d)), \\
G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), \\
G(fd,fd,T(d,a,b,c))
\end{array} \right\} \leq \theta \max \left\{ \begin{array}{c}
G(fa,fa,T(a,b,c,d)), \\
G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)), \\
G(fd,fd,T(d,a,b,c))
\end{array} \right\}.
$$

So that $T(a,b,c,d) = fa, T(b,c,d,a) = fb, T(c,d,a,b) = fc$ and $T(d,a,b,c) = fd$. Thus (a,b,c,d) is a Quadruple coincidence point of T and f . Since the pair (T,f) is ω-compatible, we have

$$\begin{aligned} f\alpha &= f^2a = f(T(a,b,c,d)) = T(fa,fb,fc,fd) = T(\alpha,\beta,\gamma,\omega) \\ f\beta &= f^2b = f(T(b,c,d,a)) = T(fb,fc,fd,fa) = T(\beta,\gamma,\omega,\alpha) \\ f\gamma &= f^2c = f(T(c,d,a,b)) = T(fc,fd,fa,fb) = T(\gamma,\omega,\alpha,\beta) \\ f\omega &= f^2d = f(T(d,a,b,c)) = T(fd,fa,fb,fc) = T(\omega,\alpha,\beta,\gamma) \end{aligned} \tag{10}$$

Now

$$\eta(\theta)G(f\alpha,f\alpha,T(\alpha,\beta,\gamma,\omega)) = 0 \le \max\left\{\begin{array}{c} G(fa,fa,f\alpha),G(fb,fb,f\beta), \\ G(fc,fc,f\gamma),G(fd,fd,f\omega), \\ G(f\alpha,f\alpha,T(\alpha,\beta,\gamma,\omega)),G(f\beta,f\beta,T(\beta,\gamma,\omega,\alpha)), \\ G(f\gamma,f\gamma,T(\gamma,\omega,\alpha,\beta)),G(f\omega,f\omega,T(\omega,\alpha,\beta,\gamma)) \end{array}\right\}.$$

Hence from (1), we have

$$\begin{aligned} G(f\alpha,f\alpha,fa) &= G(T(\alpha,\beta,\gamma,\omega),T(\alpha,\beta,\gamma,\omega),T(a,b,c,d)) \\ &\le \theta\max\left\{\begin{array}{c} G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb), \\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd), \\ G(f\alpha,f\alpha,T(\alpha,\beta,\gamma,\omega)),G(f\beta,f\beta,T(\beta,\gamma,\omega,\alpha)), \\ G(f\gamma,f\gamma,T(\gamma,\omega,\alpha,\beta)),G(f\omega,f\omega,T(\omega,\alpha,\beta,\gamma)), \\ G(fa,fa,T(a,b,c,d)),G(fb,fb,T(b,c,d,a)), \\ G(fc,fc,T(c,d,a,b)),G(fd,fd,T(d,a,b,c)) \\ G(fa,fa,T(\alpha,\beta,\gamma,\omega)),G(fb,fb,T(\beta,\gamma,\omega,\alpha)), \\ G(fc,fc,T(\gamma,\omega,\alpha,\beta)),G(fd,fd,T(\omega,\alpha,\beta,\gamma)) \end{array}\right\}. \\ &\le \theta\max\left\{\begin{array}{c} G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb), \\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd) \end{array}\right\}. \end{aligned}$$

Similarly, we have

$$G(f\beta,f\beta,fb) \le \theta\max\left\{\begin{array}{c} G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb), \\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd) \end{array}\right\}.$$

and

$$G(f\gamma,f\gamma,fc)\leq\theta\max\left\{\begin{array}{c}G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb),\\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd)\end{array}\right\}.$$

also

$$G(f\omega,f\omega,fd)\leq\theta\max\left\{\begin{array}{c}G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb),\\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd)\end{array}\right\}.$$

Thus

$$\max\left\{\begin{array}{c}G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb),\\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd),\end{array}\right\}\leq\theta\max\left\{\begin{array}{c}G(f\alpha,f\alpha,fa),G(f\beta,f\beta,fb),\\ G(f\gamma,f\gamma,fc),G(f\omega,f\omega,fd)\end{array}\right\}.$$

Hence $\alpha=fa=f\alpha,\beta=fb=f\beta,\gamma=fc=f\gamma$ and $\omega=fd=f\omega$. Hence from (10), we have $(\alpha,\beta,\gamma,\omega)$ is a common quadruple fixed point of T and f. In the following we will show the uniqueness of common quadruple fixed point in $\mathscr{P}$. For this purpose, assume that there is another quadruple fixed point $(\alpha',\beta',\gamma',\omega')$ of T,f. Now consider,

$$\eta(\theta)G(f\alpha,f\alpha,T(\alpha,\beta,\gamma,\omega))=0\leq\max\left\{\begin{array}{c}G(f\alpha,f\alpha,f\alpha'),G(f\beta,f\beta,f\beta'),\\ G(f\gamma,f\lambda,f\gamma'),G(f\omega,f\omega,f\omega'),\\ G(f\alpha,f\alpha,T(\alpha,\beta,\gamma,\omega)),G(f\beta,f\beta,T(\beta,\gamma,\omega,\alpha)),\\ G(f\gamma,f\gamma,T(\gamma,\omega,\alpha,\beta)),G(f\omega,f\omega,T(\omega,\alpha,\beta,\gamma))\end{array}\right\}$$

by (1), we have

$$\begin{aligned}G(\alpha,\alpha,\alpha') &= G\big(T(\alpha,\beta,\gamma,\omega),T(\alpha,\beta,\gamma,\omega),T(\alpha',\beta',\gamma',\omega')\big)\\ &\leq \theta\max\left\{\begin{array}{c}G(\alpha,\alpha,\alpha'),G(\beta,\beta,\beta'),\\ G(\gamma,\gamma,\gamma'),G(\omega,\omega,\omega')\end{array}\right\}.\end{aligned}$$

