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A NEW ONE PARAMETER RAYLEIGH MAXWELL DISTRIBUTION

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Abstract. In this article, a new one-parameter lifetime distribution has been suggested. The distribution is a two-component finite mixture of Rayleigh and Maxwell distribution. The various statistical properties of the distribution such as moments, skewness, kurtosis, moment generating function, and characteristics function have been discussed. Survival function and hazard function are also studied. The maximum likelihood estimate for the unknown parameter under the proposed model is derived. Finally, the model is fitted to a real-life failure data, and outcomes were compared with some standard statistical probability distributions.

Keywords: finite mixture; Rayleigh distribution; Maxwell-Boltzman distribution; life time distribution; failure data.

2010 AMS Subject Classification: 62E15, 62F10, 62F99, 62P30, 62P35.

1. Introduction

The temperature of a physical system is determined by the velocity of the particles (atoms or molecules) contributing to the system. These particles have different velocities and they also constantly changes due to collisions among themselves. Maxwell (1860) [13] showed that the energy of such particles exhibits a certain probability pattern. Boltzmann (1872) [4] later simplified the probability distribution and investigated its physical origin.

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Though the initial development of Maxwell - Boltzmann distribution revolves around gas particles, Tyagi and Bhattacharya (1989a, b) [17] [18] for the first time considered this distribution to model lifetime data. They also explored its inferential properties under the Bayesian approach and reliability function. Chaturvedi and Rani (1998) [5] generalized the distribution through transformation on gamma random variate. Poddar and Roy (2003) [15] estimated the parameters of Maxwell - Boltzmann distribution under modified linear exponential loss function. Bekker and Roux (2005) [3] studied Empirical Bayes estimation for Maxwell distribution. Dey and Maiti (2010) [8] obtained the Bayes estimator of the parameter under different loss functions. Kazmi et al. (2011) [14] obtained the maximum likelihood (ML) estimators of the location and scale parameters of the mixture of the Maxwell distribution under Type-I censoring. Al-Baldawi (2013) [1] compared the efficiency of the ML estimator of the scale parameter of Maxwell distribution with the corresponding Bayes estimator. Hossain and Huerta (2016) [10] used the Maxwell distribution in analyzing the different data sets taken from the literature. Li (2016) [12] obtained the estimators of the scale parameter of the Maxwell distribution using the Minimax, Bayesian, and ML methods. Fan (2016) [9] considered the Bayesian method to estimate the loss and risk function for the scale parameter of the Maxwell distribution. Dey et al. (2016) [7] obtained estimators of the location and scale parameters of the Two Parameter Maxwell distribution via different estimation methods. See also Arslan et al. (2017) [2], where the modified maximum likelihood (MML) estimators for the location and scale parameters of the Maxwell distribution are obtained.

The Rayleigh distribution was introduced by Lord Rayleigh (1880) [16] to study a problem in the field of acoustics. The distribution is related to several well-known distributions such as Chi-Square, Exponential, Gamma, and Weibull. The distribution has a wide range of application and hence extensive work has been done in various fields of science and technology (Johnson et. al. 1994) [11].

A physical system, for example, industrial equipment or vehicles are comprised of many individual and vital parts. All of these parts of a system may exhibit a completely different failure pattern. Considering a single probability distribution to explore the survival function of such a system may not always give us the desired outcome. A finite mixture of some known and suitable probability distributions can help in understanding the sub-populations of a system with different properties. Finite mixtures are found to be useful in various fields of physics, chemistry, biology, and social sciences.

In this study, a finite mixture of Rayleigh and Maxwell (RMM) distribution is proposed. The various statistical properties of the mixture are discussed. The parameter of the proposed mixture is estimated under the maximum likelihood method. Finally, the mixture is fitted to a real-life data set.

