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GLIMPSE OF STEINER DISTANCE PROBLEM

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Abstract. In this paper, we review in detail the Steiner Distance problem and literature related to Steiner Distance problem in Graph and trees. The relation between the diameter and radius of Steiner problem in the graph and trees will be discuss which were studied by various researchers and the structural properties related to the Steiner distance problem.

Keywords: graph; trees; Steiner distance problem; Steiner tree; Steiner distance.

2010 AMS Subject Classification: 05C05, 05C10, 05C12, 05C35, 05C75.

1. Introduction and Historical Background

The Steiner distance problem was examined by number of Mathematicians over several centuries. Fermat in 1641 inspected [1,2] given three points, a fourth is to be found, from which if three straight lines are drawn to the given points, the sum of the three lengths is minimum.

This is a restraint of the Steiner tree problem to three terminal points and a single Steiner point. A solution to the problem was formed by Italian mathematician Evangelista Torricelli

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in the same era [1]. This restriction of the Steiner tree problem is recognized as the Fermat Torricelli problem.

The first known (complete) formulation of the Euclidean Steiner tree problem was prepared in the 1800's by French Mathematician Joseph Diaz Gergonne. In the inaugural volume of his journal Annales de mathématiques pures et appliquées, more commonly known as Annales de Gergonne, Gergonne presented the problem progressively through a series of related questions. Initially, Gergonne projected what is essentially a retelling of the Fermat–Torricelli problem in a concrete setting. [1, 3] An engineer wishes to establish a communication between three cities, not located in a straight line, by means of a network composed of three branches, leading at one end to the three cities, and meeting at the other end at a single point between these three cities. The question is, how can one locate the point of intersection of the three branches of the network, so that their total length is as small as possible?

A footnote reiterated the problem in an abstract scenery. [3] One can generalize this problem by asking how to determine on a plane, a point whose sum of distances to a number of arbitrary points located in this plane is minimal. One can even extend to points located in any manner in space.

On a later page of the same volume, the first full version of the Euclidean Steiner tree problem was framed in the following setting. [3] A number of cities are located at known locations on a plane; the problem is to link them together by a system of canals whose total length is as small as possible.

The problem was later generalized on page 375 of [3]. The author of the article, listed as "Subscriber" and supposed in [1] to be Gergonne, elaborates on eleven problems related to the Euclidean Steiner tree problem. In particular, the final of these problems is translated by [1] as follows:[3] Connect any number of given points by a system of lines whose total length is as small as possible.

This is the Euclidean Steiner problem in full generality. Gergonne went on to stretch a detailed study of the problem. A condensed version of Gergonne's treatment of the problem was later published in England by "Gallicus," a Pseudonym for an unknown mathematician. See [1] for an analysis of the Gergonne's treatment of the Euclidean Steiner tree problem as well as the written work of Gallicus.

Following these initial actions of the Euclidean Steiner tree problem, the problem was discussed by Carl Fredrick Gauss and Christian Schumacher throughout a correspondence during the 1830's. In their correspondence, Gauss studied the Steiner problem restricted to 4 terminal points [1].

This restriction of the Steiner tree problem was the subject of Karl Bopp's Ph. D. dissertation in 1879 [4]. In his dissertation, Bopp demonstrated several results which were later rediscovered in modern treatments (see [5] and [6]) of the Euclidean Steiner tree problem [1]. The same topic was examined by Hoffman in 1890 [7]. Both Bopp and Hoffman were inspired by Gauss's letter and provided citations for it [1]. Hoffman's paper is of particular interest in that it delivers a short discussion of the general Steiner problem with n terminal points.

Following Hoffman's paper, there was little progress in the Steiner tree problem until 1934. In this year Jarník and Kössler observed not only the fully generalized Euclidean Steiner tree problem but also extended the problem to higher dimensional Euclidean spaces [2] (translated in [8]). According to [1], the paper was largely unnoticed by the mathematics community as it was written in Czech. Of particular significance in this paper are proofs that every Steiner point has degree at least 3 and that an angle inside of a Steiner tree has measure at least 120°.