Similarly, we can show that

$$G(\beta,\beta,\beta')\leq\theta\max\left\{\begin{array}{c}G(\alpha,\alpha,\alpha'),G(\beta,\beta,\beta'),\\ G(\gamma,\gamma,\gamma'),G(\omega,\omega,\omega')\end{array}\right\}$$

and

$$G(\gamma,\gamma,\gamma')\leq\theta\max\left\{\begin{array}{c}G(\alpha,\alpha,\alpha'),G(\beta,\beta,\beta'),\\ G(\gamma,\gamma,\gamma'),G(\omega,\omega,\omega')\end{array}\right\}.$$

also

$$G(\omega,\omega,\omega') \leq \theta \max\left\{\begin{array}{c} G(\alpha,\alpha,\alpha'),G(\beta,\beta,\beta'), \\ G(\gamma,\gamma,\gamma'),G(\omega,\omega,\omega') \end{array}\right\}.$$

Thus

$$\max\left\{\begin{array}{c} G(\alpha,\alpha,\alpha'),G(\beta,\beta,\beta'), \\ G(\gamma,\gamma,\gamma'),G(\omega,\omega,\omega') \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} G(\alpha,\alpha,\alpha'),G(\beta,\beta,\beta'), \\ G(\gamma,\gamma,\gamma'),G(\omega,\omega,\omega') \end{array}\right\}.$$

Hence $\alpha=\alpha',\beta=\beta',\gamma=\gamma'$ and $\omega=\omega'$. Thus $(\alpha,\beta,\gamma,\omega)$ is the unique common quadruple fixed point of T and f.

case(ii): If $fx_n = fx_{n+1}, fy_n = fy_{n+1}, fz_n = fz_{n+1}$ and $fw_n = fw_{n+1}$ for some n then $fx_n = T(x_n,y_n,z_n,w_n)$, $fy_n = T(y_n,z_n,w_n,x_n)$, $fz_n = T(z_n,w_n,x_n,y_n)$, $fw_n = T(w_n,x_n,y_n,z_n)$ so that (x_n,y_n,z_n,w_n) is a quadruple coincidence point of T and f . Now proceeding as in case (i) with $fx_n=\alpha$, $fy_n=\beta$, $fz_n=\gamma$ and $fw_n=\omega$, we can show that $(\alpha,\beta,\gamma,\omega)$ is the unique common quadruple fixed point of T and f .

Example 3.2: Let $\mathscr{P}=[0,1]$ and $G(x,y,z)=\max\{|x-y|,|x-z|,|y-z|\}$, $(\mathscr{P},G)$ is a complete G-metric spaces. Let $T:\mathscr{P}^4\to\mathscr{P}$ and $f:\mathscr{P}\to\mathscr{P}$ be given by$T(x,y,z,w)=\sin(\frac{x+y+z+w}{16})$ and $f(x)=\frac{x}{2}$. Then obviously, $T(\mathscr{P}^4)\subseteq f(\mathscr{P})$ and the pair (T,f) are ω-compatible and clearly for all $a,b,c,d\in\mathscr{P}$

$$\begin{aligned} \frac{2}{3}G(fa,fa,T(a,b,c,d)) &\leq G(fa,fa,T(a,b,c,d)) \\ &\leq \max\left\{\begin{array}{c} G(fa,fa,fx),G(fb,fb,fy), \\ G(fc,fc,fz),G(fd,fd,fw), \\ G(fa,fa,T(a,b,c,d)),G(fb,fb,T(b,c,d,a)), \\ G(fc,fc,T(c,d,a,b)),G(fd,fd,T(d,a,b,c)) \end{array}\right\} \end{aligned}$$

Now we have

$$\begin{aligned}
& G(T(a,b,c,d),T(a,b,c,d),,T(x,y,z,w)) \\
= & \max\{|T(a,b,c,d)-T(x,y,z,w)|\} \\
= & |\sin(\frac{a+b+c+d}{16})-\sin(\frac{x+y+z+w}{16})| \\
= & 2|\cos(\frac{a+b+c+d+x+y+z+w}{32})\sin(\frac{a+b+c+d-x-y-z-w}{32})| \\
\leq & \frac{1}{16}|a+b+c+d-x-y-z-w| \leq \frac{1}{2}|\frac{x}{2}-\sin(\frac{a+b+c+d}{16})| \\
\leq & \frac{1}{2}\max\{|fx-T(a,b,c,d)\} = \frac{1}{2}G(fx,fx,T(a,b,c,d)) \\
\leq & \frac{1}{2}\max\left\{\begin{array}{c}
G(fa,fa,fx),G(fb,fb,fy),G(fc,fc,fz),G(fd,fd,fw), \\
G(fa,fa,T(a,b,c,d)),G(fb,fb,T(b,c,d,a)), \\
G(fc,fc,T(c,d,a,b)),G(fd,fd,T(d,a,b,c)) \\
G(fx,fx,T(x,y,z,w)),G(fy,fy,T(y,z,w,x)), \\
G(fz,fz,T(z,w,x,y)),G(fw,fw,T(w,x,y,z)) \\
G(fx,fx,T(a,b,c,d)),G(fy,fy,T(b,c,d,a)), \\
G(fz,fz,T(c,d,a,b)),G(fw,fw,T(d,a,b,c))
\end{array}\right\}.
\end{aligned}$$

