Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 6, 3143-3158
https://doi.org/10.28919/jmcs/5039
ISSN: 1927-5307

SOME FIXED POINT RESULTS IN PARAMETRIC METRIC SPACE

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Abstract. In present paper, we prove fixed point theorems based on rational expressions for contraction mapping in parametric space. Moreover, we provide an example to furnish our result and usability of our result.

Keywords: contractions mapping; fixed point; parametric metric space; complete metric space; convergent.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Let $A \neq \emptyset$ and $T : A \to A$ is a mapping, then a point $p^* \in A$ s.t $Tp^* = p^*$ is a fixed point of mapping T. If $T : A \to A$ is a multi-valued map (i.e. from $A \neq \emptyset$ is a subsets of A), then point $p^* \in A$ is a fixed point of mapping T if $p^* \in Tp^*$. Most of the physical problems can be transferred to fixed point theory. Probably the origin of fixed point theory goes back to the starting of 20^{th} century as an important part of nonlinear study. Fixed point theory has mesmerized lots of researchers. In 1922, Banach (Polish mathematician) celebrated his most famous

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Received September 21, 2020

principle, which is known as Banach contraction principle [1], to find a fixed point of a map. Only lacuna of Banach contraction principle was the mapping T must be continuous throughout space. Banach contraction principle is very popular among researchers and helpful in fixed point theory. Kannan [2], rectified lacuna of Banach contraction principle and proved a fixed point theorem for operators that need not be continuous. Further, Chatterjea [3], proved a result for discontinuous mapping which is a kind of dual of Kannan mapping. A lucid survey shows that there exists a vast literature available on fixed point theory. Fixed point theorems are main concerned about existence and uniqueness of a point in a non empty set.They are applicable in iteration methods, partial differential equations, integral differential equations, variational inequalities etc. Many authors extended the Banach Contraction Principle in different directions. Since Banach contraction principle has seen, many extension and generalization in different space (see [4–19].

In last few years, different authors have developed different generalized metric space by changing triangular inequality using different approach. Some generalized metric space are D-metric, D^* metric space, b metric space, b-like metric space, partial metric space, partial b-metric space, quasi partial b metric space, Cone metric, Generalized cone metric space, etc.

Wang et al. [25], introduced and defined the expansive mapping on complete metric space and proved some fixed point theorems. Moreover, Daffer and Kaneko [26] proved some fixed point results for couple of mappings on complete metric space using expansive mapping.

Recently, the idea of a parametric metric space gave by Hussian [27], in 2014 and proved some fixed point theorems on parametric metric space. Furthermore, more information on parametric metric space are available in [28–30]. In the present paper, we prove fixed point theorems with contraction condition in parametric metric spaces.

2. Preliminaries

Definition 2.1 ([27]). *Let $A \neq \emptyset$ and let a function $T_p : A \times A \times (0,\infty) \to [0,\infty]$ then, T_p is called parametric metric space in A if*

(1) $T_p(f_*, g_*, t) = 0$ *iff* $f_* = g_*$

(2) $T_p(f_*, g_*, t) = T_p(g_*, f_*, t)$

(3) $T_p(f_*, g_*, t) \leq T_p(f_*, h_*, t) + T_p(h_*, g_*, t)$ *for all* $f_*, g_*, h_* \in A$ *and all* $t > 0$.

Pair (A, T_p) *is called parametric metric space.*

Example 2.2 ([27]). *Let* $A = R^2$ *for any* $\alpha = (\alpha_1(t), \alpha_2(t))$, $\beta = (\beta_1(t), \beta_2(t))$. *Moreover, define the function* $T_p : A \times A \times (0, \pm\infty) \to [0, \pm\infty)$ *by*

$$T_p(\alpha, \beta, t) = |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)| \quad \forall\, \alpha, \beta \in A \text{ and all } t > 0.$$

Then T_p *is parametric metric in* A *and* (A, T_p) *is parametric metric space.*