We conclude our historical overview of the Euclidean Steiner tree problem by stating two expositions which brought the problem to new altitudes in the 20th century. In 1941 Richard Courant and Herbert Robbins published their classic book What is Mathematics? [9]. Written headed for an audience of non-mathematicians, the text allows witness to the simplicity of the Steiner tree problem. It is in this publication that the problem is attributed to Steiner, though no explanation is given for the naming convention. Following the publication of the book, the problem gained traction in the mathematical community. In 1968, Gilbert and Pollack [10] published a comprehensive summary of the research for the Euclidean Steiner tree problem for a professional audience and delivered a more robust geometric framework for the problem.

2. Main Result

Steiner Tree/ Ratio Problem:

We must be familiar some important technical terms and definitions while studying Steiner distance problems in trees and graphs

(1) Let G = (V, E) is a graph with vertex set V = V (G) and edge set E = E(G) then $\mid G \mid = \mid V(G) \mid$ denote the order of G and $\parallel G \parallel = \mid E(G) \mid$ denote the size of G.

(2) If a vertex $v \in V(G)$, the degree of v, deg(v), is the number of edges incident to that vertex and δ denote the minimum degree among all vertices in V.

(3) The open neighborhood of v in G is denoted by $N_G(v)$ and is defined as the set of all vertices in G adjacent to v. The closed neighborhood of v in G is denoted by $N_G[v]$ and is defined as the union $N_G(V) \cup v$.

(4) The distance in G between two vertices $u, v \in V$, denoted $d_G(u, v)$, is the length of the shortest path in G between u and v, if there is no path between u and v in G, we say that $d_G(u, v) = \infty$.

(5) The eccentricity of a vertex v in G is defined as

$$e(v) := max\{d_G(u, v) : u \in V(G)\}$$

(6) The radius is denoted by rad(G)and is defined as

$$rad(G) : min\{e(V) : v \in V(G)\}$$

and the diameter is denoted by diam(G)and is defined as

$$diam(G) : max\{e(V) : v \in V(G)\}.$$

(7) The center of G is denoted by C(G), is the subgraph induced by all vertices $v \in V(G)$ such that e(v) = rad(G).

(8) If H is a subgraph of G and $v \in V(G)$, then the distance from v to H, denoted $d_G(v, H)$ is defined as $min\{d_G(v, u) : u \in V(H)\}$.

(9) The Steiner distance in G of a non-empty set $S \subset V(G)$ is denoted by $d_G(S)$ and is defined as the size of the smallest connected subgraph of G containing all elements of S.

The actual and clear formulation of the Steiner tree problem is as follows [1] Given a set N of n points in the plane (often called terminal points), find a system of line segments such that the union of the line segments forms a connected set containing N, and such that the total Euclidean length of the line segments is minimized.

Essentially, the resulting system of line segments must form a tree, i.e., the resulting structure is connected and comprise no cycles. A minimal spanning tree of N, MST(N), is a tree consuming only elements of N as vertices and smallest possible length. A simple method to the Euclidean Steiner tree problem would be to apply well-organized algorithms, such as Dijkstra's algorithm (See [11]), to find a minimum spanning tree of N [10]. Though, by adding extra points, called Steiner points, it is possible to find trees of potentially smaller length.