Thus all the conditions of the Theorem (3.1) are satisfied and $(0,0,0,0)$ is unique common quadruple fixed point of T and f.

Corollary 3.3: Let $(\mathscr{P},G)$ be a complete G-metric space. Suppose that $T:\mathscr{P}^4\to\mathscr{P}$ be a mapping satisfying:

$$\eta(\theta)G(x,x,T(x,y,z,w)) \leq \max\left\{\begin{array}{c}
G(x,x,p),G(y,y,q),G(z,z,r),G(w,w,s), \\
G(x,x,T(x,y,z,w)),G(y,y,T(y,z,w,x)), \\
G(z,z,T(z,w,x,y)),G(w,w,T(w,x,y,z))
\end{array}\right\}$$

implies

$$G(T(x,y,z,w),T(x,y,z,w),T(p,q,r,s)) \leq \theta \max \left\{ \begin{array}{c} G(x,x,p),G(y,y,q),G(z,z,r),G(w,w,s), \\ G(x,x,T(x,y,z,w)),G(y,y,T(y,z,w,x)), \\ G(z,z,T(z,w,x,y)),G(w,w,T(w,x,y,z)) \\ G(p,p,T(p,q,r,s)),G(q,q,T(q,r,s,p)), \\ G(r,r,T(r,s,p,q)),G(s,s,T(s,p,q,r)) \\ G(p,p,T(x,y,z,w)),G(q,q,T(y,z,w,x)), \\ G(r,r,T(z,w,x,y)),G(s,s,T(w,x,y,z)) \end{array} \right\}. \tag{11}$$

for all $x,y,z,w,p,q,r,s \in \mathscr{P}$, where $\theta \in [0,1)$ and $\eta : [0,1) \to [\frac{1}{2},1)$ defined as $\eta(\theta) = \frac{1}{1+\theta}$is a strictly decreasing function. Then there is a unique quadruple fixed point of T in $\mathscr{P}$.

3.1. APPLICATION TO INTEGRAL EQUATIONS.

In this section, we study the existence of an unique solution to an initial value problem, as an application to Corollary (3.3).

Theorem 3.1.1: Consider the initial value problem

$$x^1(t) = T(t,(x,y,z,w)(t)),\ t \in I = [0,1],\ (x,y,z,w)(0) = (x_0,y_0,z_0,w_0) \tag{12}$$

where $T : I \times \mathbb{R} \to \mathbb{R}$ with $\int_0^t T(s,(x,y,z,w)(s))ds = \max \left\{ \begin{array}{c} \int_0^t T(s,x(s))ds, \int_0^t T(s,y(s))ds, \\ \int_0^t T(s,z(s))ds, \int_0^t T(s,w(s))ds, \end{array} \right\}$
and $x_0,y_0,z_0,w_0 \in \mathbb{R}$. Then there exists unique solution in $C(I,\mathbb{R})$ for the initial value problem (12).

Proof. The integral equation corresponding to initial value problem (12) is

$$x(t) = x_0 + 3\int_0^t T(s,(x,y,z,w)(s))ds.$$

Let $\mathscr{P} = C(I,\mathbb{R})$ and $G(x,y,z) = |x-y|+|y-z|+|z-x|\ for\ x,y,z \in \mathscr{P}$, define $R : \mathscr{P}^4 \to \mathscr{P}$ by

$$R(x,y,z,w)(t) \quad = \quad \frac{x_0}{3} + \int_0^t T(s,(x,y,z,w)(s))ds.$$

Clearly for all $x, y, z, w \in \mathscr{P}$, we have

$$\frac{3}{4}G(x,x,T(x,y,z,w)) \leq \max\left\{\begin{array}{c} G(x,x,a), G(y,y,b), G(z,z,c), G(w,w,d), \\ G(x,x,T(x,y,z,w)), G(y,y,T(y,z,w,x)), \\ G(z,z,T(z,w,x,y)), G(w,w,T(w,x,y,z)) \end{array}\right\}.$$