Proof. For all $\alpha(t), \beta(t), \gamma(t) \in A$, we have

(1)
$$\begin{aligned} T_p(\alpha, \beta, t) = 0 \;\Rightarrow\quad & |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)| = 0 \\ \Rightarrow\quad & |\alpha_1(t) - \beta_1(t)| = 0 \text{ and } |\alpha_2(t) - \beta_2(t)| = 0 \\ \Rightarrow\quad & \alpha_1(t) - \beta_1(t) = 0 \text{ and } \alpha_2(t) - \beta_2(t) = 0 \\ \Rightarrow\quad & \alpha_1(t) = \beta_1(t) \text{ and } \alpha_2(t) = \beta_2(t) \end{aligned}$$

(2)
$$\begin{aligned} T_p(\alpha, \beta, t) &= |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)| \\ &= |-(\beta_1(t) - \alpha_1(t))| + |-(\beta_2(t) - \alpha_2(t))| \\ &= |\beta_1(t) - \alpha_1(t)| + |\beta_2(t) - \alpha_2(t)| \\ &= T_p(\beta_1, \alpha_1, t) + T_p(\beta_2, \alpha_2, t) \end{aligned}$$

(3)
$$\begin{aligned} T_p(\alpha, \beta, t) &= |\alpha_1(t) - \beta_1(t)| + |\alpha_2(t) - \beta_2(t)| \\ &\leq |\alpha_1(t) - \gamma_1(t)| + |\gamma_1(t) - \beta_1(t)| + |\alpha_2(t) - \gamma_2(t)| + |\gamma_2(t) - \beta_2(t)| \\ &= T_p(\alpha, \gamma, t) + T_p(\gamma, \beta, t) \end{aligned}$$

All the conditions are satisfying the property of parametric metric space. Therefore $T_p(\alpha, \beta, t)$ is a parametric metric space. □

Definition 2.3 ([27]). *Consider* $\{a_j\}$ *be a sequence in parametric metric space* (A, T_p).

(1) $\{a_j\}$ *is known as convergent to* $a \in A$ *as,*

$$\lim_{j \to \infty} a_j = 0,\ \forall\, t > 0 \text{ if } \lim_{j \to \infty} T_p(a_j, a, t) = 0$$

(2) $\{a_j\}$ *is known as Cauchy sequence in A if* $\forall\, t > 0$, *if* $\lim_{j,i\to\infty} T_p(a_j, a_i, t) = 0$.

(3) *Every Cauchy sequence* (A, T_p) *is a convergent sequence then that sequence* (A, T_p) *is called complete.*

Definition 2.4 ([27]). *Let* (A, T_p) *is a parametric metric space and a function* $T : A \to A$ *is continuous at* $a \in A$, *if for any sequence* $\{a_j\}$ *in A such that*

$$\lim_{j\to\infty} a_j = a \ \text{ then } \lim_{j\to\infty} Ta_j = Ta\,.$$

Lemma 2.5 ([27]). *Let construct a sequence* $\{m_k\}$ *in a parametric metric space* (A, T_p) *such that*

$$T_p(m_k, m_{k+1}, t) = hT_p(m_{k-1}, m_k, t),$$

where $h \in [0, 1)$ *and* $k = 1, 2, 3, \dots$.

Then $\{m_k\}$ *is a Cauchy sequence in* (A, T_p).

Verification. Let $k > l \geq 1$, it follows that

$$\begin{aligned} T_p(m_k, m_l, t) &\leq T_p(m_k, m_{k+1}, t) + T_p(m_{k+1}, m_{k+2}, t) \dots T_p(m_{l-1}, m_l, t) \\ &\leq (h^k + h^{k+1} \dots h^{k-1}) T_p(m_0, m_1, t) \ \ \forall\, t > 0, \text{ since } h < 1. \end{aligned}$$

Assume that $T_p(m_0, m_1, t) > 0$. By taking $\lim_{k,l\to+\infty}$, we get

$$\lim_{k,l\to+\infty} T_p(m_k, m_l, t) = 0\,.$$

As a result, $\{m_k\}$ is a Cauchy sequence in A. Also, if $T_p(m_0, m_1, t) = 0$ then $T_p(m_k, m_l, t) = 0\ \forall$ $k > l$.

