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TERNARY EQUIDISTANT CODES OF LENGTH $11 \leq n \leq 15$

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Abstract: We consider the problem of finding bounds on the size of ternary equidistant codes. Optimal codes have been constructed by combinatorial and computer methods. The exact values on $B_3(n,d)$ for ternary equidistant codes of length $11 \leq n \leq 15$ are presented.

Keywords: equidistant codes; bounds of codes; construction of codes.

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1. INTRODUCTION

An $(n,M,d)_q$ *equidistant code* (EC) is a set of M codewords of length n over the alphabet {0, 1 ,… q - 1} such that any two codewords differ in d positions.

An $(n,M,d,w)_q$ code is called *equidistant constant weight code* (ECWC) if all its codewords have the same weight w. Let $B_q(n,d)$ (or $B_q(n,d,w)$) denote the largest possible value of M when the other parameters are fixed. Codes with such parameters are called optimal.

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Equidistant codes have been investigated in [1], [2], [3], [4], [9] etc. In this work we continue investigations from [1]. General bounds for q-ary equidistant codes are discussed in Section 2. The obtained codes for ternary equidistant codes of length $11 \leq n \leq 15$ are presented in Section 3. The new results on $B_3(n,d)$ for $11 \leq n \leq 15$ are given in Table 1.

Presented codes are used for data security in the project "Study of the Application of New Mathematical Methods for Cardiac Data Analysis" [11].

The project contains research about semantic web presentation of digital data in the field of medical systems and building ontology for holter monitoring system. Effective organization of unstructured data are also explored, which is a suitable method of organizing data in the field of medical systems.

2. General Bounds for Equidistant Codes

Some general upper and lower bounds for equidistant codes and ECWC are given by the following theorems:

Theorem 1 $B_q(n, n) = q$.

Theorem 2 $B_q(n,d) = 1 + B_q(n,d, d)$.

Theorem 3 (the Johnson bounds for ECWC) The maximum number of codewords in a q-ary ECWC satisfy the inequalities:

$$B_q(n,d,w) \leq \frac{n}{n-w} B_q(n-1,d,w)$$
$$B_q(n,d,w) \leq \frac{n(q-1)}{w} B_q(n-1,d,w-1)$$

The simplified version of q-ary Plotkin bound [5], [8] is:

Theorem 4

$$B_q(n,d) \leq \frac{dq}{dq-n(q-1)}$$

Exact equality above is equivalent to existence of optimal equidistant code $(n, B_q(n,d),d)_q$ [10].

When $d = 3$, the bounds for $n \geq 3$ and $q > 2$ are determined [1].

Theorem 5 If $3 \leq q < 9$ and $n > 3$ then $B_q(n, 3) = 9$ and if $q \geq 9$ and $n > 3$ then $B_q(n, 3) = q$.

Theorem 6 $B_q(q+1,q,q-1) \leq (q^2+q)/2$.

Theorem 7 [10] The optimal equidistant $(n,qt,d)_q$ codes and RBIB designs ($v=qk,b,k,r,\lambda$) are equivalent to one another and their parameters are connected by the conditions $v = M$, $b = nq$, $k = t$, $r = n$, $\lambda = n - d$.

3. Computer Search And New Results

The new codes are obtained by combinatorial considerations or by computer search.

There are two main problems: code construction and determining code equivalence. For finding existence and equivalence of codes we use methods similar to those described in [6], [7], [11].

Let C be an $(n,M,d,w)_3$ ECWC for $w = d$ and $C_0 = C \cup \{0\}$ be an $(n, M_0, d)_3$ EC. Our approach is based on Theorem 2 and on the observation that an $(n,M,d,d)_3$ code C can be shortened to an $(n-1,M, d, d)_3$ code C'. Any $(n,M,d)_3$ equidistant code C contains $(n-1,M',d)_3$ codes with $M' = \left\lceil M\frac{n-d}{n}\right\rceil + 1$ codewords (Theorem 2, Theorem 3). The first problem is to construct all the $(n,M,d,w=d)_3$ codes with M codewords which contain C' as a subcode.

To obtain the results for ternary equidistant codes we use some theoretical and software tools and a computer program for the ternary case based on backtrack search. We increased the speed of the algorithm from [1] by adding new specific conditions and fixing some parts of the code.

The upper bounds for EC which we used for our research are obtained from theorems presented in Section 2 or by computer search.

Also this problem can be represented as the maximal clique problem in the graph induced by the set of all q-ary vectors of length n. To solve it, we use the socalled backtrack search. The search space is only the set of all vectors that are at distance d from every codeword of C'. Then only the distance between codewords is under control.