Now

$$\begin{aligned}
& G(R(x,y,z,w)(t), R(x,y,z,w)(t), R(a,b,c,d)(t)) \\
= \;& 2|R(x,y,z,w)(t) - R(a,b,c,d)(t)| \\
= \;& 2|\frac{x_0}{3} + \int_0^t T(s,(x,y,z,w)(s))ds - \frac{a_0}{3} + \int_0^t T(s,(a,b,c,d)(s))ds| \\
= \;& \frac{2}{3}|x(t) - a(t)| = \frac{1}{3}G(x,x,a) \\
\leq \;& \frac{1}{3}\max\left\{\begin{array}{c} G(x,x,a), G(y,y,q), G(z,z,c), G(w,w,d), \\ G(x,x,T(x,y,z,w)), G(y,y,T(y,z,w,x)), \\ G(z,z,T(z,w,x,y)), G(w,w,T(w,x,y,z)) \\ G(a,a,T(a,b,c,d)), G(b,b,T(b,c,d,a)), \\ G(c,c,T(c,d,a,b)), G(d,d,T(d,a,b,c)) \\ G(a,a,T(x,y,z,w)), G(b,b,T(y,z,w,x)), \\ G(c,c,T(z,w,x,y)), G(d,d,T(w,x,y,z)) \end{array}\right\}.
\end{aligned}$$

It follows from Corollary (3.3), we conclude that R has a unique fixed point in $\mathscr{P}$.

3.2. APPLICATION TO HOMOTOPY.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.2.1: Let $(\mathscr{P}, G)$ be complete G-metric space, U and $\overline{U}$ be an open and closed subset of $\mathscr{P}$ such that $U \subseteq \overline{U}$. Suppose $H : \overline{U}^4 \times [0,1] \to \mathscr{P}$ be an operator with following conditions are satisfying,

τ_0) $x \neq H(x,y,z,w,\kappa)$, $y \neq H(y,z,w,x,\kappa)$, $z \neq H(z,w,x,z,\kappa)$ and $w \neq H(w,x,y,z,\kappa)$ for each $x,y,z,w \in \partial U$ and $\kappa \in [0,1]$ (Here ∂U is boundary of U in $\mathscr{P}$);

τ_1) for all $x,y,z,w,a,b,c,d \in \overline{U}$ and $\kappa \in [0,1]$ such that

$$\eta(\theta)G(x,x,H(x,y,z,w,\kappa)) \leq \max\left\{\begin{array}{c} G(x,x,p),G(y,y,q),G(z,z,r),G(w,w,s), \\ G(x,x,H(x,y,z,w,\kappa)),G(y,y,H(y,z,w,x,\kappa)), \\ G(z,z,H(z,w,x,y,\kappa)),G(w,w,H(w,x,y,z,\kappa)) \end{array}\right\} \text{ implies}$$

$$G\left(\begin{array}{c} H(x,y,z,w,\kappa), \\ H(x,y,z,w,\kappa), \\ H(p,q,r,s,\kappa) \end{array}\right) \leq \theta \max\left\{\begin{array}{c} G(x,x,p),G(y,y,q),G(z,z,r),G(w,w,s), \\ G(x,x,H(x,y,z,w,\kappa)),G(y,y,H(y,z,w,x,\kappa)), \\ G(z,z,H(z,w,x,y,\kappa)),G(w,w,H(w,x,y,z,\kappa)) \\ G(p,p,H(p,q,r,s,\kappa)),G(q,q,H(q,r,s,p,\kappa)), \\ G(r,r,H(r,s,p,q,\kappa)),G(s,s,H(s,p,q,r,\kappa)) \\ G(p,p,H(x,y,z,w,\kappa)),G(q,q,H(y,z,w,x,\kappa)), \\ G(r,r,H(z,w,x,y,\kappa)),G(s,s,H(w,x,y,z,\kappa)) \end{array}\right\}.$$

(13)

where $\theta \in [0,1)$ and $\eta : [0,1) \to [\frac{1}{2},1)$ defined as $\eta(\theta) = \frac{1}{1+\theta}$is a strictly decreasing function.

τ_2) $\exists\, M \geq 0 \ni G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(a,b,c,d,\zeta)) \leq M|\kappa - \zeta|$

for every $x,y,z,w,a,b,c,d \in \overline{U}$and $\kappa,\zeta \in [0,1]$.

Then $H(.,0)$ has a quadruple fixed point $\iff$ $H(.,1)$ has a quadruple fixed point.

Proof. Let the set

$$X = \left\{\begin{array}{c} \kappa \in [0,1] : H(x,y,z,w,\kappa) = x, H(y,z,w,x,\kappa) = y \\ H(z,w,x,y,\kappa) = z, H(w,x,y,z,w,\kappa) = w, \text{ for some } x,y,z,w, \in U \end{array}\right\}.$$

Suppose that $H(.,0)$ has a quadruple fixed point in U^4, we have that $(0,0,0,0) \in X^4$. So that X is non-empty set. Now we show that X is both closed and open in $[0,1]$ and hence by the connectedness $X = [0,1]$. As a result, $H(.,1)$ has a quadruple fixed point in U^4. First we show that X closed in $[0,1]$. To see this, Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq X$ with $\kappa_n \to \kappa \in [0,1]$ as $n \to \infty$. We must show that $\kappa \in X$. Since $\kappa_n \in X$ for $n = 0,1,2,3,\cdots$, there exists sequences $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ with