2. One Parameter Rayleigh-Maxwell Distribution

Let us consider a two component mixture of Rayleigh distribution with parameter a and Maxwell-Boltzmann distribution with parameter a with their mixing components $\frac{1}{1+a}$ and $\frac{a}{1+a}$ respectively. The probability distribution of the new distribution can be written as

(1) $$f(x;a) = \frac{(1+\varepsilon x)x}{(1+a)a^2} e^{-\frac{x^2}{2a^2}}$$

The corresponding cdf is given by

(2) $$F(x) = \frac{1-e^{-z}}{1+a} + \frac{\beta a}{(1+a)} \gamma\left(\frac{3}{2}, z\right)$$

where, $\varepsilon = \sqrt{\frac{2}{\pi}} \approx 0.8$, $\beta = \frac{2}{\sqrt{\pi}} \approx 1.13$, $z = \frac{x^2}{2a^2}$ and $\gamma(a,b)$ is the lower incomplete gamma integral.

2.1. Moments of RMM Distribution.

The r^{th} raw moment is given by

(3) $$\mu_r' = \frac{2^{\frac{r}{2}} a^r}{1+a}\left[\Gamma\left(\frac{r+2}{2}\right) + \beta a \Gamma\left(\frac{r+3}{2}\right)\right] \qquad r = 1,2,3,\ldots$$

Replacing particular values of r (r = 1,2,3,4) in (3) we get the first four raw moments as

(4) $$\mu_1' = \frac{a}{1+a}(1.2533 + 1.5958a) = Mean$$

(5) $$\mu_2' = \frac{a^2}{1+a}(2+3a)$$

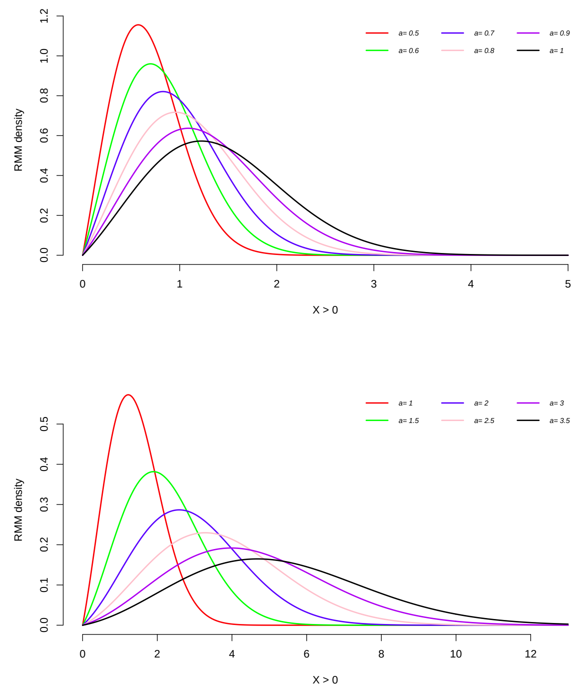

FIGURE 1. Probability density function of the RMM distribution for different values of the parameter.

$$\mu_3' = \frac{a^3}{1+a}(3.7599 + 6.3831a) \tag{6}$$

$$\mu_4' = \frac{a^4}{1+a}(8 + 15a) \tag{7}$$

The corresponding central moments are

$$\mu_2 = \frac{a^2}{(1+a)^2}(0.43a^2 + a + 0.4375) = Variance \tag{8}$$

$$\mu_3 = \frac{a^3}{(1+a)^3}[4.93a^3 + 12.20a^2 + 10.02a + 2.6840] \tag{9}$$

$$\mu_4 = \frac{a^4}{(1+a)^4}[0.63a^4 + 2.8a^3 + 4.24a^2 + 2.73a + 0.6] \tag{10}$$

