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THE MULTIPLICATIVE SECOND HYPER ZAGREB INDEX OF SOME GRAPH OPERATIONS

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Abstract. In this paper, the multiplicative second hyper Zagreb index is presented and the sharp upper bound for this index of various graph operations for example, join, composition, cartesian and corona products of graphs are derived. And we prove that the sharp upper bound is tight.

Keywords: topological indices; graph operations.

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1. INTRODUCTION

All graphs observed here are simple, connected and finite. Let $V(G), E(G)$ and $d_G(w)$ indicate the vertex set, the edge set and the degree of a vertex of a graph G respectively. A graph with p vertices and q edges is known as a (p,q) graph. We encourage the readers to see[5] for basic definitions and notations of a graph.

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A topological index is a numerical parameter which is mathematically attained from the graph structure.

Gutman et.al.,[2] introduced the first and second Zagreb indices of a graph G as follows:

$$M_1(G) = \sum_{wz\in E(G)} (d_G(w)+d_G(z)) = \sum_{w\in V(G)} d_G^2(w) \text{ and } M_2(G) = \sum_{wz\in E(G)} d_G(w)d_G(z)$$

Shirdel et.al. in [7] found Hyper-Zagreb index $HM(G)$ which is established as

$$HM(G) = \sum_{wz\in E(G)} [d_G(w)+d_G(z)]^2.$$

Also, they have computed the hyper - Zagreb index of the cartesian product, composition, join and disjunction of graphs.

A forgotten topological index F-index [4] is defined for a graph G as

$$F(G) = \sum_{w\in V(G)} d_G^3(w) = \sum_{wz\in E(G)} [d_G^2(w)+d_G^2(z)]$$

Farahani et.al [3] defined the second hyper Zagerb as

$$HM_2(G) = \sum_{wz\in E(G)} [d_G(w)d_G(z)]^2.$$

Here we introduce a second forgotten topological index F_2 which is defined for a graph G as

$$F_2(G) = \sum_{w\in V(G)} d_G^4(w).$$

V.R.Kulli [6] introduced the first and second Gourava indices and defined as

$$GO_1(G) = \sum_{wz\in E(G)} (d_G(w)+d_G(z))+(d_G(w)d_G(z))$$

and

$$GO_2(G) = \sum_{wz\in E(G)} d_G(w)d_G(z)(d_G(w)+d_G(z))$$

Todeschine et al $[9,10]$ presented the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$$\prod_1 = \prod_1(G) = \prod_{w\in V(G)} d_G(w)^2 = \prod_{wz\in E(G)} \big[d_G(w)+d_G(z)\big]$$
$$\text{and } \prod_2 = \prod_2(G) = \prod_{wz\in E(G)} d_G(w)d_G(z)\big]$$

Recently, Akbar [1] has introduced the multiplicative hyper Zagreb index, denoted by

$$\prod HM(G) = \prod_{uv\in E(G)} (d_G(w)+d_G(z))^2$$

Also in this paper, the upper bounds on the multiplicative hyper Zagreb index of the cartesian, corona product, composition, join and disjunction of graphs.

In this paper, we introduce a new graph invariant namely multiplicative second hyper Zagreb index, denoted by

$$\prod HM_2(G) = \prod_{uv\in E(G)} (d_G(w)d_G(z))^2$$

In this paper, we compute the sharp upper bound for the multiplicative second hyper Zagreb index of the graph operations for example, join, composition, cartesian and corona products and prove that our bound is tight.

2. Preliminaries

Lemma 2.1. [5, 8]

(a) $d_{G_1+G_2}(w) = \begin{cases} d_{G_1}(w)+V(G_2), & w\in V(G_2) \\ d_{G_2}(w)+V(G_1), & w\in V(G_2) \end{cases}$

(b) $d_{G_1[G_2]}(w,z) = V(G_2)d_{G_1}(w)+d_{G_2}(z)$

(c) $d_{G_1\Box G_2}((w_i,z_j)) = d_{G_1}(w_i)+d_{G_2}(z_j)$, *where* $(w_i,z_j)\in V(G_1\Box G_2)$.

(d)

$$d_{G_1\odot G_2}(w) = \begin{cases} d_{G_1}(w)+p_2 & if\ w\in V(G_1) \\ d_{G_1}(w)+p_2 & if\ w\in V(G_{2,i})\ for\ some\ 0\le i\le p_1-1, \end{cases}$$

where $u\in V(G_1\odot G_2)$ $G_{2,i}$ *is the ith copy of the graph* G_2 *in* $G_1\odot G_2$.