$x_n = H(x_n,y_n,z_n,w_n,\kappa_n)$, $y_n = H(y_n,z_n,w_n,x_n,\kappa_n)$, $z_n = H(z_n,w_n,x_n,y_n,\kappa_n)$

and $w_n = H(w_n,x_n,y_n,z_n,\kappa_n)$

Consider

$$
\begin{aligned}
& G(x_n, x_n, x_{n+1}) \\
= \quad & G\left(H(x_n, y_n, z_n, w_n, \kappa_n), H(x_n, y_n, z_n, w_n, \kappa_n), H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_{n+1})\right) \\
\leq \quad & G\left(\begin{array}{c} H(x_n, y_n, z_n, w_n, \kappa_n), H(x_n, y_n, z_n, w_n, \kappa_n), \\ H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_n) \end{array}\right) \\
& + G\left(\begin{array}{c} H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_n), H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_n), \\ H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_{n+1}) \end{array}\right) \\
\leq \quad & G\left(H(x_n, y_n, z_n, w_n, \kappa_n), H(x_n, y_n, z_n, w_n, \kappa_n), H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_n)\right) \\
& + M|\kappa_n - \kappa_{n+1}|.
\end{aligned}
$$

Letting $n \to \infty$, we get

$$
\begin{aligned}
& \lim_{n\to\infty} G(x_n, x_n, x_{n+1}) \\
\leq \quad & \lim_{n\to\infty} G\left(H(x_n, y_n, z_n, w_n, \kappa_n), H(x_n, y_n, z_n, w_n, \kappa_n), H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_n)\right) + 0.
\end{aligned}
$$

Since

$$
\eta(\theta) G\left(x_n, x_n, H(x_n, y_n, z_n, w_n, \kappa)\right) \leq \max \left\{ \begin{array}{c} G\left(x_n, x_n, x_{n+1}\right), G\left(y_n, y_n, y_{n+1}\right), \\ G\left(z_n, z_n, z_{n+1}\right), G\left(w_n, w_n, w_{n+1}\right), \\ G\left(x_n, x_n, H(x_n, y_n, z_n, w_n, \kappa)\right), \\ G\left(y_n, y_n, H(y_n, z_n, w_n, x_n, \kappa)\right), \\ G\left(z_n, z_n, H(z_n, w_n, x_n, y_n, \kappa)\right), \\ G\left(w_n, w_n, H(w_n, x_n, y_n, z_n, \kappa)\right) \end{array} \right\}
$$

from (13), we have that

$$
\begin{aligned}
& \lim_{n\to\infty} G(x_n, x_n, x_{n+1}) \\
\leq \quad & \lim_{n\to\infty} G\left(H(x_n, y_n, z_n, w_n, \kappa_n), H(x_n, y_n, z_n, w_n, \kappa_n), H(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}, \kappa_n)\right)
\end{aligned}
$$

$$\leq \lim_{n\to\infty} \theta \max \left\{ \begin{array}{c} G(x_n,x_n,x_{n+1}), G(y_n,y_n,y_{n+1}), G(z_n,z_n,z_{n+1}), G(w_n,w_n,w_{n+1}), \\ G(x_n,x_n,H(x_n,y_n,z_n,w_n,\kappa_n)), G(y_n,y_n,H(y_n,z_n,w_n,x_n,\kappa_n)), \\ G(z_n,z_n,H(z_n,w_n,x_n,y_n,\kappa_n)), G(w_n,w_n,H(w_n,x_n,y_n,z_n,\kappa_n)), \\ G(x_{n+1},x_{n+1},H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n)), \\ G(y_{n+1},y_{n+1},H(y_{n+1},z_{n+1},w_{n+1},x_{n+1},\kappa_n)), \\ G(z_{n+1},z_{n+1},H(z_{n+1},w_{n+1},x_{n+1},y_{n+1},\kappa_n)), \\ G(w_{n+1},w_{n+1},H(w_{n+1},x_{n+1},y_{n+1},z_{n+1},\kappa_n)), \\ G(x_{n+1},x_{n+1},H(x_n,y_n,z_n,w_n,\kappa_n)), G(y_{n+1},y_{n+1},H(y_n,z_n,w_n,x_n,\kappa_n)), \\ G(z_{n+1},z_{n+1},H(z_n,w_n,x_n,y_n,\kappa_n)), G(w_{n+1},w_{n+1},H(w_n,x_n,y_n,z_n,\kappa_n)) \end{array} \right\}.$$

$$\leq \lim_{n\to\infty} \theta \max \left\{ G(x_n,x_n,x_{n+1}), G(y_n,y_n,y_{n+1}), G(z_n,z_n,z_{n+1}), G(w_n,w_n,w_{n+1}) \right\}.$$

Therefore, we have

$$\lim_{n\to\infty} \max \left\{ \begin{array}{c} G(x_n,x_n,x_{n+1}), G(y_n,y_n,y_{n+1}), \\ G(z_n,z_n,z_{n+1}), G(w_n,w_n,w_{n+1}) \end{array} \right\} \leq \lim_{n\to\infty} \theta \max \left\{ \begin{array}{c} G(x_n,x_n,x_{n+1}), G(y_n,y_n,y_{n+1}), \\ G(z_n,z_n,z_{n+1}), G(w_n,w_n,w_{n+1}) \end{array} \right\}.$$