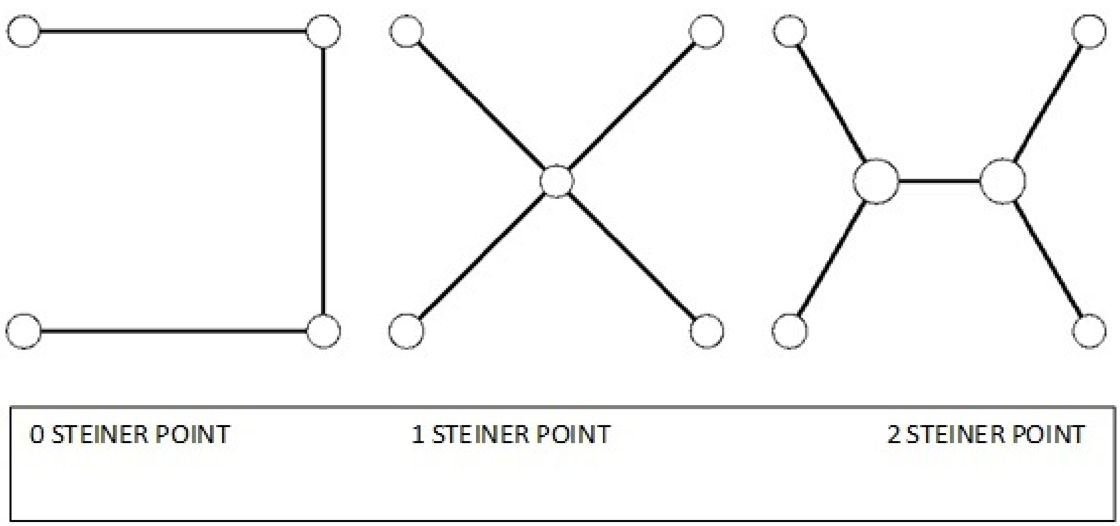

FIGURE 1. Steiner point.

For example, if we consider the 4 points at the corner of a unit square, one can find a smaller tree spanning N by including a Steiner point at the center of the square. Even more, by including two Steiner points, one can find a still smaller tree spanning N.

The smallest tree spanning N without restricting the number of Steiner points is called a Steiner tree for N, and denoted by ST(N).

The Steiner tree problem can be extended to metric spaces outside the Euclidean plane. A metric space is a set X with a distance function d, mapping X^2 to the non-negative real numbers, satisfying a finite set of axioms. See [12] for the entire treatment of metric spaces. Certainly, Steiner tree problems have been studied with respect to weighted graphs [10], the rectangular metric space [13], as well as general metric spaces [15]. Several extensions of the Steiner tree problem have been examined (See [14]), but we limit ourselves to the Steiner tree problem in this discussion.

The trouble in finding effective solutions to the Steiner tree problem extends to its complexity. Both the Euclidean and rectangular Steiner tree problem have been made known to be NP-hard [13]. In fact, a version of the Steiner tree problem in graphs was involved in Karp's original list of NP-complete problems [16].

Given the computational difficulty of finding a Steiner tree for a set of points, important effort has been put into finding good approximations of Steiner trees (See [15] and [10]). Remember that a minimal spanning tree of a set N is a smallest tree connecting N containing no terminal points other than N, i.e. the smallest tree containing N and no Steiner points. Let d(MST(N)) denote the weight of a minimal spanning tee of N. Then, if d(ST(N)) denotes the weight of a Steiner tree of N, one can analyze the parameter

$$\rho(X,d) := \max_{N\subseteq X} \frac{d(MST(N))}{d(ST(N))}$$

Here, $p(X,d)$ gives the maximum ratio of the weight of a minimal spanning tree of N to the weight of a Steiner tree of N over all $N \subseteq X$. This ratio is known as the Steiner ratio problem. For any metric space (X, d), it is known that $p(X,d) \leq 2$ and that this bound is tight for certain graphs [58]. The most widespread version of the Steiner ratio problem is the Euclidean Steiner ratio problem which enquires for the value of p in the context of the plane with the Euclidean metric. As of yet, the ratio is unidentified. Though a longstanding conjecture of Gilbert and Pollock appears to be reasonable to this.

Conjecture[10]Suppose that N is a finite set of point in the plane. Then,

$$\max_{N\subseteq X} \frac{d(MST(N))}{d(ST(N))} = \frac{2}{\sqrt{3}}$$

Should this bound be accurate, it is sharp. This can be simply seen by three points in the plane forming an equilateral triangle with sides of unit length.

(1) To make the minimal spanning tree of N, we need two edges of length.

