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ITERATIVE METHOD OF EXTRAGRADIENT TYPE IN HILBERT SPACES

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Abstract. This article is concerned with hybrid implicit extragradient methods for variational inequality problems with constraints of a family of nonexpansive mappings, and a system of variational inequalities. One analyzes the convergence of the hybrid methods and obtains convergence theorems of solutions without the aid of compactness in Hilbert spaces.

Keywords: common solution; convergence analysis; fixed point; monotone operator; variational inequality.

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1. Introduction

Throughout this work, one always suppose that H is a real infinite dimensional Hilbert space and C is a nonempty set of in H. Let $S : C \to H$ be a nonlinear single-valued mapping and denote by $\mathrm{Fix}(S)$ the fixed-point set of mapping S, that is, $\mathrm{Fix}(S) = \{x \in C : x = Sx\}$. S is said to be an asymptotically nonexpansive mapping if $\|T^n x - T^n y\| \le (1+\theta_n)\|x-y\|$, $\forall n \ge 1,\ x, y \in C$, where $\{\theta_n\} \subset [0, +\infty)$ is a sequence such that $\lim_{n\to\infty} \theta_n = 0$. In particular, T is said to be nonexpansive provided that $\theta_n = 0$. S is said to be an averaged mapping if it can written as $S = (1-\alpha)I + \alpha R$, where $\alpha \in (0,1)$ is a real constant, I is the identity mapping of H and

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$R : C \to C$ is a nonexpansive mapping. Recall that the classical monotone variational inequality problem (VI) is to find $x^* \in C$ such that

$$\langle Sx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The set of all solutions the inequality is denoted by VI(C,A). From the viewpoint of computation, lots of authors are concerned with robust iterative algorithms for solving the variational inequality in infinite dimensional spaces; see, e.g., [1, 2, 3, 4, 5, 6, 7]. Among these fast iterative algorithms, algorithms of extragradient type introduced and studied by Korpelevich [8] are under spotlight of many investigators since they are efficient for non cocoercive mappings. Recall that S is called monotone if $\langle Sx - Sy, x - y \rangle \geq 0$, $\forall x, y \in C$. It is called η-strongly monotone if $\langle Sx - Sy, x - y \rangle \geq \eta \|x - y\|^2$, $\forall x, y \in C$, where $\eta > 0$ is real number. Moreover, it is called α-inverse-strongly monotone (or α-cocoercive) if $\langle Sx - Sy, x - y \rangle \geq \alpha \|Sx - Sy\|^2$, $\forall x, y \in C$, where $\alpha > 0$ is a positive number. Obviously, each inverse-strongly monotone mapping is Lipschitzian monotone, and each strongly monotone and Lipschitzian mapping is inverse-strongly monotone but the converse is not true. Let $B_1, B_2 : C \to H$ be two nonlinear single-valued mappings. One considers the following system of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C. \end{cases} \tag{1.2}$$

This system include several problems, such as, variational inequality problems, complementarity problems, convex quadratic programming and fixed-point problems; see, e.g., [9, 10, 11, 12, 13, 14]. In particular, if $B_1 = B_2 = S$ and $x^* = y^*$, then problem (1.2) become the classical variational inequality (1.1), which solution set is denoted by VI(C,A). Note that, problem (1.2) can be transformed into a fixed-point problem in the following way.

Recently Cai et al. [12] introduced a viscosity implicit sequence $\{x_n\}$ for solving a hierarchical variational inequality (HVI) over the common solution set $\Omega = \mathrm{GSVI}(C, B_1, B_2) \cap \mathrm{Fix}(T)$ of the system (1.2) and the fixed-point problem of T

$$\begin{cases} u_n = s_n x_n + (1-s_n) y_n, \\ z_n = P_C(u_n - \mu_2 B_2 u_n), \\ y_n = P_C(z_n - \mu_1 B_1 z_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \rho F) T^n y_n] \quad \forall n \geq 0, \end{cases}$$

where $\mu_1 \in (0, 2\alpha), \mu_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{s_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = \lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \|T^{n+1} y_n - T^n y_n\| < \infty$, $0 < \varepsilon \leq s_n \leq 1$ and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$. They established a solution theorem in norm. Recently, lots of authors investigated the common solution with the aid of nearest point projections under mild conditions; see, e.g., [15, 16, 17, 18, 19, 20].

On the other hand, common fixed-point problems, which find more applications in signal processing and image reconstructions, are now under spotlight of researchers. Let $\{T_i\}_{i=1}^{N}$ be N nonexpansive mappings on H such that the common fixed-point set $\Omega = \cap_{i=1}^{N} \mathrm{Fix}(T_i)$ is not empty. In 2015, Bnouhachem et al. [13] introduced the following iterative algorithm

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) T_N^n T_{N-1}^n \cdots T_1^n x_n, \\ x_{n+1} = \alpha_n \rho f(y_n) + \gamma_n x_n + ((1-\gamma_n) I - \alpha_n \mu F) y_n \quad \forall n \geq 0, \end{cases}$$

where $T_i^n = (1-\delta_n^i) I + \delta_n^i T_i$ and $\delta_n^i \in (0,1)$ for $i = 1, 2, ..., N$, $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \rho < \frac{\nu}{\tau}$, with $\nu = \mu(\eta - \frac{\mu\kappa^2}{2})$ $\limsup_{n\to\infty} \gamma_n < 1$, $\liminf_{n\to\infty} \gamma_n > 0$, $\lim_{n\to\infty} |\delta_{n-1}^i - \delta_n^i| = \lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [\sigma, 1)$ and $\lim_{n\to\infty} \beta_n = \beta < 1$. They proved the strong convergence of sequence $\{x_n\}$. The limit of $\{x_n\}$ also solve a monotone variational inequality with contractive mapping f. For the works on the iterative methods for common element problems, one refers to [21, 22, 23, 24, 25, 26] and the references cited therein.