It follows that $\lim_{n\to\infty} G(x_n,x_n,x_{n+1}) = 0$, $\lim_{n\to\infty} G(y_n,y_n,y_{n+1}) = 0$, $\lim_{n\to\infty} G(z_n,z_n,z_{n+1}) = 0$ and $\lim_{n\to\infty} G(w_n,w_n,w_{n+1}) = 0$. Now, we shall show that $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ are Cauchy sequences in the G-metric space $(\mathscr{P},G)$. Assume the contrary, that is, one of the sequences$\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ is not a Cauchy sequence, there exists $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$\begin{aligned} & G(x_{m_k},x_{m_k},x_{n_k}) \geq \varepsilon, \quad G(y_{m_k},y_{m_k},y_{n_k}) \geq \varepsilon \\ & G(z_{m_k},z_{m_k},z_{n_k}) \geq \varepsilon, \quad G(w_{m_k},w_{m_k},w_{n_k}) \geq \varepsilon. \end{aligned} \tag{14}$$

and

$$\begin{aligned} & G\left(x_{m_k},x_{m_k},x_{n_{k-1}}\right) < \varepsilon, \quad G\left(y_{m_k},y_{m_k},y_{n_{k-1}}\right) < \varepsilon \\ & G\left(z_{m_k},z_{m_k},z_{n_{k-1}}\right) < \varepsilon, \quad G\left(w_{m_k},w_{m_k},w_{n_{k-1}}\right) < \varepsilon. \end{aligned} \tag{15}$$

By use of the rectangle inequality and (14), (15), we have

$$\varepsilon \leq G(x_{m_k},x_{m_k},x_{n_k}) \leq G\left(x_{m_k},x_{m_k},x_{m_{k+1}}\right) + G\left(x_{m_{k+1}},x_{m_{k+1}},x_{n_k}\right)$$

Letting $k \to \infty$, we get

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty} G\left(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k}\right) \\
&\leq \lim_{k\to\infty} G\left(H(x_{m_{k+1}}, y_{m_{k+1}}, z_{m_{k+1}}, w_{m_{k+1}}, \kappa_{m_{k+1}}), H(x_{m_{k+1}}, y_{m_{k+1}}, z_{m_{k+1}}, w_{m_{k+1}}, \kappa_{m_{k+1}}), H(x_{n_k}, y_{n_k}, z_{n_k}, w_{n_k}, \kappa_{n_k})\right) \\
&\leq \lim_{n\to\infty} \theta \max \left\{ \begin{array}{c}
G\left(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k}\right), G\left(y_{m_{k+1}}, y_{m_{k+1}}, y_{n_k}\right), \\
G\left(z_{m_{k+1}}, z_{m_{k+1}}, z_{n_k}\right), G\left(w_{m_{k+1}}, w_{m_{k+1}}, w_{n_k}\right), \\
G\left(x_{m_{k+1}}, x_{m_{k+1}}, H(x_{m_{k+1}}, y_{m_{k+1}}, z_{m_{k+1}}, w_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(y_{m_{k+1}}, y_{m_{k+1}}, H(y_{m_{k+1}}, z_{m_{k+1}}, w_{m_{k+1}}, x_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(z_{m_{k+1}}, z_{m_{k+1}}, H(z_{m_{k+1}}, w_{m_{k+1}}, x_{m_{k+1}}, y_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(w_{m_{k+1}}, w_{m_{k+1}}, H(w_{m_{k+1}}, x_{m_{k+1}}, y_{m_{k+1}}, z_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(x_{n_k}, x_{n_k}, H(x_{n_k}, y_{n_k}, z_{n_k}, w_{n_k}, \kappa_{n_k})\right), \\
G\left(y_{n_k}, y_{n_k}, H(y_{n_k}, z_{n_k}, w_{n_k}, x_{n_k}, \kappa_{n_k})\right), \\
G\left(z_{n_k}, z_{n_k}, H(z_{n_k}, w_{n_k}, x_{n_k}, y_{n_k}, \kappa_{n_k})\right), \\
G\left(w_{n_k}, w_{n_k}, H(w_{n_k}, x_{n_k}, y_{n_k}, z_{n_k}, \kappa_{n_k})\right), \\
G\left(x_{n_k}, x_{n_k}, H(x_{m_{k+1}}, y_{m_{k+1}}, z_{m_{k+1}}, w_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(y_{n_k}, y_{n_k}, H(y_{m_{k+1}}, z_{m_{k+1}}, w_{m_{k+1}}, x_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(z_{n_k}, z_{n_k}, H(z_{m_{k+1}}, w_{m_{k+1}}, x_{m_{k+1}}, y_{m_{k+1}}, \kappa_{m_{k+1}})\right), \\
G\left(w_{n_k}, w_{n_k}, H(w_{m_{k+1}}, x_{m_{k+1}}, y_{m_{k+1}}, z_{m_{k+1}}, \kappa_{m_{k+1}})\right)
\end{array} \right\}. \\
&\leq \lim_{n\to\infty} \theta \max \left\{ G\left(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k}\right), G\left(y_{m_{k+1}}, y_{m_{k+1}}, y_{n_k}\right), G\left(z_{m_{k+1}}, z_{m_{k+1}}, z_{n_k}\right), G\left(w_{m_{k+1}}, w_{m_{k+1}}, w_{n_k}\right) \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty} \max \left\{ \begin{array}{c} G\left(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k}\right), G\left(y_{m_{k+1}}, y_{m_{k+1}}, y_{n_k}\right), \\ G\left(z_{m_{k+1}}, z_{m_{k+1}}, z_{n_k}\right), G\left(w_{m_{k+1}}, w_{m_{k+1}}, w_{n_k}\right) \end{array} \right\} \\
&\leq \theta \lim_{k\to\infty} \max \left\{ \begin{array}{c} G\left(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k}\right), G\left(y_{m_{k+1}}, y_{m_{k+1}}, y_{n_k}\right), \\ G\left(z_{m_{k+1}}, z_{m_{k+1}}, z_{n_k}\right), G\left(w_{m_{k+1}}, w_{m_{k+1}}, w_{n_k}\right) \end{array} \right\}.
\end{aligned}$$