Lemma 2.2 (Arithmetic geometric Inequality)**.** *Let* $y_1,y_2,\dots,y_n$ *be non-negative numbers. Then* $\dfrac{y_1+y_2+\cdots+y_n}{n} \ge \sqrt[n]{y_1y_2\cdots y_n}$

3. THE MULTIPLICATIVE SECOND HYPER ZAGREB INDEX OF JOIN OF GRAPHS

Theorem 3.1. *Let $G_i, i=1,2$ be a $(p_i,q_i)-$ graph. Then*

$$\prod HM_2(G_1+G_2) \leq \left[\frac{\begin{array}{r}HM_2(G_1)+p_2^2HM(G_1)+p_2^4q_1+2p_2GO_2(G_1)\\+2p_2^2M_2(G_1)+2p_2^3M_1(G_1)\end{array}}{q_1}\right]^{q_1}$$

$$\times\left[\frac{\begin{array}{r}HM_2(G_2)+p_1^2HM(G_2)+p_1^4q_2\\+2p_1GO_2(G_2)+2p_1^2M_2(G_2)+2p_1^3M_1(G_2)\end{array}}{q_2}\right]^{q_2}$$

$$\times\left[\frac{\begin{array}{r}M_1(G_1)M_1(G_2)+p_1^2p_2M_1(G_1)+p_1p_2^2M_1(G_2)\\+4p_1q_2M_1(G_1)+p_1^3p_2^3+p_2q_1M_1(G_2)+4p_1^2p_2^2q_2\\+4p_1^2p_2^2q_1+16p_1p_2q_1q_2\end{array}}{p_1p_2}\right]^{p_1p_2}$$

Proof : From the definition of the second hyper Zagreb index,

$$\begin{aligned}\prod HM_2(G_1+G_2) &= \prod_{wz\in E(G_1+G_2)}\left[d^2_{G_1+G_2}(w)\,d^2_{G_1+G_2}(z)\right]\\ &= \prod_{wz\in E(G_1)}\left[d^2_{G_1+G_2}(w)d^2_{G_1+G_2}(z)\right]\\ &\times \prod_{wz\in E(G_2)}\left[d^2_{G_1+G_2}(w)\,d^2_{G_1+G_2}(z)\right]\\ &\times \prod_{w\in V(G_1)}\prod_{z\in V(G_2)}\left[d^2_{G_1+G_2}(w)\,d^2_{G_1+G_2}(z)\right]\\ &= A\times B\times C\end{aligned}$$

where A, B and C indicate the products of the above terms in order.

Now we calculate A.

$$\begin{aligned}
A &= \prod_{wz\in E(G_1)} \left[d^2_{G_1+G_2}(w) d^2_{G_1+G_2}(z)\right] \\
&= \prod_{wz\in E(G_1)} \left[(d_{G_1}(w)+p_2)^2 (d_{G_1}(z)+p_2)^2\right] \\
&\leq \left[\frac{\sum\limits_{wz\in E(G_1)} \left[(d_{G_1}(w)+p_2)^2 (d_{G_1}(z)+p_2)^2\right]}{q_1}\right]^{q_1} \\
&= \left[\frac{\sum\limits_{wz\in E(G_1)} \left[d^2_{G_1}(w)+p_2^2+2p_2 d_{G_1}(w)\right]\left[d^2_{G_1}(z)+p_2^2+2p_2 d_{G_1}(z)\right]}{q_1}\right]^{q_1} \\
&= \left[\frac{\begin{array}{r} HM_2(G_1)+p_2^2 HM(G_1)+p_2^4 q_1+2p_2 GO_2(G_1) \\ +2p_2^2 M_2(G_1)+2p_2^3 M_1(G_1)\end{array}}{q_1}\right]^{q_1}
\end{aligned}$$

Next we calculate B.