The purpose of this work is to introduce and analyze hybrid implicit extragradient methods for solving variational inequality problems with constraints of a family of nonexpansive mappings, and a symmetrical system of variational inequalities. One analyzes the convergence of the hybrid methods and obtains convergence theorems of solutions without the aid of compactness in Hilbert spaces. One also solves common fixed point problems of nonexpansive and strictly pseudocontractive mappings in Hilbert spaces.

2. Preliminaries

One lists some essential tools for the proof of our main results.

In this case, we say that T is α-averaged. It is easy to see that the averaged mapping T is also nonexpansive and $\mathrm{Fix}(T) = \mathrm{Fix}(R)$.

Lemma 2.1. [13] *If the self-mappings $\{T_i\}_{i=1}^N$ defined on C are averaged and have a common fixed point, then $\cap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 \cdots T_N)$.*

Lemma 2.2. [27] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the conditions: $a_{n+1} \le (1-\lambda_n)a_n + \lambda_n \gamma_n \ \forall n \ge 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=1}^\infty \lambda_n = \infty$, and (ii) $\limsup_{n\to\infty} \gamma_n \le 0$ or $\sum_{n=1}^\infty |\lambda_n \gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3. [28] *Let $\lambda \in (0,1]$, $T : C \to H$ be a nonexpansive mapping, and the mapping $T^\lambda : C \to H$ be defined by $T^\lambda x := Tx - \lambda \mu F(Tx) \ \forall x \in C$, where $F : H \to H$ is κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e., $\|T^\lambda x - T^\lambda y\| \le (1-\lambda\tau)\|x-y\| \ \forall x,y \in C$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1]$.*

Lemma 2.4. *Let the mappings $B_1, B_2 : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : C \to C$ be defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$. If $0 \le \mu_1 \le 2\alpha$ and $0 \le \mu_2 \le 2\beta$, then $G : C \to C$ is nonexpansive.*

Proof. Since B_1 is α-inverse-strongly monotone and B_2 is β-inverse-strongly monotone, one has

$$\begin{aligned}
\|(I-\mu_1 B_1)u - (I-\mu_1 B_1)v\|^2 &\le \|u-v\|^2 - 2\mu_1 \langle u-v, B_u - B_1 v\rangle + \mu_1^2 \|B_u - B_1 v\|^2 \\
&\le \|u-v\|^2 - \mu_1(2\alpha - \mu_1)\|B_u - B_1 v\|^2 \\
&\le \|u-v\|^2.
\end{aligned}$$

On finds that $I - \mu_1 B_1$ is nonexpansive, so is $I - \mu_2 B_2$. This shows $G : C \to C$ is nonexpansive. □

Lemma 2.5. [29] *Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be asymptotically nonexpansive such* *that* $\mathrm{Fix}(T) \neq \emptyset$. *Then* $I-T$ *is demiclosed at zero, i.e., if* $\{x_n\} \subset C$ *converges weakly to some* $x \in C$, *and* $\{(I-T)x_n\}$ *converges strongly to zero, then* $(I-T)x = 0$, *where* I *is the identity mapping of* X.

Lemma 2.6. [30] *Let* $T : C \to H$ *be a* ζ*-strict pseudocontraction. Define* $S : C \to H$ *by* $Sx = \lambda Tx + (1-\lambda)x \ \forall x \in C$. *Then as* $\lambda \in [\zeta, 1)$, S *is a nonexpansive mapping with* $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.

3. Main Results

In this section, one always let the feasible set C be a convex and closed, and assume that the following condition hold.

$T : C \to C$ is an asymptotically nonexpansive mapping with $\{\theta_n\}$ and $\{T_i\}_{i=1}^N$ are N nonexpansive self-mappings on C, and $B_1, B_2 : C \to H$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

$\Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) \neq \emptyset$, where $T_0 := T$, $\mathrm{GSVI}(C, B_1, B_2) := \mathrm{Fix}(G)$ and $G := P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)$ for constants $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$ and $F : C \to H$ is κ-Lipschitzian and η-strongly monotone such that $\nu\delta < \tau := 1 - \sqrt{1-\rho(2\eta - \rho\kappa^2)}$ for $\nu \geq 0$ and $\rho \in (0, \frac{2\eta}{\kappa^2})$.

$T_i^n := (1-\delta_n^i)I + \delta_n^i T_i$ where $\delta_n^i \in (0,1)\ \forall n \geq 1, i = 1, 2, ..., N$.

$\{s_n\} \subset (0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ such that

(i) $\alpha_n + \gamma_n \leq 1\ \forall n \geq 1$;

(ii) $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, $\sum_{n=1}^\infty |\delta_{n+1}^i - \delta_n^i| < \infty$ for $i = 1, 2, ..., N$;

(iv) $0 < \varepsilon \leq s_n \leq 1$, $\sum_{n=1}^\infty |s_{n+1} - s_n| < \infty$;

(v) $\{\beta_n\} \subset [\sigma, 1)$, $0 < \lim_{n\to\infty} \beta_n = \beta < 1$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$;

(vi) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty$.

Theorem 3.1. *labelth1 For any given* $x_1 \in C$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = P_C(I-\mu_1 B_1)P_C(I-\mu_2 B_2)(s_n x_n + (1-s_n)z_n), \\ y_n = \beta_n x_n + (1-\beta_n)T_N^n T_{N-1}^n \cdots T_1^n z_n, \\ x_{n+1} = P_C[\alpha_n \nu f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \rho F)T^n y_n] \quad \forall n \geq 1. \end{cases}$$

Then $x_n \to x^* \in \Omega$ *provided* $\sum_{n=1}^{\infty} \|T^{n+1}y_n - T^n y_n\| < \infty$, *where* $x^* \in \Omega$ *is a unique solution to* $\langle (\rho F - \nu f)x^*, p - x^* \rangle \geq 0 \; \forall p \in \Omega$.