It follows that $\lim_{k\to\infty} G\left(x_{m_{k+1}}, x_{m_{k+1}}, x_{n_k}\right) = 0$, $\lim_{k\to\infty} G\left(y_{m_{k+1}}, y_{m_{k+1}}, y_{n_k}\right) = 0$, $\lim_{k\to\infty} G\left(z_{m_{k+1}}, z_{m_{k+1}}, z_{n_k}\right) = 0$ and $\lim_{k\to\infty} G\left(w_{m_{k+1}}, w_{m_{k+1}}, w_{n_k}\right) = 0$. Therefore, $\varepsilon = 0$. which is a contradiction. Hence $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ are Cauchy sequences in the G-metric space $(\mathscr{P}, G)$ and by the completeness of $(\mathscr{P}, G)$, there exist $a, b, c, d \in \mathscr{P}$ with

$$\lim_{n\to\infty} x_{n+1} = a \quad \lim_{n\to\infty} y_{n+1} = b \quad \lim_{n\to\infty} z_{n+1} = c \quad \lim_{n\to\infty} w_{n+1} = d.$$

Since

$$\eta(\theta)G(a,a,H(a,b,c,d,\kappa)) \leq \max\left\{\begin{array}{c} G(a,a,x_n), G(b,b,y_n), G(c,c,z_n), G(d,d,w_n), \\ G(a,a,H(a,b,c,d,\kappa)), G(b,b,H(b,c,d,a,\kappa)), \\ G(c,c,H(c,d,a,b,\kappa)), G(d,d,H(d,a,b,c,\kappa)) \end{array}\right\}$$

we have

$$\begin{aligned} G(a,a,H(a,b,c,d,\kappa)) &\leq \lim_{n\to\infty} G(H(x_n,y_n,z_n,w_n,\kappa),H(x_n,y_n,z_n,w_n,\kappa),H(a,b,c,d,\kappa)) \\ &\leq \lim_{n\to\infty} \theta\max\left\{\begin{array}{c} G(x_n,x_n,a), G(y_n,y_n,b), G(z_n,z_n,c), G(w_n,w_n,d), \\ G(a,a,H(a,b,c,d,\kappa)), G(b,b,H(b,c,d,a,\kappa)), \\ G(c,c,H(c,d,a,b,\kappa)), G(d,d,H(d,a,b,c,\kappa)) \\ G(x_n,x_n,H(x_n,y_n,z_n,w_n,\kappa)), G(y_n,y_n,H(y_n,z_n,w_n,x_n,\kappa)), \\ G(z_n,z_n,H(z_n,w_n,x_n,y_n,\kappa)), G(w_n,w_n,H(w_n,x_n,y_n,z_n,\kappa)) \\ G(a,a,H(x_n,y_n,z_n,w_n,\kappa)), G(b,b,H(y_n,z_n,w_n,x_n,\kappa)), \\ G(c,c,H(z_n,w_n,x_n,y_n,\kappa)), G(d,d,H(w_n,x_n,y_n,z_n,\kappa)) \end{array}\right\}. \\ &\leq \theta\max\left\{\begin{array}{c} G(a,a,H(a,b,c,d,\kappa)), G(b,b,H(b,c,d,a,\kappa)), \\ G(c,c,H(c,d,a,b,\kappa)), G(d,d,H(d,a,b,c,\kappa)) \end{array}\right\}. \end{aligned}$$

Therefore,

$$\max\left\{\begin{array}{c} G(a,a,H(a,b,c,d,\kappa)), G(b,b,H(b,c,d,a,\kappa)), \\ G(c,c,H(c,d,a,b,\kappa)), G(d,d,H(d,a,b,c,\kappa)) \end{array}\right\} \leq \theta\max\left\{\begin{array}{c} G(a,a,H(a,b,c,d,\kappa)), \\ G(b,b,H(b,c,d,a,\kappa)), \\ G(c,c,H(c,d,a,b,\kappa)), \\ G(d,d,H(d,a,b,c,\kappa)) \end{array}\right\}.$$