$$\begin{aligned}
B &= \prod_{wz\in E(G_2)} \left[d^2_{G_1+G_2}(w) d^2_{G_1+G_2}(z)\right] \\
&= \prod_{wz\in E(G_2)} \left[(d_{G_2}(w)+p_1)^2 (d_{G_2}(z)+p_1)^2\right] \\
&\leq \left[\frac{\sum\limits_{wz\in E(G_2)} \left[(d_{G_2}(w)+p_1)^2 (d_{G_2}(z)+p_1)^2\right]}{q_2}\right]^{q_2} \\
&= \left[\frac{\sum\limits_{wz\in E(G_2)} \left[d^2_{G_2}(w)+p_1^2+2p_1 d_{G_2}(w)\right]\left[d^2_{G_2}(z)+p_1^2+2p_1 d_{G_2}(z)\right]}{q_2}\right]^{q_2}
\end{aligned}$$

$$= \left[\frac{\begin{array}{r} HM_2(G_2) + p_1^2 HM(G_2) + p_1^4 q_2 + 2p_1 GO_2(G_2) \\ +2p_1^2 M_2(G_2) + 2p_1^3 M_1(G_2) \end{array}}{q_2} \right]^{q_2}$$

Finally, we compute C.

$$\begin{aligned} C &= \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} \left[d^2_{G_1+G_2}(w) + d^2_{G_1+G_2}(z) \right] \\ &= \prod_{w \in V(G_1)} \prod_{z \in V(G_2)} \left[(d_{G_1}(w) + p_2)^2 (d_{G_2}(z) + p_1)^2 \right] \\ &\leq \left[\frac{\sum\limits_{w \in V(G_1)} \sum\limits_{z \in V(G_2)} [(d_{G_1}(w) + p_2)^2 (d_{G_2}(z) + p_1)^2}{p_1 p_2} \right]^{p_1 p_2} \\ &= \left[\frac{\sum\limits_{w \in V(G_1)} \sum\limits_{z \in V(G_2)} \left[d^2_{G_1}(w) + p_2^2 + 2p_2 d_{G_1}(w) \right] \left[d^2_{G_2}(z) + p_1^2 + 2p_1 d_{G_2}(z) \right]}{p_1 p_2} \right]^{p_1 p_2} \\ &= \left[\frac{\begin{array}{r} M_1(G_1) M_1(G_2) + p_1^2 p_2 M_1(G_1) + p_1 p_2^2 M_1(G_2) + 4p_1 q_2 M_1(G_1) \\ +p_1^3 p_2^3 + p_2 q_1 M_1(G_2) + 4p_1^2 p_2^2 q_2 + 4p_1^2 p_2^2 q_1 + 16 p_1 p_2 q_1 q_2 \end{array}}{p_1 p_2} \right]^{p_1 p_2} \end{aligned}$$

Now using A, B and C we get the deired result. □

Lemma 3.2. *Let $G_i, (i = 1, 2)$ be two regular graphs of degree r_i.*

Let $G_i, (i = 1, 2)$ be a $(p_i, q_i) -$ graph. Then

$$\begin{aligned} \prod HM_2(G_1 + G_2) = (r_1 + p_2)^{4q_1} \times (r_2 + p_1)^{4q_2} \\ \times [(r_1 + p_2)^2 (r_2 + p_1)^2]^{p_1 p_2} \end{aligned}$$

Proof :

$$\prod HM_2(G_1 + G_2) = \prod_{wz \in E(G_1+G_2)} \left[d^2_{G_1+G_2}(w) \, d^2_{G_1+G_2}(z) \right]$$

$$= \prod_{wz\in E(G_1)} \left[d^2_{G_1+G_2}(w)d^2_{G_1+G_2}(z)\right]$$

$$\times \prod_{wz\in E(G_2)} \left[d^2_{G_1+G_2}(w)\,d^2_{G_1+G_2}(z)\right]$$

$$\times \prod_{w\in V(G_1)} \prod_{z\in V(G_2)} \left[d^2_{G_1+G_2}(w)\,d^2_{G_1+G_2}(z)\right]$$

$$= \prod_{wz\in E(G_1)} (r_1+p_2)^2(r_1+p_2)^2 \prod_{wz\in E(G_2)} (r_2+p_1)^2(r_2+p_1)^2$$

$$\prod_{wz\in E(G_1)} \prod_{wz\in E(G_2)} (r_1+p_2)^2(r_2+p_1)^2$$

$$= (r_1+p_2)^{4q_1} \times (r_2+p_1)^{4q_2}$$

$$\times \left[(r_1+p_2)^2(r_2+p_1)^2\right]^{p_1p_2} \tag{1}$$

□

Remark 3.3. We find the upper bound of Lemma 3.2 when G is a regular graph of degree r with p vertices and q edges. Here
$q = \frac{pr}{2}, M_1(G) = pr^2, M_2(G) = qr^2, F(G) = 2qr^2, F_2(G) = 2qr^3$,
$HM(G) = 4qr^2 HM_2(G) = qr^4, GO_2(G) = 2qr^3$

Corollary 3.4. *Let $G_i, (i = 1,2)$ be two regular graphs of degree r_i.*
Let $G_i, (i = 1,2)$ be a $(p_i, q_i)-$ graph. Then

$$\prod HM_2(G_1+G_2) \leq (r_1+p_2)^{4q_1} \times (r_2+p_1)^{4q_2}$$

$$\times \left[(r_1+p_2)^2(r_2+p_1)^2\right]^{p_1p_2} \tag{2}$$

From (1) and (2) the bound is tight.