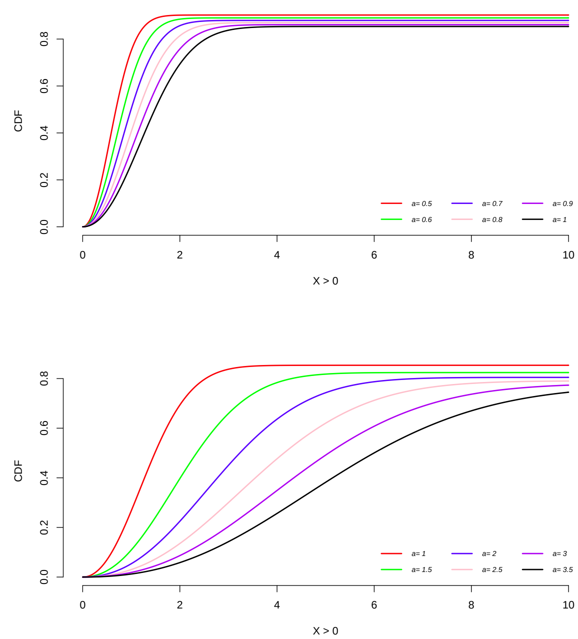

FIGURE 2. Cumulative distribution function of the RMM distribution for different values of the parameter.

2.1.1. *Skewness and Kurtosis.* The skewness and kurtosis of RMM distribution were found out to be

$$Skewness = \frac{\mu_3^2}{\mu_2^3} = \frac{[4.93a^3 + 12.20a^2 + 10.02a + 2.6840]^2}{[0.43a^2 + a + 0.4375]^3} \tag{11}$$

$$Kurtosis = \frac{\mu_4}{\mu_2^2} = \frac{[0.63a^4 + 2.8a^3 + 4.24a^2 + 2.73a + 0.6]}{[0.43a^2 + a + 0.4375]^2} \tag{12}$$

2.1.2. *Harmonic Mean.* The harmonic mean of RMM distribution is

$$\begin{aligned} \text{Harmonic Mean}(\frac{1}{H}) &= E[\frac{1}{X}] \\ &= \int_0^\infty \frac{1}{x} f(x)dx \\ &= \frac{1.25 + 0.8a}{a(1+a)} \end{aligned} \tag{13}$$

2.1.3. *Mode of RMM distribution.* The mode of a distribution is obtained by solving the equation

$$\frac{\partial}{\partial x} log[f(x)] = 0 \tag{14}$$

On simplification, the equation to obtain mode of RMM distribution reduces to

$$0.8x^3 + x^2 - 1.6a^2x - a^2 = 0 \tag{15}$$

2.2. Generating Functions.

2.2.1. *Moment Generating Function.*

The moment generating function (mgf) of RMM distribution is

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \int_0^\infty e^{tx} f(x)dx \\ &= \int_0^\infty \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} f(x)dx \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{2^{r/2}a^r}{1+a} \left[\Gamma(\frac{r+2}{2}) + \frac{2a}{\sqrt{\pi}}\Gamma(\frac{r+3}{2})\right] \end{aligned} \tag{16}$$

2.2.2. *Characteristic Function.*

The characteristic function (cf) of RMM distribution is

$$\begin{aligned} \Phi_X(t) &= E[e^{itX}] \\ &= \int_0^\infty e^{itx} f(x)dx \\ &= \int_0^\infty \sum_{r=0}^{\infty} \frac{(it)^r x^r}{r!} f(x)dx \\ &= \sum_{r=0}^{\infty} \left(\frac{(it)^r}{r!}\right) \frac{2^{r/2}a^r}{1+a} \left[\Gamma(\frac{r+2}{2}) + \frac{2a}{\sqrt{\pi}}\Gamma(\frac{r+3}{2})\right] \end{aligned} \tag{17}$$

2.3. Survival Function.

The survival function of our proposed RMM model is given by

$$\begin{aligned} S(x) &= 1 - F(x) \\ &= \frac{a + e^{-z} - ac\,\gamma(3/2, z)}{1+a} \end{aligned} \tag{18}$$

Where,
$c = \sqrt{\frac{2}{\pi}}$, $z = \frac{x^2}{2a^2}$ and $\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$, lower incomplete gamma integral.