It follows that $H(a,b,c,d,\kappa) = a, H(b,c,d,a,\kappa) = b, H(c,d,a,b,\kappa) = c$ and $H(d,a,b,c,\kappa) = d$. Thus $\kappa \in X$. Hence X is closed in $[0,1]$. Let $\kappa_0 \in X$, then there exist $x_0,y_0,z_0,w_0 \in U$ with $x_0 = H(x_0,y_0,z_0,w_0,\kappa_0)$, $y_0 = H(y_0,z_0,w_0,x_0,\kappa_0)$, $z_0 = H(z_0,w_0,x_0,y_0,\kappa_0)$ and $w_0 = H(w_0,x_0,y_0,z_0,\kappa_0)$. Since U is open, then there exist $r > 0$ such that $B_G(x_0,x_0,r) \subseteq U$. Choose $\kappa \in (\kappa_0 - \varepsilon, \kappa_0 + \varepsilon)$ such that

$|\kappa-\kappa_0| \leq \frac{1}{M^n} < \frac{\varepsilon}{2}$, then for $x \in \overline{B_G(x_0,x_0,r)} = \{x \in X / G(x,x,x_0) \leq r + G(x_0,x_0,x_0)\}$. Also

$$\eta(\theta)G(x,x,H(x_0,y_0,z_0,w_0,\kappa)) \leq \max\left\{\begin{array}{c} G(x,x,x_0), G(y,y,y_0), G(z,z,z_0), G(w,w,w_0), \\ G(x,x,H(x_0,y_0,z_0,w_0,\kappa)), G(y,y,H(y_0,z_0,w_0,x_0,\kappa)), \\ G(z,z,H(z_0,w_0,x_0,y_0,\kappa)), G(w,w,H(w_0,x_0,y_0,z_0,\kappa)) \end{array}\right\}$$

Now we have

$$\begin{aligned} & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0) \\ = \; & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(x_0,y_0,z_0,w_0,\kappa_0)) \\ \leq \; & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(x,y,z,w,\kappa_0)) \\ & +G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)) \\ \leq \; & M|\kappa-\kappa_0| + G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)) \\ \leq \; & \frac{1}{M^{n-1}} + G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0) \\ & \leq G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)). \\ \leq \; & \theta \max\left\{\begin{array}{c} G(x,x,x_0), G(y,y,y_0), G(z,z,z_0), G(w,w,w_0), \\ G(x,x,H(x,y,z,w,\kappa)), G(y,y,H(y,z,w,x,\kappa)), \\ G(z,z,H(z,w,x,y,\kappa)), G(w,w,H(w,x,y,z,\kappa)) \\ G(x_0,x_0,H(x_0,y_0,z_0,w_0,\kappa)), G(y_0,y_0,H(y_0,z_0,w_0,x_0,\kappa)), \\ G(z_0,z_0,H(z_0,w_0,x_0,y_0,\kappa)), G(w_0,w_0,H(w_0,x_0,y_0,z_0,\kappa)) \\ G(x_0,x_0,H(x,y,z,w,\kappa)), G(y_0,y_0,H(y,z,w,x,\kappa)), \\ G(z_0,z_0,H(z,w,x,y,\kappa)), G(w_0,w_0,H(w,x,y,z,\kappa)) \end{array}\right\}. \\ \leq \; & \theta \max\left\{ G(x,x,x_0), G(y,y,y_0), G(z,z,z_0), G(w,w,w_0), \right\}. \end{aligned}$$

Therefore, we have

$$\max\left\{\begin{array}{c} G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0) \\ G(H(y,z,w,x,\kappa),H(y,z,w,x,\kappa),y_0) \\ G(H(z,w,x,y,\kappa),H(z,w,x,y,\kappa),z_0) \\ G(H(w,x,y,z,\kappa),H(w,x,y,z,\kappa),w_0) \end{array}\right\} \leq \theta \max\left\{\begin{array}{c} G(x,x,x_0), G(y,y,y_0), \\ G(z,z,z_0), G(w,w,w_0) \end{array}\right\}.$$

$$\leq \quad \theta \max\left\{\begin{array}{c} r+G(x_0,x_0,x_0), r+G(y_0,y_0,y_0), \\ r+G(z_0,z_0,z_0), r+G(w_0,w_0,w_0) \end{array}\right\}.$$

Thus for each fixed $\kappa \in (\kappa_0-\varepsilon,\kappa_0+\varepsilon)$, $H(.,\kappa): \overline{B_G(x_0,x_0,r)} \to \overline{B_G(x_0,x_0,r)}$, $H(.,\kappa): \overline{B_G(y_0,y_0,r)} \to \overline{B_G(y_0,y_0,r)}$, $H(.,\kappa): \overline{B_G(z_0,z_0,r)} \to \overline{B_G(z_0,z_0,r)}$ and $H(.,\kappa): \overline{B_G(w_0,w_0,r)} \to \overline{B_G(w_0,w_0,r)}$. Then all conditions of Theorem $(3.2.1)$ are satisfied. Thus we conclude that $H(.,\kappa)$ has a quadruple fixed point in $\overline{U}^4$. But this must be in U^4. Since (τ_0) holds. Thus, $\kappa \in X$ for any $\kappa \in (\kappa_0-\varepsilon,\kappa_0+\varepsilon)$. Hence $(\kappa_0-\varepsilon,\kappa_0+\varepsilon) \subseteq X$. Clearly X is open in [0, 1]. For the reverse implication, we use the same strategy.

4. CONCLUSION

We ensured the existence and uniqueness of a common Quadruple fixed point for two mappings in the class of G-metric spaces via suzuki-type contractions. Two illustrated applications have been provided.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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