4. The Multiplicative Second Hyper Zagreb Index of Composition of Graphs

Theorem 4.1. *Let $G_i, i=1,2$ be a $(p_i,q_i)-$ graph. Then*

$$\prod HM_2(G_1[G_2]) \leq \left[\frac{\begin{array}{l} p_2^4q_2F_2(G_1)+p_2^2M_1(G_1)HM(G_2)+p_1HM_2(G_2) \\ +2p_2^3F(G_1)M_1(G_2)+2p_2^2M_1(G_1)M_2(G_2)+4p_2q_1GO_2(G_2)\end{array}}{p_1q_2}\right]^{p_1q_2}$$

$$\times\left[\frac{\begin{array}{l} p_2^6HM_2(G_1)+p_2^3M_1(G_2)F(G_1)+q_1(M_1(G_2))^2+4p_2^4q_2GO_2(G_2) \\ +16p_2^2q_2^2M_2(G_1)+4p_2q_2M_1(G_1)M_1(G_2)\end{array}}{q_1p_2^2}\right]^{q_1p_2^2}$$

Proof :

$$\begin{aligned}
\prod HM_2(G_1[G_2]) &= \prod_{(w,k)(z,l)\in E(G_1[G_2])}\left[d^2_{G_1[G_2]}(w,k)\,d^2_{G_1[G_2]}(z,l)\right] \\
&= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left[d^2_{G_1[G_2]}(w,k)d^2_{G_1[G_2]}(z,l)\right] \\
&\times \prod_{k\in V(G_2)}\prod_{l\in V(G_2)}\prod_{wz\in E(G_1)}[d^2_{G_1[G_2]}(w,k)d^2_{G_1[G_2]}(z,l)] \\
&= A\times B,
\end{aligned}$$

where A and B indicate the products of the above terms in order.

Now we compute A.

$$\begin{aligned}
A &= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left[d^2_{G_1[G_1]}(w,k)\,d^2_{G_1[G_2]}(w,l)\right] \\
&= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left[[p_2d_{G_1}(w)+d_{G_2}(k)]^2[p_2d_{G_1}(w)+d_{G_2}(l)]^2\right] \\
&\leq \left[\frac{\sum\limits_{w\in V(G_1)}\sum\limits_{kl\in E(G_2)}\left[[p_2d_{G_1}(w)+d_{G_2}(k)]^2[p_2d_{G_1}(w)+d_{G_2}(k)]^2\right]}{p_1q_2}\right]^{p_1q_2}
\end{aligned}$$

$$= \left[\frac{\sum\limits_{w\in V(G_1)} \sum\limits_{kl\in E(G_2)} \left[p_2^2 d_{G_1}^2(w) + d_{G_2}^2(k) + 2p_2 d_{G_1}(w) d_{G_2}(k)\right] \left[p_2^2 d_{G_1}^2(w) + d_{G_2}^2(l) + 2p_2 d_{G_1}(w) d_{G_2}(l)\right]}{p_1 q_2} \right]^{p_1 q_2}$$

$$= \left[\frac{p_2^4 q_2 F_2(G_1) + p_2^2 M_1(G_1) HM(G_2) + p_1 HM_2(G_2) + 2p_2^3 F(G_1) M_1(G_2) + 2p_2^2 M_1(G_1) M_2(G_2) + 4p_2 q_1 GO_2(G_2)}{p_1 q_2} \right]^{p_1 q_2}$$

$$B = \prod_{k\in V(G_2)} \prod_{l\in V(G_2)} \prod_{wz\in E(G_1)} \left[d_{G_1[G_1]}^2(w,k)\, d_{G_1[G_2]}^2(z,l)\right]$$

$$= \prod_{k\in V(G_2)} \prod_{l\in V(G_2)} \prod_{wz\in E(G_1)} \left[[p_2 d_{G_1}(w) + d_{G_2}(k)]^2 [p_2 d_{G_1}(z) + d_{G_2}(l)]^2\right]$$