Proof. Set $u_n = s_n x_n + (1-s_n)z_n$ and $v_n = P_C(I - \mu_2 B_2)u_n$. From our conditions on the parameters, one may assume, without loss of generality, that $\{\gamma_n\} \subset [a,b] \subset (0,1)$ and $\theta_n \leq \frac{\alpha_n(\tau - \nu\delta)}{2} \; \forall n \geq 1$. Hence

$$\alpha_n \nu\delta + \gamma_n + (1-\gamma_n - \alpha_n\tau)(1+\theta_n) \leq 1 - \alpha_n(\tau - \nu\delta) + \theta_n \leq 1 - \frac{\alpha_n(\tau-\nu\delta)}{2}.$$

Observe that $G : C \to C$ is defined as $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, where $\mu_1 \in (0, 2\alpha)$ and $\mu_2 \in (0, 2\beta)$. Lemma 2.4 shows that G is nonexpansive. It can be readily seen that there exists a unique element $u_n \in C$ such that $u_n = s_n x_n + (1-s_n)Gu_n$. So, the hybrid implicit extragradient method can be rewritten as

$$\begin{cases} u_n = s_n x_n + (1-s_n)Gu_n, \\ y_n = \beta_n x_n + (1-\beta_n)T_N^n T_{N-1}^n \cdots T_1^n G u_n, \\ x_{n+1} = P_C[\alpha_n \nu f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \rho F)T^n y_n] \quad \forall n \geq 1. \end{cases}$$

One claims that $P_\Omega(\nu f + I - \rho F)$ is a contraction. An application of Lemma 2.3, we have

$$\begin{aligned} \|P_\Omega(\nu f + I - \rho F)x - P_\Omega(\nu f + I - \rho F)y\| &\leq \nu\|f(x) - f(y)\| + \|(I - \rho F)x - (I-\rho F)y\| \\ &\leq \nu\delta\|x-y\| + (1-\tau)\|x-y\| = [1 - (\tau - \nu\delta)]\|x-y\| \; \forall x,y \in C, \end{aligned}$$

which implies that $P_\Omega(\nu f + I - \rho F)$ is a contraction. So, $x^* = P_\Omega(\nu f + I - \rho F)x^*$. Thus, there exists a unique solution $x^* \in \Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2)$ to

$$\langle (\rho F - \nu f)x^*, p - x^* \rangle \geq 0 \quad \forall p \in \Omega.$$

Next, we divide the rest of the proof into several steps.

Step 1. We show $\{x_n\}$ is bounded. Indeed, taking an arbitrary $p \in \Omega$, one has $Gp = p$, $Tp = p$ and $T_i p = p$ for $i = 1, \ldots, N$. Since $G : C \to C$ is nonexpansive, one obtains from that $\|u_n - p\| \leq s_n\|x_n - p\| + (1-s_n)\|u_n - p\|$. Hence $\|u_n - p\| \leq \|x_n - p\| \quad \forall n \geq 1$. Then, according to the relationship $\cap_{i=1}^N \mathrm{Fix}(T_i) = \cap_{i=1}^N \mathrm{Fix}(T_i^n) = \mathrm{Fix}(T_N^n T_{N-1}^n \cdots T_1^n)$, we get from Lemma 2.1 that

$$\begin{aligned} \|y_n - p\| &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|T_N^n T_{N-1}^n \cdots T_1^n G u_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|Gu_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Since $\alpha_n+\gamma_n\le 1$ leads to $0<\frac{\alpha_n}{1-\gamma_n}\le 1$, from Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1}-p\| &\le \|\alpha_n(\nu f(x_n)-\rho Fp)+\gamma_n(x_n-p)+((1-\gamma_n)I-\alpha_n\rho F)T^n y_n\\
&\quad -((1-\gamma_n)I-\alpha_n\rho F)p\|\\
&\le \alpha_n\nu\delta\|x_n-p\|+\alpha_n\|(\nu f-\rho F)p\|+\gamma_n\|x_n-p\|\\
&\quad +(1-\gamma_n)\|(I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n y_n-(I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)p\|\\
&\le \alpha_n\nu\delta\|x_n-p\|+\alpha_n\|(\nu f-\rho F)p\|\\
&\quad +\gamma_n\|x_n-p\|+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)\|y_n-p\|\\
&\le [\alpha_n\nu\delta+\gamma_n+(1-\gamma_n-\alpha_n\tau)(1+\theta_n)]\|x_n-p\|+\alpha_n\|(\nu f-\rho F)p\|\\
&\le [1-\tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n-p\|+\tfrac{\alpha_n(\tau-\nu\delta)}{2}\cdot\tfrac{2\|(\nu f-\rho F)p\|}{\tau-\nu\delta}\\
&\le \max\{\tfrac{2\|(\nu f-\rho F)p\|}{\tau-\nu\delta},\|x_n-p\|\}.
\end{aligned}$$

By induction, we get $\|x_n-p\|\le\max\{\frac{2\|(\nu f-\rho F)p\|}{\tau-\nu\delta},\|x_1-p\|\}$. Thus, $\{x_n\}$ is a bounded vector sequence.

Step 2. We show that $x_n-x_{n+1}\to 0$ and $y_n-y_{n+1}\to 0$.