Hence $\{m_k\}$ is Cauchy sequence in A.

3. Main Results

Here, we prove some fixed point results for continuous function as well as satisfy the contraction conditions by considering the self-mapping on parametric metric space.

Theorem 3.1. *Let (A,T_p) is a complete parametric metric space and mapping $T : A \to A$ is a continuous then it satisfied the condition:*

$$T_p(Ta,Tb,t) \leq \alpha_1 T_p(a,b,t)+\alpha_2[T_p(a,Ta,t)+T_p(b,Tb,t)] + \alpha_3[T_p(a,Tb,t)+T_p(b,Ta,t)]+\alpha_4\left[\frac{T_p(a,b,t)T_p(a,Tb,t)}{T_p(a,b,t)+T_p(b,Tb,t)}\right] + \alpha_5\left[\frac{T_p(a,Tb,t)T_p(b,Tb,t)}{T_p(a,b,t)+T_p(b,Tb,t)}\right] \tag{3.1}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 \geq 0$ with $\alpha_1+2\alpha_2+4\alpha_3+\alpha_4+\alpha_5 < 1$ for all, and $t > 0$. Then T has a unique fixed point.

Proof. Let m_0 be an initial point and $\{m_j\}$ is a sequence such that $m_j = Tm_{j-1} = T^j m_0$. If there is a point $m_0 \in A$ such that $m_j = m_{j+1}$, then m_j is a fixed point. Therefore, there is no need to proceed further. Otherwise $m_j \neq m_{j+1}$. Using the inequality (3.2), we have

$$\begin{aligned}
T_p(a,b,t) &= T_p(Tm_j,Tm_{j+1},t)\\
&\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2[T_p(m_j,Tm_j,t)+T_p(m_{j+1},Tm_{j+1},t)]\\
&\quad +\alpha_3[T_p(m_j,Tm_{j+1},t)+T_p(m_{j+1},Tm_j,t)]\\
&\quad +\alpha_4\left[\frac{T_p(m_j,m_{j+1},t)T_p(m_j,Tm_{j+1},t)}{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},Tm_{j+1},t)}\right]\\
&\quad +\alpha_5\frac{T_p(m_j,Tm_{j+1},t)T_p(m_{j+1},Tm_{j+1},t)}{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},Tm_{j+1},t)}\\
&\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2[T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)]\\
&\quad +\alpha_3[T_p(m_j,m_{j+2},t)+T_p(m_{j+1},m_{j+1},t)]\\
&\quad +\alpha_4\left[\frac{T_p(m_j,m_{j+1},t)T_p(m_j,m_{j+2},t)}{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)}\right]\\
&+\alpha_5\frac{T_p(m_j,m_{j+2},t)T_p(m_{j+1},m_{j+2},t)}{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)}\\
&\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2[T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)]\\
&+\alpha_3[T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)+T_p(m_{j+1},m_{j+2},t)+T_p(m_j,m_{j+1},t)]\\
&+\alpha_4 T_p(m_j,m_{j+1},t)+\alpha_5 T_p(m_{j+1},m_{j+2},t)
\end{aligned}$$

$$\leq \alpha_1 T_p(m_j, m_{j+1}, t) + \alpha_2 T_p(m_j, m_{j+1}, t)$$

$$+ \alpha_2 T_p(m_{j+1}, m_{j+2}, t) + 2\alpha_3 T_p(m_j, m_{j+1}, t)$$

$$+ 2\alpha_3 T_p(m_{j+1}, m_{j+2}, t) + \alpha_4 T_p(m_j, m_{j+1}, t) + \alpha_5 T_p(m_{j+1}, m_{j+2}, t)$$

$$T_p(m_{j+1}, m_{j+2}, t) \leq \frac{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4}{1 - (\alpha_2 + 2\alpha_3 + \alpha_5)} T_p(m_j, m_{j+1}, t)$$

Let $h = \dfrac{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4}{1 - (\alpha_2 + 2\alpha_3 + \alpha_5)} < 1$, as $\alpha_1 + 2\alpha_2 + 4\alpha_3 + \alpha_4 + \alpha_5 < 1$.