$$\leq \left[\frac{\sum\limits_{k\in V(G_2)} \sum\limits_{l\in V(G_2)} \sum\limits_{wz\in E(G_1)} \left[[p_2 d_{G_1}(w) + d_{G_2}(k)]^2 [p_2 d_{G_1}(z) + d_{G_2}(l)]^2\right]}{p_2^2 q_1} \right]^{p_2^2 q_1}$$

$$= \left[\frac{\sum\limits_{k\in V(G_2)} \sum\limits_{l\in V(G_2)} \sum\limits_{wz\in E(G_1)} \left[p_2^2 d_{G_1}^2(p) + d_{G_2}^2(k) + 2p_2 d_{G_1}(w) d_{G_2}(k)\right] \left[p_2^2 d_{G_1}^2(z) + d_{G_2}^2(l) + 2n_2 d_{G_1}(z) d_{G_2}(l)\right]}{q_1 p_2^2} \right]^{q_1 p_2^2}$$

$$= \left[\frac{p_2^6 HM_2(G_1) + p_2^3 M_1(G_2) F(G_1) + q_1 (M_1(G_2))^2 + 4p_2^4 q_2 GO_2(G_2) + 16 p_2^2 q_2^2 M_2(G_1) + 4p_2 q_2 M_1(G_1) M_1(G_2)}{q_1 p_2^2} \right]^{q_1 p_2^2}$$

Using A and B, we get the required result. □

Lemma 4.2. *Let $G_i, i=1,2$ be two regular graphs of degree r_i and let $G_i, i=1,2$ be a (p_i,q_i)-graph. Then $\prod HM_2\left(G_1\left[G_2\right]\right)=\left(p_2 r_1+r_2\right)^{4\left(p_1 q_2+p_2^2 q_1\right)}$.*

Proof :

$$
\begin{aligned}
\prod HM_2(G_1[G_2]) &= \prod_{w\in V(G_1)} \prod_{kl\in E(G_2)} \left[d^2_{G_1[G_2]}(w,k) d^2_{G_1,[G_2]}(w,l)\right] \\
&\times \prod_{k\in V(G_2)} \prod_{l\in V(G_2)} \prod_{wz\in E(G_1)} \left[d^2_{G_1[G_2]}(w,k) d^2_{G_1[G_2]}(z,l)\right] \\
&= \prod_{w\in V(G_1)} \prod_{kl\in E(G_2)} (p_2r_1+r_2)^2(p_2r_1+r_2)^2 \\
&\times \prod_{k\in V(G_2)} \prod_{l\in V(G_2)} \prod_{wz\in E(G_1)} (p_2r_1+r_2)^2(p_2r_1+r_2)^2 \\
&= (p_2r_1+r_2)^{4p_1q_2} \times (p_2r_1+r_2)^{4p_2^2q_1} \\
&= (p_2r_1+r_2)^{4\left(p_1q_2+p_2^2q_1\right)} \qquad (3)
\end{aligned}
$$

∎

Corollary 4.3. *Let $G_i, (i=1,2)$ be two regular graphs of degree r_i. Let $G_i, (i=1,2)$ be a $(p_i,q_i)-$ graph. Then*

$$
\prod HM_2\left(G_1\left[G_2\right]\right) \leq \left(p_2 r_1+r_2\right)^{4\left(p_1 q_2+p_2^2 q_1\right)} \qquad (4)
$$

From (3) and (4) our bound is tight.

5. The Multiplicative Second Hyper Zagreb Index of Cartesian Product of Graphs

Theorem 5.1. *Let $G_i, i=1,2$ be a $(p_i,q_i)-$graph. Then*

$$
\prod HM_2(G_1\square G_2) \leq \left[\frac{\begin{array}{c} q_2F_2(G_1)+2F(G_1)M_1(G_2)+4M_1(G_1)M_2(G_2) \\ +M_1(G_1)F(G_2)+p_1HM_2(G_2)+4q_1GO_2(G_2)\end{array}}{p_1q_2}\right]^{p_1q_2}
$$

$$\times\left[\frac{\begin{array}{c}p_1F_2(G_2)+2F(G_2)M_1(G_1)+4M_1(G_2)M_2(G_1)\\ +M_1(G_2)F(G_1)+p_2HM_2(G_1)+4q_2GO_2(G_1)\end{array}}{p_2q_1}\right]^{p_2q_1}$$