Indeed, we estimate

$$\begin{aligned}
&\|y_{n+1}-y_n\|\\
&=\|(1-\beta_n)(T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}-T_N^nT_{N-1}^n\cdots T_1^nz_n)\\
&\quad -(\beta_{n+1}-\beta_n)T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}+\beta_n(x_{n+1}-x_n)+(\beta_{n+1}-\beta_n)x_{n+1}\|\\
&\le \beta_n\|x_{n+1}-x_n\|+(1-\beta_n)\|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}-T_N^nT_{N-1}^n\cdots T_1^nz_n\|\\
&\quad +|\beta_{n+1}-\beta_n|\|x_{n+1}-T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}\|\\
&\le \beta_n\|x_{n+1}-x_n\|+(1-\beta_n)[\|z_{n+1}-z_n\|+\|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}-T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\|]\\
&\quad +|\beta_{n+1}-\beta_n|\|x_{n+1}-T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}\|.
\end{aligned}$$

It follows from the definition of T_i^{n+1} that

$$\begin{aligned}
&\|T_2^{n+1}T_1^{n+1}z_{n+1}-T_2^nT_1^nz_{n+1}\|\\
&\le \|T_2^{n+1}T_1^{n+1}z_{n+1}-T_2^{n+1}T_1^nz_{n+1}\|+\|T_2^{n+1}T_1^nz_{n+1}-T_2^nT_1^nz_{n+1}\|\\
&\le \|T_1^{n+1}z_{n+1}-T_1^nz_{n+1}\|+\|T_2^{n+1}T_1^nz_{n+1}-T_2^nT_1^nz_{n+1}\|\\
&\le |\delta_{n+1}^1-\delta_n^1|(\|z_{n+1}\|+\|T_1z_{n+1}\|)+|\delta_{n+1}^2-\delta_n^2|(\|T_1^nz_{n+1}\|+\|T_2T_1^nz_{n+1}\|),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\|T_3^{n+1}T_2^{n+1}T_1^{n+1}z_{n+1} - T_3^nT_2^nT_1^nz_{n+1}\| \\
&\le \|T_2^{n+1}T_1^{n+1}z_{n+1} - T_2^nT_1^nz_{n+1}\| + \|(1-\delta_{n+1}^3)T_2^nT_1^nz_{n+1} \\
&\quad +\delta_{n+1}^3T_3T_2^nT_1^nz_{n+1} - (1-\delta_n^3)T_2^nT_1^nz_{n+1} - \delta_n^3T_3T_2^nT_1^nz_{n+1}\| \\
&\le |\delta_{n+1}^1 - \delta_n^1|(\|z_{n+1}\| + \|T_1z_{n+1}\|) + |\delta_{n+1}^2 - \delta_n^2|(\|T_1^nz_{n+1}\| \\
&\quad +\|T_2T_1^nz_{n+1}\|) + |\delta_{n+1}^3 - \delta_n^3|(\|T_2^nT_1^nz_{n+1}\| + \|T_3T_2^nT_1^nz_{n+1}\|).
\end{aligned}$$

By induction on N, we have

$$\begin{aligned}
&\|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} - T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\| \\
&\le |\delta_{n+1}^1 - \delta_n^1|(\|z_{n+1}\| + \|T_1z_{n+1}\|) + |\delta_{n+1}^2 - \delta_n^2|(\|T_1^nz_{n+1}\| + \|T_2T_1^nz_{n+1}\|) \\
&\quad +\cdots + |\delta_{n+1}^N - \delta_n^N|(\|T_{N-1}^n\cdots T_1^nz_{n+1}\| + \|T_NT_{N-1}^n\cdots T_1^nz_{n+1}\|).
\end{aligned}$$

Since $\sum_{n=1}^\infty \sum_{i=1}^N |\delta_n^i - \delta_{n+1}^i| < \infty$, one asserts that

$$\sup_{n\ge1}\{\sum_{i=1}^N(\|T_{i-1}^n\cdots T_1^nz_{n+1}\| + \|T_iT_{i-1}^n\cdots T_1^nz_{n+1}\|)\} \le M_0 \quad \text{for some } M_0 > 0,$$

with $T_0^n := I$, and hence

$$\sum_{n=1}^\infty \|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} - T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\| \le \sum_{n=1}^\infty\sum_{i=1}^N |\delta_{n+1}^i - \delta_n^i|M_0 < \infty.$$

On the other hand, since $\alpha_n + \gamma_n \le 1$ implies $0 < \frac{\alpha_n}{1-\gamma_n} \le 1$, we deduce that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\le \|\alpha_{n+1}\nu f(x_{n+1}) - \alpha_n\nu f(x_n) + ((1-\gamma_{n+1})I - \alpha_{n+1}\rho F)T^{n+1}y_{n+1} \\
&\quad -((1-\gamma_n)I - \alpha_n\rho F)T^ny_n + \gamma_{n+1}x_{n+1} - \gamma_nx_n\| \\
&= \|\alpha_{n+1}\nu(f(x_{n+1}) - f(x_n)) + (1-\gamma_{n+1})[(I - \tfrac{\alpha_{n+1}}{1-\gamma_{n+1}}\rho F)T^{n+1}y_{n+1} \\
&\quad -(I - \tfrac{\alpha_{n+1}}{1-\gamma_{n+1}}\rho F)T^ny_n] + (\alpha_{n+1} - \alpha_n)(\nu f(x_n) - \rho FT^ny_n) \\
&\quad +(\gamma_{n+1} - \gamma_n)(x_n - T^ny_n) + \gamma_{n+1}(x_{n+1} - x_n)\| \\
&\le \alpha_{n+1}\nu\delta\|x_{n+1} - x_n\| + (1-\gamma_{n+1} - \alpha_{n+1}\tau)(1+\theta_{n+1})\|y_{n+1} - y_n\| \\
&\quad +\|T^{n+1}y_n - T^ny_n\| + (|\alpha_{n+1} - \alpha_n| + |\gamma_{n+1} - \gamma_n|)M_1 + \gamma_{n+1}\|x_{n+1} - x_n\|,
\end{aligned}$$

where $\sup_{n\ge1}\{\|\nu f(x_n) - \rho FT^ny_n\| + \|x_n - T^ny_n\|\} \le M_1$ for some $M_1 > 0$. Also, we observe that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \|s_{n+1}x_{n+1} + (1-s_{n+1})z_{n+1} - s_nx_n - (1-s_n)z_n\| \\
&\le s_{n+1}\|x_{n+1} - x_n\| + (1-s_{n+1})\|z_{n+1} - z_n\| + |s_{n+1} - s_n|\|x_n - z_n\|,
\end{aligned}$$

which immediately yields $\|z_{n+1}-z_n\| \le \|x_{n+1}-x_n\| + \frac{|s_{n+1}-s_n|}{\varepsilon}\|x_n-z_n\|$. From thees estimations, one has