For the case of q-ary codes, the corresponding graph where the search is realized is defined as follows. Vertices are vectors of length n over the alphabet of size q, and two vertices are connected by an edge if and only if the Hamming distance between the corresponding vectors is exactly d.

What we have to find in the graph thus obtained is the quantity $B_q(n, d)$, the size of the largest clique in this graph.

The exact values for $B_3(n,d)$ for ternary equidistant codes of are given in Table 1. The obtained new values in the table are denoted by bold typeface.

More details of classification results for ternary equidistant codes for $n \leq 10$ are given in [1].

The exact values for $n = d$ follow from Theorem 1. The exact values for $d = 3$ in Table 1 are obtained in [1]. The codewords of the $(4,9,3)_3$ code are (up to equivalence): (0000, 0111, 0222, 1012, 1120, 1201, 2021, 2102, 2210). There exists a family of unique optimal equidistant codes with parameters $(n,9,3)_3$ for $n \geq 3$ [1].

Some values in Table 1 for $6 \leq d \leq 15$ are derived by the next theorems and propositions:

Theorem 8 If $n > 6$ then $B_3(n,n-1) = 3$ and if n > 12 then $B_3(n,n-2) = 3$.

Proof: It is easy to prove that in these cases we could have only codes consisting of 3 codewords. It follows from Theorem 4 that for $d = n - 1$ and $n > 6$ we have that $B_3(n,n-1) \leq 3 + x$ where $0 < x < 1$ so $B_3(n,n-1) = 3$. Also, for $d = n -2$ and $n > 12$ we have that $B_3(n,n-2) \leq 3 + x$ where $0 < x < 1$ so $B_3(n,n-2) = 3$.

There exist families of optimal equidistant codes with parameters $(n,3,n-1)_3$ for $n > 6$ and $(n,3,n-2)_3$ for $n > 12$.

Proposition 9 There exist 2 inequivalent optimal ECs with parameters $(n,3,n-1)_3$ for $10 < n \leq 15$ and 3 inequivalent optimal ECs with parameters $(n,3,n-2)_3$ for $12 < n \leq 15$ (up to equivalence).

The inequivalent $(11,3,10)_3$ ECs are

(00000000000, 01111111111, 02222222222)

and

(00000000000, 01111111111, 10122222222).

The inequivalent $(13,3,11)_3$ ECs are

(0000000000000, 0011111111111, 0022222222222), (0000000000000, 0011111111111, 0101222222222)

and

(0000000000000, 0011111111111, 1100112222222).

Proposition 10 There are optimal ECs with parameters

- $(11,15,6)_3$;
- $(n,18,6)_3$ for $12 \leq n \leq 15$.

Proposition 11 There are optimal ECs with parameters

- $(11,11,7)_3$;
- $(n,12,7)_3$ for $12 \leq n \leq 14$;
- $(15,14,7)_3$.

Proposition 12 There are optimal ECs with parameters

- $(11,12,8)_3$;
- $(n,13,8)_3$ for $12 \leq n \leq 14$;
- $(15,16, 8)_3$.

Proposition 13 There are optimal ECs with parameters

- $(11,4,9)_3$;
- $(12,9,9)_3$;
- $(n,27,9)_3$ for $13 \leq n \leq 15$.

Proposition 14 There are optimal ECs with parameters

- $(12,4,10)_3$;
- $(13,6.10)_3$;
- $(n,12,10)_3$ for $14 \leq n \leq 15$.

Proposition 15 Then are optimal ECs with parameters

- $(14,6,11)_3$:
- $(15,10,11)_3$.

Proposition 16 There is optimal EC with parameters $(15,6,12)_3$.

Some of the codes in the previous propositions are explicitly listed below.

(12,18,6)₃:

000000000000, 000000111111, 000000222222, 000011001122, 000011112200, 000011220011, 000101010212, 000101121020, 000101202101, 001001012021, 001001120102, 001001201210, 010001021201, 010001102012, 010001210120, 100001022110, 100001100221, 100001211002

(11,11,7)₃:

00000000000, 00001111111, 00110011222, 01011102022, 01110120101, 10102100122, 10111201001, 11020001121, 11200111002, 12011010102, 21101010021

(12,12,7)₃:

000000000000, 000001111111, 000110011222, 001011102022, 001110120101, 010102100122, 010111201001, 011020001121, 011200111002, 012011010102, 021101010021, 101101001102

(11,12,8)₃:

00000000000, 00011111111, 00101222222, 01222001122, 10222112200, 12012020212, 12121100021, 12200221101, 21020202211, 21102121010, 21211210002