Proof :

$$\begin{aligned}\prod HM_2(G_1\Box G_2) &\leq \prod_{(w,k)(z,l)\in E(G_1\Box G_2)}\left[d^2_{G_1\Box G_2}(w,k)d^2_{G_1\Box G_2}(z,l)\right]\\ &= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left[d^2_{G_1\Box G_2}(w,k)d^2_{G_1\Box G_2}(z,l)\right]\\ &\times \prod_{k\in V(G_2)}\prod_{wz\in E(G_1)}\left[d^2_{G_1\Box G_2}(w,k)d^2_{G_1\Box G_2}(z,l)\right]\\ &= A\times B\end{aligned}$$

where A and B indicate the products of the above terms in order.

Now we calculate A.

$$\begin{aligned}A &= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left[d^2_{G_1\Box G_2}(w,k)d^2_{G_1\Box G_2}(z,l)\right]\\ &= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left[[d_{G_1}(w)+d_{G_2}(k)]^2[d_{G_1}(w)+d_{G_2}(l)]^2\right]\\ &\leq \left[\frac{\sum_{w\in V(G_1)}\sum_{kl\in E(G_2)}[d_{G_1}(w)+d_{G_2}(k)]^2[d_{G_1}(w)+d_{G_2}(l)]^2}{p_1q_2}\right]^{p_1q_2}\\ &= \left[\frac{\begin{array}{r}\sum_{w\in V(G_1)}\sum_{kl\in E(G_2)}\left[d^2_{G_1}(w)+d^2_{G_2}(k)+2d_{G_1}(w)d_{G_2}(k)\right]\\ \left[d^2_{G_1}(k)+d^2_{G_2}(l)+2d_{G_1}(w)d_{G_2}(l)\right]\end{array}}{p_1q_2}\right]^{p_1q_2}\end{aligned}$$

$$= \left[\frac{\begin{array}{l} q_2F_2(G_1)+2F(G_1)M_1(G_2)+4M_1(G_1)M_2(G_2)+M_1(G_1)F(G_2) \\ +p_1HM_2(G_2)+4q_1GO_2(G_2) \end{array}}{p_1q_2}\right]^{p_1q_2}$$

Now we compute B.

$$\begin{aligned}
B &= \prod_{k\in V(G_2)}\prod_{wz\in E(G_1)} \left[d^2_{G_1\Box G_2}(w,k)d^2_{G_1\Box G_2}(z,l)\right] \\
&= \prod_{k\in V(G_2)}\prod_{wz\in E(G_1)} \left[[d_{G_1}(w)+d_{G_2}(k)]^2[d_{G_1}(z)+d_{G_2}(k)]^2\right] \\
&\leq \left[\frac{\sum\limits_{k\in V(G_2)}\sum\limits_{wz\in E(G_1)}[d_{G_1}(w)+d_{G_2}(k)]^2[d_{G_1}(z)+d_{G_2}(k)]^2}{p_2q_1}\right]^{p_2q_1} \\
&= \left[\frac{\begin{array}{r} \sum\limits_{k\in V(G_2)}\sum\limits_{wz\in E(G_1)}\left[d^2_{G_1}(w)+d^2_{G_2}(k)+2d_{G_1}(w)d_{G_2}(k)\right] \\ \left[d^2_{G_1}(z)+d^2_{G_2}(k)+2d_{G_1}(z)d_{G_2}(k)\right] \end{array}}{p_2q_1}\right]^{p_2q_1} \\
&= \left[\frac{\begin{array}{l} q_1F_2(G_2)+2F(G_2)M_1(G_1)+4M_1(G_2)M_2(G_1)+M_1(G_2)F(G_1) \\ +p_2HM_2(G_1)+4q_2GO_2(G_1) \end{array}}{p_2q_1}\right]^{p_2q_1}
\end{aligned}$$

Using A and B we get the desired result. □

Lemma 5.2. *Let $G_{i_1} i = 1,2$ be two regular graphs of degree r_i and let $G_i ;= 1,2$ be a $(p_i,q_i)-$ graph. Then $\prod HM_2(G_1\Box G_2) = (r_1+r_2)^{4(p_1q_2+p_2q_1)}$*