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\le \beta_n\|x_{n+1}-x_n\| + (1-\beta_n)[\|x_{n+1}-x_n\| + \tfrac{|s_{n+1}-s_n|}{\varepsilon}\|x_n-z_n\| + \|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} \\
&\quad -T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\|] + |\beta_{n+1}-\beta_n|\|x_{n+1}-T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1}\| \\
&\le \|x_{n+1}-x_n\| + (|s_{n+1}-s_n| + |\beta_{n+1}-\beta_n|)M_2 + \|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} \\
&\quad -T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\|,
\end{aligned}$$

where $\sup_{n\ge1}\{\frac{\|x_n-z_n\|}{\varepsilon} + \|x_n - T_N^nT_{N-1}^n\cdots T_1^nz_n\|\} \le M_2$ for some $M_2>0$. Further, one also has

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\le [\alpha_{n+1}\nu\delta + \gamma_{n+1} + (1-\gamma_{n+1}-\alpha_{n+1}\tau)(1+\theta_{n+1})]\|x_{n+1}-x_n\| \\
&\quad +(1-\gamma_{n+1}-\alpha_{n+1}\tau)(1+\theta_{n+1})[(|s_{n+1}-s_n| + |\beta_{n+1}-\beta_n|)M_2 \\
&\quad +\|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} - T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\|] \\
&\quad +\|T^{n+1}y_n - T^ny_n\| + (|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n|)M_1 \\
&\le [1-\tfrac{\alpha_{n+1}(\tau-\nu\delta)}{2}]\|x_{n+1}-x_n\| + M_2(|s_{n+1}-s_n| + |\beta_{n+1}-\beta_n|) + (|\alpha_{n+1}-\alpha_n| \\
&\quad +|\gamma_{n+1}-\gamma_n|)M_1 + \|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} - T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\| + \|T^{n+1}y_n - T^ny_n\|.
\end{aligned}$$

Since $\{\frac{\alpha_n(\tau-\nu\delta)}{2}\}\subset[0,1]$, $\sum_{n=1}^\infty \frac{\alpha_n(\tau-\nu\delta)}{2} = \infty$, and

$$\begin{aligned}
\sum_{n=1}^\infty &[M_2(|s_{n+1}-s_n| + |\beta_{n+1}-\beta_n|) + (|\alpha_{n+1}-\alpha_n| + |\gamma_{n+1}-\gamma_n|)M_1 \\
&+\|T_N^{n+1}T_{N-1}^{n+1}\cdots T_1^{n+1}z_{n+1} - T_N^nT_{N-1}^n\cdots T_1^nz_{n+1}\| + \|T^{n+1}y_n - T^ny_n\|] < \infty
\end{aligned}$$

and the assumptions (ii), (iv), (v), (vi)), applying Lemma 2.2, we conclude that $\|x_{n+1}-x_n\|\to0$ as $n\to\infty$. So, $\|y_{n+1}-y_n\|\to0$ as $n\to\infty$.

Step 3. We show that $x_n - Gx_n \to 0$.

Indeed, we denote $q := P_C(p-\mu_2B_2p)$, and note that $v_n = P_C(u_n-\mu_2B_2u_n)$ and $z_n = P_C(v_n - \mu_1B_1v_n)$. Then $z_n = Gu_n$, $\|v_n-q\|^2 \le \|u_n-p\|^2 - \mu_2(2\beta-\mu_2)\|B_2u_n - B_2p\|^2$ and $\|z_n-p\|^2 \le \|v_n-q\|^2 - \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1q\|^2$. These two inequalities lead to

$$\|z_n-p\|^2 \le \|x_n-p\|^2 - \mu_2(2\beta-\mu_2)\|B_2u_n-B_2p\|^2 - \mu_1(2\alpha-\mu_1)\|B_1v_n-B_1q\|^2.$$

Let $w_n := \alpha_n \nu f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \rho F)T^n y_n$. From the convexity of $\|\cdot\|^2$, we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \le \|\alpha_n(\nu f(x_n) - \rho F T^n y_n) + \gamma_n(x_n - p) + (1-\gamma_n)(T^n y_n - p)\|^2 \\
&\le \gamma_n\|x_n-p\|^2 + (1-\gamma_n)\|T^n y_n - p\|^2 + 2\alpha_n\langle \nu f(x_n) - \rho F T^n y_n, w_n - p\rangle \\
&\le \gamma_n\|x_n-p\|^2 + (1-\gamma_n)[\beta_n\|x_n-p\|^2 + (1-\beta_n)\|z_n-p\|^2 + \theta_n(2+\theta_n)\|y_n-p\|^2] \\
&\quad +2\alpha_n\langle \nu f(x_n) - \rho F T^n y_n, w_n - p\rangle \\
&\le \|x_n-p\|^2 - (1-\gamma_n)(1-\beta_n)[\mu_2(2\beta-\mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1q\|^2] \\
&\quad +\theta_n(2+\theta_n)\|y_n-p\|^2 + 2\alpha_n\|\nu f(x_n) - \rho F T^n y_n\|\|w_n-p\| \\
&\le \|x_n-p\|^2 - (1-\gamma_n)(1-\beta_n)[\mu_2(2\beta-\mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1q\|^2] \\
&\quad +\theta_n(2+\theta_n)M_3 + 2\alpha_n M_3,
\end{aligned}$$