Therefore,

$$T_p(m_{j+1}, m_{j+2}, t) \leq h T_p(m_j, m_{j+2}, t).$$

Similarly,

$$T_p(m_j, m_{j+1}, t) \leq h T_p(m_{j-1}, m_j, t)$$

$$T_p(m_{j+1}, m_{j+2}, t) \leq h.h T_p(m_{j-1}, m_j, t)$$

$$\leq h^2 T_p(m_{j-1}, m_j, t).$$

Using iteration up to j times,

$$T_p(m_j, m_{j+1}, t) \leq h^j T_p(m_0, m_1, t),$$

where $0 \leq h \leq 1$ and $t > 0$.

$\Rightarrow h^j \to 0$ as $j \to \infty$. Using Lemma 2.5 sequence $\{m_j\}$ is Cauchy sequence. So $\exists\ \mu \in j$ such that $m_j \to \mu$ as $j \to \infty$.

Next, we will show that μ is a fixed point of T. For that $m_j \to \mu$ as $j \to \infty$. By means of continuity into T, we have

$$\lim_{j \to \infty} Tm = T\mu$$

$$\lim_{j \to \infty} m_{j+1} = T\mu$$

Then, $T\mu = \mu$, then μ is a fixed point of T.

For uniqueness, let μ and ρ be the two fixed point of T for $\mu \neq \rho$, we have

$$T_p(\mu, \rho, t) \leq \alpha_1 T_p(\mu, \rho, t) + \alpha_2[T_p(\mu, T\rho, t) + T_p(\rho, T\rho, t)] + \alpha_3[T_p(\mu, T\rho, t) + T_p(\rho, T\mu, t)]$$

$$+\alpha_4\left[\frac{T_p(\mu,\rho,t)T_p(\mu,T\rho,t)}{T_p(\mu,\rho,t)+T_p(\rho,T\rho,t)}\right]+\alpha_5\frac{T_p(\mu,T\rho,t)T_p(\rho,T\rho,t)}{T_p(\mu,\rho,t)+T_p(\rho,T\rho,t)}$$

$$T_p(\mu,\rho,t)\leq\alpha_1T_p(\mu,\rho,t)+\alpha_2[T_p(\mu,\rho,t)+T_p(\rho,\rho,t)]+\alpha_3[T_p(\mu,\rho,t)+T_p(\rho,\mu,t)]$$
$$+\alpha_4\left[\frac{T_p(\mu,\rho,t)T_p(\mu,\rho,t)}{T_p(\mu,\rho,t)+T_p(\rho,\rho,t)}\right]+\alpha_5\frac{T_p(\mu,\rho,t)T_p(\rho,\rho,t)}{T_p(\mu,\rho,t)+T_p(\rho,\rho,t)}.$$

As μ and ρ are fixed point of T.

Therefore, by above equation we have,

$$T_p(\mu,\mu,t)=0 \text{ and } T_p(\rho,\rho,t)=0.$$

So, above equation become

$$T_p(\mu,\rho,t)\leq[\alpha_1+\alpha_3+\alpha_4]T_p(\mu,\rho,t)+\alpha_3T_p(\rho,\mu,t). \tag{3.2}$$

Similarly,

$$T_p(\rho,\mu,t)\leq[\alpha_1+\alpha_3+\alpha_4]T_p(\rho,\mu,t)+\alpha_3T_p(\mu,\rho,t). \tag{3.3}$$

Subtract (3.3) from (3.2)

$$T_p(\mu,\rho,t)-T_p(\rho,\mu,t)|\leq|(\alpha_1+\alpha_3+\alpha_4)-\alpha_3|T_p(\mu,\rho,t)-T_p(\rho,\mu,t)|$$
$$\leq|\alpha_1+\alpha_4|\,|T_p(\mu,\rho,t)-T_p(\rho,\mu,t)|. \tag{3.4}$$