Proof :

$$\begin{aligned}
\prod HM_2(G_1\Box G_2) = &\prod_{w\in V(G_1)}\prod_{kl\in E(G_2)} \left[d^2_{G_1\Box G_2}(w,k)\,d^2_{G_1\Box G_2}(w,l)\right] \\
&\times \prod_{k\in V(G_2)}\prod_{wz\in E(G_1)} [d^2_{G_1\Box G2}(w,k)+d_{G_1^2\Box G_2}(z,k)]
\end{aligned}$$

$$= \prod_{w\in V(G_1)} \prod_{kl\in E(G_2)} (r_1+r_2)^2 (r_1+r_2)^2$$

$$\times \prod_{k\in V(G_2)} \prod_{wz\in E(G_2)} (r_1+r_2)^2 (r_1+r_2)^2$$

$$= (r_1+r_2)^{4p_1q_2} \times (r_1+r_2)^{4p_2q_1}$$

$$= (r_1+r_2)^{4(p_1q_2+p_2q_1)} \tag{5}$$

□

Corollary 5.3. *Let $G_i, (i=1,2)$ be two regular graphs of degree r_i. Let $G_i, (i=1,2)$ be a $(p_i,q_i)-$ graph. Then*

$$\prod HM_2(G_1 \square G_2) \leq (r_1+r_2)^{4(p_1q_2+p_2q_1)} \tag{6}$$

From (5) and (6) the bound is tight.

6. The multiplicative second hyper Zagreb index of corona product of graphs

Theorem 6.1. *Let $G_i, i=1,2$ be a (p_i,q_i)-graph. Then*

$$\prod HM_2(G_1 \odot G_2) \leq \left[\frac{\begin{array}{r} p_2^4 q_1 + 2p_2^3 M_1(G_1) + 4p_2^2 M_2(G_1) + p_2^2 F(G_1) \\ +HM_2(G_1) + 2p_2 GO_2(G_1) \end{array}}{q_1} \right]^{q_1}$$

$$\times \left[\frac{\begin{array}{r} q_2 + 2M_1(G_2) + 4M_2(G_2) + F(G_2) \\ +HM_2(G_2) + 2GO_2(G_2) \end{array}}{q_2} \right]^{p_1q_2}$$

$$\left[\frac{\begin{array}{r}M_1(G_1)M_1(G_2)+4q_2M_1(G_2)+p_2M_1(G_1)+p_2M_1(G_2)(p_1p_2+4q_1)\\ +16q_1q_2p_2+4p_2^2q_1+p_1p_2^2(p_2+4q_2)\end{array}}{p_1p_2}\right]^{p_1p_2}$$

Proof :

$$\begin{aligned}\prod HM_2(G_1\odot G_2) &= \prod_{wz\in E(G_1)}\left((d_{G_1}(w)+p_2)^2(d_{G_1}(z)+p_2)^2\right)\\ &\times \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left((d_{G_2}(k)+1)^2(d_{G_2}(l)+1)^2\right)\\ &\times \prod_{w\in V(G_1)}\prod_{k\in V(G_2)}\left((d_{G_1}(w)+p_2)^2(d_{G_2}(k)+1)^2\right)\\ &= A\times B\times C\end{aligned}$$

where A,B and C are the products of the about terms in order.

Now calculate A,

$$\begin{aligned}A &= \prod_{wz\in E(G_1)}\left((d_{G_1}(w)+p_2)^2(d_{G_1}(z)+p_2)^2\right)\\ &\leq \left[\frac{\sum\limits_{wz\in E(G_1)}(d_{G_1}(w)+p_2)^2(d_{G_1}(z)+p_2)^2}{q_1}\right]^{q_1}\\ &= \left[\frac{\sum\limits_{wz\in E(G_1)}[d_{G_1}^2(u)+p_2^2+2p_2d_{G_1}(w)][d_{G_1}^2(z)+p_2^2+2p_2d_{G_1}(z)]}{q_1}\right]^{q_1}\\ &= \left[\frac{p_2^4q_1+2p_2^3M_1(G_1)+4p_2^2M_2(G_1)+p_2^2F(G_1)+HM_2(G_1)+2p_2GO_2(G_1)}{q_1}\right]^{q_1}\end{aligned}$$

Next compute B.