where $\sup_{n\ge 1}\{\|y_n-p\|^2 + \|\nu f(x_n) - \rho F T^n y_n\|\|w_n-p\|\} \le M_3$ for some $M_3 > 0$. This immediately yields

$$\begin{aligned}
&(1-\gamma_n)(1-\beta_n)[\mu_2(2\beta-\mu_2)\|B_2u_n - B_2p\|^2 + \mu_1(2\alpha-\mu_1)\|B_1v_n - B_1q\|^2] \\
&\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \theta_n(2+\theta_n)M_3 + 2\alpha_n M_3 \\
&\le (\|x_n-p\| + \|x_{n+1}-p\|)\|x_n - x_{n+1}\| + \theta_n(2+\theta_n)M_3 + 2\alpha_n M_3.
\end{aligned}$$

Since $\mu_1 \in (0,2\alpha)$, $\mu_2 \in (0,2\beta)$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\theta_n = 0$ and $\liminf_{n\to\infty}(1-\gamma_n)(1-\beta_n) \ge (1-b)(1-\beta) > 0$, we obtain from $x_n - x_{n+1} \to 0$ that

$$\lim_{n\to\infty}\|B_2u_n - B_2p\| = \lim_{n\to\infty}\|B_1v_n - B_1q\| = 0.$$

On the other hand, one has

$$2\|z_n-p\|^2 \le 2\mu_1\|B_1q - B_1v_n\|\|z_n-p\| + \|v_n-q\|^2 + \|z_n-p\|^2 - \|v_n - z_n + p - q\|^2,$$

which implies that

$$\|z_n-p\|^2 \le 2\mu_1\|B_1q - B_1v_n\|\|z_n-p\| + \|v_n-q\|^2 - \|v_n - z_n + p - q\|^2.$$

Similarly, we obtain

$$\|v_n-q\|^2 \le 2\mu_2\|B_2p - B_2u_n\|\|v_n-q\| + \|u_n-p\|^2 - \|u_n - v_n + q - p\|^2$$

and

$$\begin{aligned}
\|z_n-p\|^2 \ &\le \|x_n-p\|^2 - \|u_n - v_n + q - p\|^2 - \|v_n - z_n + p - q\|^2 \\
&\quad +2\mu_1\|B_1q - B_1v_n\|\|z_n-p\| + 2\mu_2\|B_2p - B_2u_n\|\|v_n-q\|.
\end{aligned}$$

These lead to

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \le \gamma_n\|x_n-p\|^2+(1-\gamma_n)[\beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2\\
&+\theta_n(2+\theta_n)\|y_n-p\|^2]+2\alpha_n\langle \nu f(x_n)-\rho FT^n y_n, w_n-p\rangle\\
&\le \|x_n-p\|^2-(1-\gamma_n)(1-\beta_n)[\|u_n-v_n+q-p\|^2+\|v_n-z_n+p-q\|^2]\\
&+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|\\
&+\theta_n(2+\theta_n)\|y_n-p\|^2+2\alpha_n\|\nu f(x_n)-\rho FT^n y_n\|\|w_n-p\|\\
&\le \|x_n-p\|^2-(1-\gamma_n)(1-\beta_n)[\|u_n-v_n+q-p\|^2+\|v_n-z_n+p-q\|^2]\\
&+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+\theta_n(2+\theta_n)M_3+2\alpha_n M_3.
\end{aligned}
$$

This yields that

$$
\begin{aligned}
&(1-\gamma_n)(1-\beta_n)[\|u_n-v_n+q-p\|^2+\|v_n-z_n+p-q\|^2]\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\mu_1\|B_1q-B_1v_n\|\|z_n-p\|\\
&+2\mu_2\|B_2p-B_2u_n\|\|v_n-q\|+\theta_n(2+\theta_n)M_3+2\alpha_n M_3.
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n=0,\ \lim_{n\to\infty}\theta_n=0$ and $\liminf_{n\to\infty}(1-\gamma_n)(1-\beta_n)\ge(1-b)(1-\beta)>0$, we infer $x_n-x_{n+1}\to 0$ that $\lim_{n\to\infty}\|u_n-v_n+q-p\|=0$ and $\lim_{n\to\infty}\|v_n-z_n+p-q\|=0$. So it follows that $\|u_n-Gu_n\|=\|u_n-z_n\|\le\|u_n-v_n+q-p\|+\|v_n-z_n+p-q\|\to 0$ as $n\to\infty$). Since $\|u_n-z_n\|=\|s_nx_n+(1-s_n)z_n-z_n\|=s_n\|x_n-z_n\|$ implies

$$
\|x_n-z_n\|=\frac{\|u_n-z_n\|}{s_n}\le\frac{\|u_n-z_n\|}{\varepsilon}\to 0 \quad (n\to\infty),
$$

we have $\|u_n-x_n\|=\|s_nx_n+(1-s_n)z_n-x_n\|=(1-s_n)\|z_n-x_n\|\to 0\ (n\to\infty)$, which attains

$$
\|x_n-Gx_n\|\le\|x_n-u_n\|+\|u_n-Gu_n\|+\|Gu_n-Gx_n\|\le 2\|x_n-u_n\|+\|u_n-Gu_n\|\to 0 \quad (n\to\infty).
$$

Step 4. We show that $x_n-T_N^nT_{N-1}^n\cdots T_1^nx_n\to 0$ and $x_n-Tx_n\to 0$.