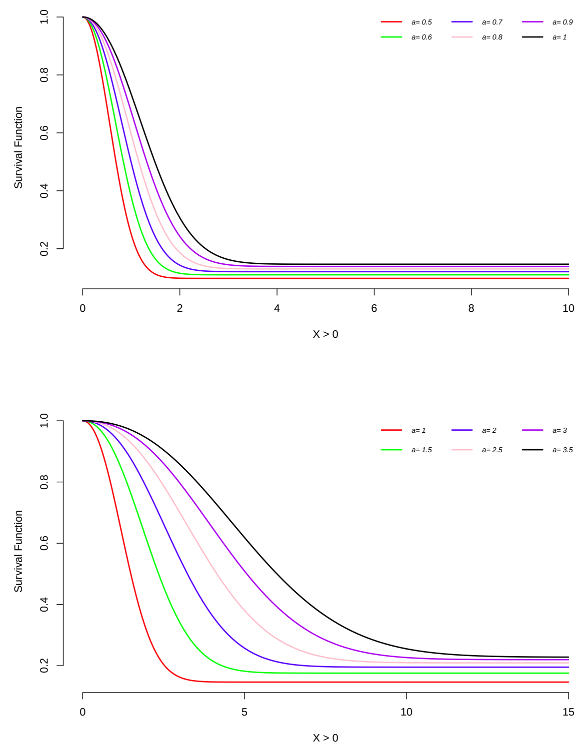

FIGURE 3. Survival function of the RMM distribution for different values of the parameter.

2.4. Hazard Function. The hazard function of RMM distribution is obtained as

$$\begin{aligned} h(x) &= \frac{f(x)}{S(x)} = \frac{f(x)}{1-F(x)} \\ &= \frac{x(1+cx)e^{-z}}{a^2[a+e^{-z}-ac\,\gamma(3/2,z)]}; \qquad x>0 \end{aligned} \tag{19}$$

The reverse hazard rate function of RMM distribution is

$$\begin{aligned} \Phi(x) &= \frac{f(x)}{F(x)} \\ &= \frac{x(1+cx)e^{-z}}{a^2[1-e^{-z}+ac\,\gamma(3/2,z)]}; \qquad x>0 \end{aligned} \tag{20}$$

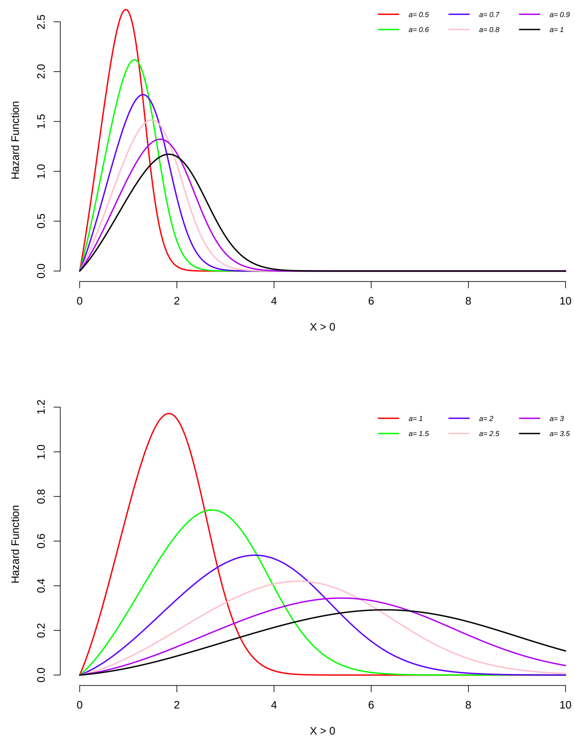

FIGURE 4. Hazard function of the RMM distribution.