Here, $|\alpha_1+\alpha_4|<1$, above inequality hold.

$$\Rightarrow T_p(\mu,\rho,t)-T_p(\rho,\mu,t)=0 \tag{3.5}$$

From (3.2), (3.3) and (3.5), we have

$$T_p(\mu,\rho,t)=0 \text{ and } T_p(\rho,\mu,t)=0$$
$$\Rightarrow \quad \mu=\rho$$

This completes the proof. □

Theorem 3.2. *Let complete parametric metric space is (A,T_p) and $T:A\to A$ be a continuous then it satisfying the condition:*

$$T_p(Ta,Tb,t)\leq\alpha_1T_p(a,b,t)+\alpha_2[T_p(a,Ta,t)+T_p(b,Tb,t)\left[\frac{T_p(a,b,t)+T_p(b,Tb,t)}{T_p(a,Tb,t)}\right]$$

$$+\alpha_3[T_p(a,Tb,t)+T_p(b,Ta,t)]\left[\frac{T_p(a,b,t)+T_p(b,Tb,t)+T_p(a,Tb,t)}{T_p(a,Tb,t)}\right] \tag{3.6}$$

where $\alpha_1,\alpha_2,\alpha_3 \geq 0$ *with* $\alpha_1+2\alpha_2+8\alpha_3<1$ *for all* $a,b\in A$ *and* $t>0$. *Then* T *has unique fixed point.*

Proof. Let m_0 be an initial point and a sequence $\{m_j\}$ such that $m_j = Tm_{j-1} = T^j m_0$. If there is a point $m_0 \in A$ such that $m_j = m_{j+1}$, then m_j is a fixed point. Therefore, there is no need to proceed further. Otherwise $m_j \neq m_{j+1}$. Using the inequality (3.6), we have

$$\begin{aligned}
T_p(a,b,t) = T_p(Tm_j,Tm_{j+1},t) &\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2[T_p(m_j,Tm_j,t)\\
&\quad +T_p(m_{j+1},Tm_{j+1},t)\left[\frac{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},Tm_{j+1},t)}{T_p(m_j,Tm_{j+1},t)}\right]\\
&\quad +\alpha_3[T_p(m_j,Tm_{j+1},t)+T_p(m_{j+1},Tm_j,t)]\\
&\quad \cdot\left[\frac{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},Tm_{j+1},t)+T_p(m_j,Tm_{j+1},t)}{T_p(m_j,Tm_{j+1},t)}\right]\\
&\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2[T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)\\
&\quad \cdot\left[\frac{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)}{T_p(m_j,m_{j+2},t)}\right]\\
&\quad +\alpha_3[T_p(m_j,m_{j+2},t)+T_p(m_{j+1},m_{j+1},t)]\\
&\quad \cdot\left[\frac{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)+T_p(m_j,m_{j+2},t)}{T_p(m_j,m_{j+2},t)}\right]\\
&\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2[T_p(m_j,m_{j+1},t)+(T_p(m_{j+1},m_{j+2},t)]\\
&\quad \cdot\left[\frac{T_p(m_j,m_{j+1},t)+T_p(m_{j+1},m_{j+2},t)}{T_p(m_j,m_{j+2},t)}\right]\\
&\quad +\alpha_3[T_p(m_j,m_{j+2},t)+T_p(m_{j+1},m_{j+2},t)\\
&\quad \cdot +T_p(m_j,m_{j+1},t)]\left[2\frac{T_p(m_j,m_{j+1},t)}{T_p(m_j,m_{j+2},t)}\right]\\
&\leq \alpha_1 T_p(m_j,m_{j+1},t)+\alpha_2 T_p(m_j,m_{j+1},t)+\alpha_2 T_p(m_{j+1},m_{j+2},t)\\
&\quad +2\alpha_3 T_p(m_j,m_{j+1},t)+2\alpha_3 T_p(m_{j+1},m_{j+2},t)\\
&\quad +2\alpha_3 T_p(m_j,m_{j+1},t)+2\alpha_3 T_p(m_{j+1},m_{j+2},t)
\end{aligned}$$

$$T_p(m_{j+1},m_{j+2},t) \leq \frac{\alpha_1+\alpha_2+4\alpha_3}{1-4\alpha_3-\alpha_2}T_p(m_j,m_{j+1},t).$$