Observe that

$$
\begin{aligned}
&\|y_n-p\|^2=\|\beta_n(x_n-p)+(1-\beta_n)(T_N^nT_{N-1}^n\cdots T_1^nz_n-p)\|^2\\
&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T_N^nT_{N-1}^n\cdots T_1^nz_n\|^2\\
&\le\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-T_N^nT_{N-1}^n\cdots T_1^nz_n\|^2,
\end{aligned}
$$

which yields

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \le \gamma_n\|x_n-p\|^2+(1-\gamma_n)[\|y_n-p\|^2+\theta_n(2+\theta_n)\|y_n-p\|^2] \\
&+2\alpha_n\langle \nu f(x_n)-\rho FT^n y_n, w_n-p\rangle \\
&\le \|x_n-p\|^2-(1-\gamma_n)\beta_n(1-\beta_n)\|x_n-T_N^nT_{N-1}^n\cdots T_1^n z_n\|^2+\theta_n(2+\theta_n)M_3+2\alpha_n M_3.
\end{aligned}$$

This immediately implies that

$$\begin{aligned}
&(1-\gamma_n)\beta_n(1-\beta_n)\|x_n-T_N^nT_{N-1}^n\cdots T_1^n z_n\|^2 \\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\theta_n(2+\theta_n)M_3+2\alpha_n M_3.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\theta_n=0$ and $\liminf_{n\to\infty}(1-\gamma_n)\beta_n(1-\beta_n)\ge(1-b)\beta(1-\beta)>0$, we infer from $x_n-x_{n+1}\to 0$ that $\lim_{n\to\infty}\|x_n-T_N^nT_{N-1}^n\cdots T_1^n z_n\|=0$. We observe that $\|y_n-x_n\|=(1-\beta_n)\|T_N^nT_{N-1}^n\cdots T_1^n z_n-x_n\|$ and

$$\begin{aligned}
&\|x_n-T_N^nT_{N-1}^n\cdots T_1^n x_n\| \\
&\le \|x_n-T_N^nT_{N-1}^n\cdots T_1^n z_n\|+\|T_N^nT_{N-1}^n\cdots T_1^n z_n-T_N^nT_{N-1}^n\cdots T_1^n x_n\| \\
&\le \|x_n-T_N^nT_{N-1}^n\cdots T_1^n z_n\|+\|z_n-x_n\|.
\end{aligned}$$

Hence, $\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\|x_n-T_N^nT_{N-1}^n\cdots T_1^n x_n\|=0$. We also note that $\|x_n-T^n y_n\|\le\|x_n-x_{n+1}\|+\alpha_n\|\nu f(x_n)-\rho FT^n y_n\|+\gamma_n\|x_n-T^n y_n\|$, which yields

$$\|x_n-T^n y_n\| \quad \le \tfrac{1}{1-b}[\|x_n-x_{n+1}\|+\alpha_n\|\nu f(x_n)-\rho FT^n y_n\|]\to 0 \quad (n\to\infty).$$

Consequently, we obtain from the above two inequalities that

$$\|y_n-T^n y_n\|\le\|y_n-x_n\|+\|x_n-T^n y_n\|\to 0 \quad (n\to\infty).$$

In view of this and $\sum_{n=1}^{\infty}\|T^{n+1}y_n-T^n y_n\|<\infty$, we get

$$\begin{aligned}
\|y_n-Ty_n\| &\le \|y_n-T^n y_n\|+\|T^n y_n-T^{n+1}y_n\|+\|T^{n+1}y_n-Ty_n\| \\
&\le (2+\theta_1)\|y_n-T^n y_n\|+\|T^n y_n-T^{n+1}y_n\|\to 0 \quad (n\to\infty).
\end{aligned}$$

which implies that as $n\to\infty$,

$$\|x_n-Tx_n\|\le\|x_n-y_n\|+\|y_n-Ty_n\|+\|Ty_n-Tx_n\|\le(2+\theta_1)\|x_n-y_n\|+\|y_n-Ty_n\|\to 0.$$

Step 5. We show that

$$\limsup_{n\to\infty}\langle \nu f(x^*)-\rho F(x^*), x_n-x^*\rangle\le 0,$$

where $x^* = P_\Omega(\nu f + I - \rho F)x^*$.

Indeed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \nu f(x^*) - \rho F(x^*), x_n - x^*\rangle = \lim_{k\to\infty}\langle \nu f(x^*) - \rho F(x^*), x_{n_k} - x^*\rangle.$$

Since $\{x_n\}$ is a bounded sequence in C, one assumes that $x_{n_k} \rightharpoonup \bar{x} \in C$. Also, since $G : C \to C$ is nonexpansive and $T : C \to C$ is asymptotically nonexpansive, utilizing Lemma 2.5 and $x_{n_k} \rightharpoonup \bar{x}$ that $\bar{x} \in \mathrm{Fix}(G) \cap \mathrm{Fix}(T) = \mathrm{Fix}(T) \cap \mathrm{GSVI}(C, B_1, B_2)$. Now, let the mapping $W : C \to C$ be defined as $Wx := \beta x + (1-\beta)T_N^n T_{N-1}^n \cdots T_1^n x$, with $\sigma \le \beta < 1$. It follows that W is a nonexpansive mapping and $\mathrm{Fix}(W) = \cap_{i=1}^N \mathrm{Fix}(T_i)$. Noticing $\|Wx_n - x_n\| = (1-\beta)\|T_N^n T_{N-1}^n \cdots T_1^n x_n - x_n\|$, we get $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$. Utilizing Lemma 2.5 again, we deduce from $(I-W)x_n \to 0$ and $x_{n_k} \rightharpoonup \bar{x}$ that $\bar{x} \in \mathrm{Fix}(W) = \cap_{i=1}^N \mathrm{Fix}(T_i)$. Consequently, $\bar{x} \in \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2) = \Omega$. Therefore

$$\begin{aligned}\limsup_{n\to\infty}\langle \nu f(x^*) - \rho F(x^*), x_n - x^*\rangle &= \lim_{k\to\infty}\langle \nu f(x^*) - \rho F(x^*), x_{n_k} - x^*\rangle \\ &= \langle \nu f(x^*) - \rho F(x^*), \bar{x} - x^*\rangle \le 0.\end{aligned}$$

So it follows from $x_n - x_{n+1} \to 0$ that $\limsup_{n\to\infty}\langle \nu f(x^*) - \rho F(x^*), x_{n+1} - x^*\rangle \le 0$.