$$\begin{aligned}B &= \prod_{w\in V(G_1)}\prod_{kl\in E(G_2)}\left((d_{G_2}(k)+1)^2(d_{G_2}(l)+1)^2\right)\\ &\leq \left[\frac{\sum\limits_{w\in V(G_1)}\sum\limits_{kl\in E(G_2)}(d_{G_2}(k)+1)^2(d_{G_2}(l)+1)^2}{p_1q_2}\right]^{p_1q_2}\end{aligned}$$

$$= \left[\frac{\sum\limits_{w\in V(G_1)} \sum\limits_{kl\in E(G_2)} \Big[1+2(d_{G_2}(k)+d_{G_2}(l))+4d_{G_2}(k)d_{G_2}(l) \\ (d^2_{G_2}(k)+d^2_{G_2}(l))+d^2_{G_2}(k)d^2_{G_2}(l) \\ +2(d^2_{G_2}(k)d_{G_2}(l)+d_{G_2}(k)d^2_{G_2}(l))\Big]}{p_1q_2} \right]^{p_1q_2}$$

$$= \left[\frac{q_2+2M_1(G_2)+4M_2(G_2)+F(G_2)+HM_2(G_2)+2GO_2(G_2)}{q_2}\right]^{p_1q_2}$$

Finally, compute C

$$C = \prod_{w\in V(G_1)} \prod_{k\in V(G_2)} \big(d_{G_1}(w)+p_2)^2(d_{G_2}(k)+1)^2\big)$$

$$\leq \left[\frac{\sum\limits_{w\in V(G_1)} \sum\limits_{k\in V(G_2)} \big(d_{G_1}(w)+p_2)^2(d_{G_2}(k)+1)^2\big)}{p_1p_2}\right]^{p_1p_2}$$

$$= \left[\frac{\sum\limits_{w\in V(G_1)} \sum\limits_{k\in V(G_2)} \Big(d^2_{G_1}(w)+2p_2d_{G_1}(w)+p_2^2)(d^2_{G_2}(k)+2d_{G_2}(k)+1)\Big)}{p_1p_2}\right]^{p_1p_2}$$

$$= \left[\frac{M_1(G_1)M_1(G_2)+4q_2M_1(G_2)+p_2M_1(G_1)+p_2M_1(G_2)(p_1p_2+4q_1) \\ +16q_1q_2p_2+4p_2^2q_1+p_1p_2^2(p_2+4q_2)}{p_1p_2}\right]^{p_1p_2}$$

Now multiplying A, B and C we get the required result. □

Lemma 6.2. *Let $G_i, i=1,2$ be two regular graph of degree r_i, and let $G_i, i=1,2$ be a $(p_i,q_i)-$ graph. Then*

$$\prod HM_2(G_1\odot G_2) = (r_1+p_2)^{4q_1}\times(r_2+1)^{4p_1q_2}\times((r_1+p_2)^2(r_2+1)^2)^{p_1p_2}$$

Proof :

$$\prod HM_2(G_1\odot G_2) = \prod_{wz\in E(G_1)} \big((d_{G_1}(w)+p_2)^2(d_{G_1}(z)+p_2)^2\big)$$

$$\times \prod_{w\in V(G_1)} \prod_{kl\in E(G_2)} \left((d_{G_2}(k)+1)^2(d_{G_2}(l)+1)^2\right)$$

$$\times \prod_{w\in V(G_1)} \prod_{k\in V(G_2)} \left((d_{G_1}(w)+p_2)^2(d_{G_2}(k)+1)^2\right)$$

$$= \prod_{wz\in E(G_1)} (r_1+p_2)^2(r_1+p_2)^2$$

$$\times \prod_{uwinV(G_1)} \prod_{kl\in E(G_2)} (r_2+1)^2(r_2+1)^2$$

$$\times \prod_{w\in V(G_1)} \prod_{k\in V(G_2)} (r_1+p_2)^2(r_2+1)^2$$

$$= (r_1+p_2)^{4q_1} \times (r_2+1)^{4p_1q_2} \times ((r_1+p_2)^2(r_2+1)^2)^{p_1p_2} \tag{7}$$

□

Corollary 6.3. *Let $G_i,(i=1,2)$ be two regular graphs of degree r_i.*
Let $G_i,(i=1,2)$ be a $(p_i,q_i)-$ graph. Then

$$\prod HM_2(G_1 \odot G_2) \leq (r_1+p_2)^{4q_1} \times (r_2+1)^{4p_1q_2} \times ((r_1+p_2)^2(r_2+1)^2)^{p_1p_2} \tag{8}$$

From (7) and (8) the bound is tight.

7. Conclusion

In this paper,we have defined the multiplicative second hyper Zagreb index and derived the sharp upper bound for this index of various graph operations lke join, composition, cartesian and corona producst of graphs are derived. And we have proved that the sharp upper bound is tight.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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