(2) To make the Steiner tree of N, we add a single Steiner point at the center of the triangle.

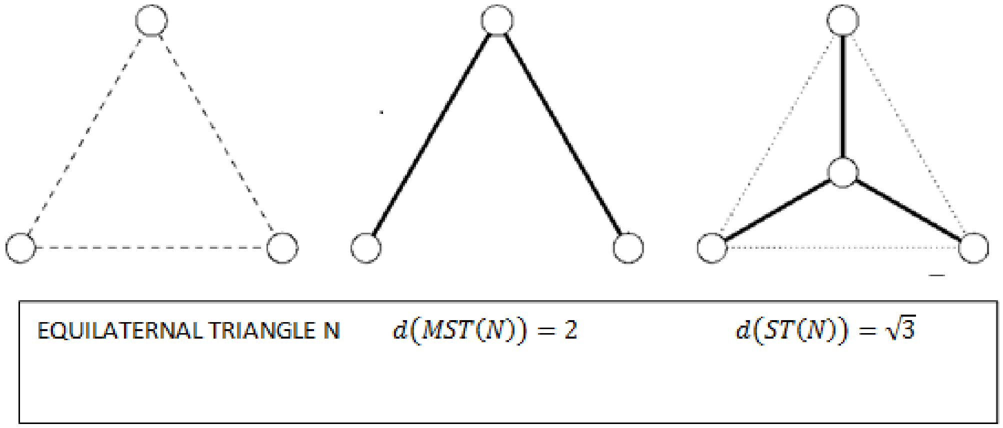

FIGURE 2. Steiner point.

A claimed proof of Conjecture was published in 1992 [18]. This proof, though, was shown to be incorrect by Ivanov and Tuzhilin ten years later [17]. While providing proof of Conjecture would certainly be impressive.

Definition 2.1. *Steiner k-eccentricity for an integer $k \geq 2$, the Steiner k-eccentricity of a vertex v in G is denoted by $e_k(v)$ and is defined as the maximum Steiner distance of all vertex subsets of G of size k containing v.*

$$e_k(v) = \max_{S \subset V(G), |S|=k} \{d(S) : v \in S\}..$$

Definition 2.2. *Steiner k-radius for an integer $k \geq 2$, the Steiner k-radius is denoted by $srad_k(G)$ and is then defined as*

$$srad_k(G) := min\{e_k(v) : v \in G\},$$

Definition 2.3. *Steiner k-diameter for an integer $k \geq 2$, the Steiner k-diameter is denoted by $sdiam_k(G)$ and is then defined as*

$$sdiam_k(G) := min\{e_k(v) : v \in G\},$$

Definition 2.4. *The Steiner k-center is denoted by $C_k(G)$, is the subgraph induced by all vertices v with $e_k(v) = srad_k(G)$.*

Remark: Suppose that H is a connected subgraph of G containing S with $\| H \| = d(S)$. Then,

(1) The subgraph H is a tree. Such a tree is called a Steiner tree of S.

(2) The set of end vertices (vertices of degree 1) of H must be a subset of S.

(3) If G is a connected graph and $v \in V(G)$, then $e_2(v) = e(v), srad_2(G) = rad(G), sdiam_2(G) = diam(G)$, and $C_2(G) = C(G)$.

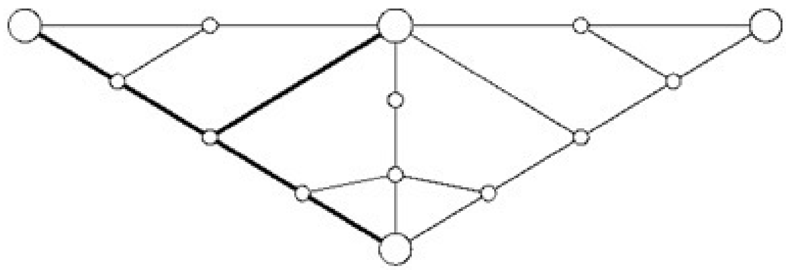

FIGURE 3. Steiner point.