Step 6. We show that $x_n \to x^*$.

Indeed, since $\alpha_n \nu\delta + \gamma_n + (1-\gamma_n-\alpha_n\tau)(1+\theta_n) \le 1 - \frac{\alpha_n(\tau-\nu\delta)}{2}$ $\forall n \ge 1$, it follows from $x_{n+1} = P_C w_n$ that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 = \langle x_{n+1} - w_n, x_{n+1} - x^*\rangle + \langle w_n - x^*, x_{n+1} - x^*\rangle \\ &\le \langle w_n - x^*, x_{n+1} - x^*\rangle \\ &\le [\alpha_n\nu\delta\|x_n - x^*\| + \gamma_n\|x_n - x^*\| + \|((1-\gamma_n)I - \alpha_n\rho F)T^n y_n \\ &\quad -((1-\gamma_n)I - \alpha_n\rho F)x^*\|]\|x_{n+1} - x^*\| + \alpha_n\langle(\nu f - \rho F)x^*, x_{n+1} - x^*\rangle \\ &\le [\alpha_n\nu\delta\|x_n - x^*\| + \gamma_n\|x_n - x^*\| + (1-\gamma_n)(1 - \tfrac{\alpha_n}{1-\gamma_n}\tau)(1+\theta_n)\|y_n - x^*\|]\|x_{n+1} - x^*\| \\ &\quad +\alpha_n\langle(\nu f - \rho F)x^*, x_{n+1} - x^*\rangle \\ &\le [1 - \tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle(\nu f - \rho F)x^*, x_{n+1} - x^*\rangle \\ &\le \tfrac{1}{2}[1 - \tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n - x^*\|^2 + \tfrac{1}{2}\|x_{n+1} - x^*\|^2 + \alpha_n\langle(\nu f - \rho F)x^*, x_{n+1} - x^*\rangle,\end{aligned}$$

which hence yields

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le [1 - \tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n - x^*\|^2 + 2\alpha_n\langle(\nu f - \rho F)x^*, x_{n+1} - x^*\rangle \\ &= [1 - \tfrac{\alpha_n(\tau-\nu\delta)}{2}]\|x_n - x^*\|^2 + \tfrac{\alpha_n(\tau-\nu\delta)}{2} \cdot \tfrac{4\langle(\nu f - \rho F)x^*, x_{n+1} - x^*\rangle}{\tau-\nu\delta}.\end{aligned}$$

Applying Lemma 2.2, we conclude that $\|x_n - x^*\| \to 0$ as $n \to \infty$. This completes the proof. □

Remark 3.1. *Compared with the corresponding results in Cai et al.* [12], *Bnouhachem et al.* [13] *and Ceng and Wen* [14], *our results improve and extend them in the following aspects. The mappings are extended from nonexpansive mappings to asymptotically nonexpansive mappings. The signal variational inequality is extended to the system of variational inequalities. Our algorithm, which more robust and efficient, is based on a viscosity descent method, which link our problem with another monotone variational inequality with mapping f. In addition, there is no compact assumptions and the restrictions are much mild.*

Under the conditions of Theorem 1, one can show that the sequence $\{x_n\}$ generated by

$$\begin{cases} u_n = s_n x_n + (1-s_n)z_n, \\ z_n = P_C(I-\mu_1 B_1)P_C(u_n - \mu_2 B_2 u_n), \\ y_n = \beta_n x_n + (1-\beta_n)T_N^n T_{N-1}^n \cdots T_1^n z_n, \\ x_{n+1} = P_C[\alpha_n \nu f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \rho F)(\zeta T y_n + (1-\zeta)y_n)] \quad \forall n \geq 1, \end{cases}$$

converges to $x^* \in \Omega = \cap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{GSVI}(C, B_1, B_2)$, where $x^* \in \Omega$ is a unique solution to the $\langle (\rho F - \nu f)x^*, p - x^* \rangle \geq 0\ \forall p \in \Omega$ in norm.

Let the mapping $S : C \to C$ be defined by $Sx := \zeta T x + (1-\zeta)x\ \forall x \in C$. Then $\lambda = \zeta \in [\zeta, 1)$ and $T : C \to C$ is a ζ-strict pseudocontraction. By virtue of Lemma 2.6, we know that $S : C \to C$ is a nonexpansive mapping with $\theta_n = 0$ and $\mathrm{Fix}(S) = \mathrm{Fix}(T)$. In this situation, the above iterative scheme can be rewritten as

$$\begin{cases} u_n = s_n x_n + (1-s_n)z_n, \\ z_n = P_C(I-\mu_1 B_1)v_n = P_C(u_n - \mu_2 B_2 u_n), \\ y_n = \beta_n x_n + (1-\beta_n)T_N^n T_{N-1}^n \cdots T_1^n z_n, \\ x_{n+1} = P_C[\alpha_n \nu f(x_n) + \gamma_n x_n + ((1-\gamma_n)I - \alpha_n \rho F)S y_n] \quad \forall n \geq 1. \end{cases}$$

By the similar arguments to those in the proof of Theorem 3.1, we can obtain the desired result.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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