TABLE 1. Descriptive statistics for different values of the parameter

Parameter	Mean	Variance	Harmonic Mean	Skewness	Kurtosis
0.5	0.6837	0.1161	2.2000	113.0907	3.1266
0.6	0.8290	0.1677	1.8021	118.1770	3.1309
0.7	0.9760	0.2286	1.5210	123.0796	3.1361
0.8	1.1244	0.2988	1.3125	127.7937	3.1418
0.9	1.2740	0.3783	1.1520	132.3193	3.1478
1.0	1.4246	0.4669	1.0250	136.6599	3.1541
1.5	2.1882	1.0458	0.6533	155.8035	3.1845
2.0	2.9633	1.8478	0.4750	171.3434	3.2109
2.5	3.7449	2.8699	0.3714	184.1038	3.2327
3.0	4.5305	4.1105	0.3042	194.7252	3.2506
3.5	5.3189	5.5685	0.2571	203.6830	3.2654

3. Inferential Procedures

The likelihood equation corresponding to (1) is

$$L(x;a)=\left\{\frac{1}{(1+a)a^2}\right\}^n\prod_{i=1}^{n}x_i\,(1+c\,x_i)\,e^{-\sum_{i=1}^{n}\frac{x_i^2}{2a^2}} \tag{21}$$

Where, c=$\sqrt{\dfrac{2}{\pi}}\backsimeq 0.8$

The log likelihood function is

$$logL=-2n\,log(a)-n\,log(1+a)+\sum_{i=1}^{n}log\,x_i+\sum_{i=1}^{n}log\,(1+c\,x_i)-\frac{1}{2a^2}\sum_{i=1}^{n}x_i^2 \tag{22}$$

Differentiating (22) w.r.t. a and equating to zero we get

$$3a^3+2a^2-(1+a)T_x=0 \tag{23}$$

Where, $T_x=\dfrac{1}{n}\sum_{i=1}^{n}x_i^2$

Solving (23) for ***a*** numerically, we can get the maximum likelihood estimate of the parameter.

4. Application

The proposed model is fitted to a data set related to the number of miles to the first major motor failure of 191 buses operated by a large city bus company (Davis [6]). The outcomes are compared with Rayleigh, Maxwell-Boltzman, Gamma, Chi-square, and Exponential distributions. To compare the performance of our proposed model with other distributions, different discrimination criteria such as AIC, BIC, AICC, HQIC, and CAIC are constructed under the log-likelihood function. Table (2) presents the data related to the motor failure of 191 buses. Table (3) presents the discriminating criteria under different distributions.

TABLE 2. Data related to motor failure of 191 city buses

Distance Interval (in Thousands of Miles)	Number of Failures
0 - 20	6
20 - 40	11
40 - 60	16
60 - 80	25
80 - 100	34
100 - 120	46
120 - 140	33
140 - 160	16
160 - 180	2
180+	2
Total	191

TABLE 3. Results of AIC, AICC, HQIC and CAIC for different probability distribution considering the data related to motor failure of city buses

Test	RMM	Rayleigh	Maxwell	Gamma	Chi-Square	Exponential
AIC	1945.70	1963.46	1947.30	1977.73	3377.02	2130.36
BIC	1948.95	1966.72	1950.56	1984.23	3380.27	2133.61
AICC	1945.72	1963.49	1949.32	1977.79	3377.04	2130.38
HQIC	1947.01	1964.78	1950.62	1980.36	3378.33	2131.68
CAIC	1949.95	1967.72	1953.56	1986.23	3381.27	2134.61

Our proposed model is performing better in explaining the data set than the remaining distributions since the values of AIC, AICC, HQIC, and CAIC are less compare to Rayleigh, Maxwell - Boltzman, Gamma, Chi-square and Exponential distribution.

5. Conclusion

The superior performance of our proposed model can be confirmed from the different discrimination criteria since the best model is the one that gives the minimum values of those criteria. The distribution can be used in cases where we observed a high rate of failure as we move towards the mode of the data and then failure rate decreases drastically. The various statistical properties of the proposed model were also discussed. Further extension of the proposed model was also possible and will be studied in future work.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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