(11,4,9)₃:

00000000000, 00111111111,

11001122222, 12222200112

(12,9,9)₃:

000000000000, 000111111111, 000222222222, 111000111222, 111111222000, 111222000111, 222000222111, 222111000222, 222222111000

(13,27,9)₃:

0000000000000, 0000111111111, 0000222222222, 0111000111222, 0111111222000, 0111222000111, 0222000222111, 0222111000222, 0222222111000, 1012012012012, 1012120120120, 1012201201201, 1120012120201, 1120120201012, 1120201012120, 1201012201120, 1201120012201, 1201201120012, 2021021021021, 2021102102102, 2021210210210, 2102021102210, 2102102210021, 2102210021102, 2210021210102, 2210102021210, 2210210102021

(12,4,10)₃:

000000000000, 001111111111,

110011222222, 222222001122

(13,6,10)₃:

0000000000000, 0001111111111, 0010122222222, 1112200011122, 1121211202200, 1122022120011

(14,12,10)₃:

00000000000000, 00001111111111, 00010122222222, 01112200011122, 01121212200201, 01222020122011, 10212221101200, 12100201022211, 12111120210010, 21220101212100, 22021202121020, 22202122000121

(14,6,11)₃:

00000000000000, 00011111111111,

01100112222222, 11122220001112,

12211221220020, 12222002112201

(15,10,11)₃:

000000000000000, 000011111111111, 000101222222222, 011222000111222, 101222112022001, 111012220202110, 122020121120220, 212110201012021, 222102022101101, 222211010220012

***(15,6,12)₃*:**

000000000000000, 000111111111111, 111000111222222, 111222222000111, 222111222222000, 222222000111222

Remark: All $(n+k,M,d)_3$ codes in the previous propositions, which codewords are not explicitly listed are obtained from $(n,M,d)_3$ codes by construction *I*.

Construction *I*: From the $(n,M,d)_q$ code *A* we construct an $(n+k,M,d)_q$ code in the following way:

$$\left\{\left(\underbrace{0\ldots0}_{k},a\right)|a\in A\right\}$$

n/d	3	4	5	6	7	8	9	10	11	12	13	14	15
3	3_g												
4	9_g	3											
5	9_g	6	3										
6	9_g	7	4	3									
7	9_g	8	7	3	3								
8	9_g	8	8	9	3	3							
9	9_g	8	8	12	6	3	3						
10	9_g	8	8	15	10	6	3	3					
11	$\mathbf{9}_g$	$\mathbf{8}_I$	$\mathbf{8}_I$	**15**	**11**	**12**	**4**	$\mathbf{3}_d$	$\mathbf{3}_1$				
12	$\mathbf{9}_g$	$\mathbf{8}_I$	$\mathbf{8}_I$	**18**	**12**	**13**	$\mathbf{9}_c$	**4**	$\mathbf{3}_d$	$\mathbf{3}_1$			
13	$\mathbf{9}_g$	$\mathbf{8}_I$	**9**	$\mathbf{18}_I$	$\mathbf{12}_I$	$\mathbf{13}_I$	$\mathbf{27}_z$	$\mathbf{6}_c$	$\mathbf{3}_d$	$\mathbf{3}_d$	$\mathbf{3}_1$		
14	$\mathbf{9}_g$	$\mathbf{8}_I$	$\mathbf{9}_I$	$\mathbf{18}_I$	$\mathbf{12}_I$	$\mathbf{13}_I$	$\mathbf{27}_I$	**12**	$\mathbf{6}_c$	$\mathbf{3}_d$	$\mathbf{3}_d$	**3**	
15	$\mathbf{9}_g$	$\mathbf{8}_I$	$\mathbf{9}_I$	$\mathbf{18}_I$	**14**	**16**	$\mathbf{27}_I$	$\mathbf{12}_I$	**10**	$\mathbf{6}_c$	$\mathbf{3}_d$	$\mathbf{3}_d$	**3**

TABLE 1: $B_3(n,d)$ FOR $11 \leq n \leq 15$

Key to Table 1:

1 - Theorem 1; g - Theorem 5; z - Theorem 7; d - Theorem 8; c - concatenation; *I* - from Construction *I*; no index - from computer search.

4. CONCLUSION

We consider the problem of finding bounds on the size of ternary equdistant codes for $11 \leq n \leq 15$. We present some combinatorial constructions and computer methods that are used to find new optimal equidistant codes. Some of the new codes are explicitly listed. Presented codes are used for data security in the specialized medical system.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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