The edges of a Steiner tree for three nodes are in bold. This Steiner tree contains 5 edges which implies d(H) = 5. This example illustrates that a Steiner tree is not unique as the shortest path also contains 5 edges.

Generalization of the problem:

Number of results has been generalized for the Steiner distance problem of distance problems, we try to give some of them as follows

Theorem 2.1. *If G is a connected graph of order n and minimum degree δ, then*

$$sdiam_2(G) \leq \frac{3n}{\delta+1} + O(1)$$

which shows the relationship between the diameter of connected graph and its minimum degree, and its generalization to k is,

Theorem 2.2. *Suppose that G is a connected graph of order n and minimum degree δ, then, if $2 \leq k \leq n$ is an integer,*

$$sdiam_k(G) \leq \frac{3n}{\delta+1} + 3k$$

Ali, Dankelmann, and Mukewmbi [28] later improved this bound and gave even better bounds for graphs which are triangle-free or contain no 4-cycles.

Theorem 2.3. *Suppose that G is a connected graph with complement* $\bar{G}$ *. If* $sdiam_2(G) > 3$*, then* $sdiam_2(\bar{G}) > 3$*.*
The generalization of this to an extension of a result of Ore [25] to find an upper bound on the Steiner k-diameter of G by the cycle with n vertices is as,

Theorem 2.4. *Let G be a connected graph with complement* $\bar{G}$ *and* $sdiam_k(G) = k+l$ *where* $1 < l < k-1$ *, then*

$$sdiam_k(G) + sdiam_k(\bar{G}) < 3k.$$

Mao [26] provided general bounds and obtained sharp results for

$$sdiam_3(G) + sdiam_3(\bar{G}) \text{ and}$$

$$sdiam_3(G) \bullet sdiam_3(\bar{G}).$$

to bound $sdiam_k(G)$ *the generalization is*

Theorem 2.5. *Suppose that G is a connected graph of order* $\mid G \mid= n$ *and* $sdiam_k(G) = d_k$*,then*

$$\parallel G \parallel \leq d_k + \binom{k-1}{2} + \binom{n-d_k-1}{2} + (n-d_k-1)(k+1)$$

furthermore, if G is 2-connected, then

$$sdiam_k(G) \geq \left[\frac{n(k-1)}{k}\right]$$

Ali [27] observed the Steiner k-diameter in relation to the girth, length of a shortest cycle, of a graph as,

Theorem 2.6. *Suppose that G is a connected graph of order n, girth g and minimum degree* $\delta \leq 3$*. Let* $2 \leq k \leq n$ *be an integer.*
1.If g is odd then

$$sdiam_k(G) \leq g\frac{n}{K} + (g-1)k - 2g + 1$$

where $K = 1+\delta\frac{(\delta+1)^{(g-1)/2}-1}{\delta-2}$

2.If g is even, then

$$sdiam_k(G) \leq g\frac{n}{L} + (g-1)k - 2g + 2$$

where $L = 2\delta\frac{(\delta+1)^{(g-1)/2}-1}{\delta-2}$

Theorem 2.7. *Suppose that G is a connected graph of order n and* $2 \leq k \leq n$*, then,*

$$k-1 \leq \mu_k(G) \leq \frac{k-1}{k+1}(n+1)$$

where $\mu_k(G) := \frac{W_k(G)}{\binom{\mid v(G) \mid}{k}}$ *is Steiner K-average distance and*
$W_k(G) := \sum_{(s \subset V(G), |s|=k)} d(S)$ *for an integer* $k > 2$*, the Steiner k-Wiener index for k = 2*

Theorem 2.8. *Suppose that G is a connected graph of order n, then,*

$$1 \leq (G) \leq \frac{1}{3}(n+1)$$

3. Conclusion

The glimpse of NP problem with the help of Steiner distance problem as a tree and graph given in brief to motivate the research in this direction. Historical background and technical terminologies related to Steiner distance problem has been covered and it will be used for future research in the same.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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