Let $h = \dfrac{\alpha_1 + \alpha_2 + 4\alpha_3}{1 - 4\alpha_3 - \alpha_2}$ as $h < 1$.

Therefore, $T_p(m_{j+1}, m_{j+2}, t) \leq hT_p(m_j, m_{j+1}, t)$.

Similarly, $T_p(m_j, m_{j+1}, t) \leq hT_p(m_{j-1}, m_j, t)$.

$$h^j \to 0 \text{ as } j \to \infty$$

$$\begin{aligned} T_p(m_{j+1}, m_{j+2}, t) &\leq h.hT_p(m_{j-1}, m_j, t) \\ &\leq h^2 T_p(m_{j-1}, m_j, t). \end{aligned}$$

Using iteration up to j times,

$$T_p(m_j, m_{j+1}, t) \leq h^j T_p(m_0, m_1, t)$$

where $0 \leq h \leq 1$ and $t > 0$.

$\Rightarrow h^j \to 0$ as $j \to \infty$. Using Lemma 2.5 sequence $\{m_j\}$ is Cauchy sequence. So $\exists\ \mu \in m$ such that

$$m_j \to \mu \text{ as } j \to \infty.$$

Now, we will prove μ is a fixed point of T.

Since $m_j \to \mu$ as $j \to \infty$. By means of continuity into T, we have

$$\lim_{j \to \infty} Tm = T\mu$$

$$\lim_{j \to \infty} m_{j+1} = T\mu$$

Then $T\mu = \mu$, then μ is a fixed point of T.

For uniqueness, let us consider μ and ρ be the two fixed point of T for $\mu \neq \rho$, we have

$$\begin{aligned} T_p(\mu, \rho, t) \leq \alpha_1 T_p(\mu, \rho, t) + \alpha_2 [T_p(\mu, T\mu, t) + T_p(\rho, T\rho, t) \left[\frac{T_p(\mu, \rho, t) + T_p(\rho, T\rho, t)}{T_p(\mu, T\rho, t)} \right] \\ + \alpha_3 [T_p(\mu, T\rho, t) + T_p(\mu, T\mu, t)] \left[\frac{T_p(\mu, \rho, t) + T_p(\rho, T\rho, t) + T_p(\mu, T\rho, t)}{T_p(\mu, T\rho, t)} \right] \end{aligned}$$

Here, μ and ρ are fixed point of T.

Therefore, by given condition, we have

$$T_p(\mu, \mu, t) = 0 \text{ and } T_p(\rho, \rho, t) = 0.$$

So, above equation become

$$T_p(\mu, f_2{}^*, t) \leq \alpha_1 T_p(\mu, f_2{}^*, t) + 2\alpha_3 T_p(\mu, f_2{}^*, t) + 2\alpha_3 T_p(f_2{}^*, \mu, t)$$

and

$$(\mu, \rho, t) \leq (\alpha_1 + 2\alpha_3) T_p(\mu, \rho, t) + 2\alpha_3 T_p(\rho, \mu, t). \tag{3.7}$$

Similarly,

$$T_p(\rho, \mu, t) \leq (\alpha_1 + 2\alpha_3) T_p(\rho, \mu, t) + 2\alpha_3 T_p(\mu, \rho, t). \tag{3.8}$$

Subtract above two equations, we get

$$\begin{aligned} |T_p(\mu, \rho, t) - T_p(\rho, \mu, t)| &\leq |\alpha_1 + 2\alpha_3 - 2\alpha_3| \, |T_p(\mu, \rho, t) - T_p(\rho, \mu, t)| \\ &\leq |\alpha_1| \, |T_p(\mu, \rho, t) - T_p(\rho, \mu, t)|. \end{aligned} \tag{3.9}$$

Clearly, $|\alpha_1| < 1$. So, above inequality holds.
If

$$T_p(\mu, \rho, t) - T_p(\rho, \mu, t) = 0. \tag{3.10}$$

From (3.7), (3.9) and (3.10), we have

$$T_p(\mu, \rho, t) = 0 \text{ and } T_p(\rho, \mu, t) = 0$$

$$\Rightarrow \quad \mu = \rho$$

This completes the proof. T has unique fixed point. □

Example 3.3. *Consider (A, T_p) be a complete parametric metric space and $T : R^+ \to R^+$ be a mapping, since*

$$T_p(x^*, y^*, t) = t|x^* - y^*|,$$

such that

$$x^*{}_\delta = 1 + \frac{1}{\delta} \ \textit{and} \ y^*{}_\delta = 1 + \frac{2}{\delta}$$

Therefore,

$$T_p(x^*{}_\delta, y^*{}_\delta, t) = t|x^*{}_\delta - y^*{}_\delta|$$

$$= t\left|1+\frac{1}{\delta}-1-\frac{2}{\delta}\right|$$
$$= t\left|-\frac{1}{\delta}\right|$$
$$= t\frac{1}{\delta}$$

$$\lim_{\delta\to\infty} T_p(x^*_\delta, y^*_\delta, t) = \lim_{\delta\to\infty} t\frac{1}{\delta} = t.0 = 0$$

$$\lim_{\delta\to\infty} T_p(x^*_\delta, y^*_\delta, t) \to 0$$

As both, $x^*{}_\delta = 1+\frac{1}{\delta}$ *and* $y^*{}_\delta = 1+\frac{2}{\delta}$ *tend to* 1 *as* $\delta \to \infty$. *Hence 1 is the fixed point. Hence, it satisfy all the condition of complete parametric metric space for* $t > 0$.

For Theorem 3.1: $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{9}$, $\alpha_4 = \frac{17}{12}$, $\alpha_5 = \frac{1}{18}$ and
Theorem 3.2: $\alpha_1 = \frac{1}{8}$, $\alpha_2 = \frac{1}{16}$, $\alpha_3 = \frac{1}{32}$

Theorem 3.4. *Let* (A, T_p) *be a complete parametric metric space and* $t > 0$. *Let* $S, T : A \to A$ *is a mapping then it satisfies the condition:*

(1) $T(A) \subseteq S(A)$

(2) *S, T is continuous and*

(3) $T_p(Sx, Ty) \leq \alpha_1 T_p(x,y,t) + \alpha_2[T_p(x,Sx,t) + T_p(y,Ty,t)\left[\frac{T_p(x,y,t)+T_p(y,Ty,t)}{T_p(x,Ty,t)}\right]$

(3.11)
$$+\alpha_3[T_p(x,Ty,t)+T_p(y,Sx,t)]\left[\frac{\{T_p(x,y,t)+T_p(y,Ty,t)+T_p(x,Ty,t)\}^2}{T_p(x,Ty,t)^2}\right]$$

where $\alpha_1+\alpha_2+\alpha_3 \geq 0$ *with* $\alpha_1+2\alpha_2+12\alpha_3 < 1 \ \forall\ x,y \in A$ *and* $t > 0$. *Then prove that S, T has a common unique fixed point.*

Proof. Let $m_0 \in A$ be any arbitrary point. And the sequence $\{m_j\}_{j\in N}$, we have

$$m_1 = S(m_0), m_2 = T(m_1)\ldots m_{2j+1} = Sm_{2j}, m_{2j} = T(m_{2j-1}),$$

also

$$T_p(m_{2j+1}, m_{2j+2}, t) = T_p(Sm_{2j}, Tm_{2j+1}, t)$$

$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t) + \alpha_2 [T_p(m_{2j}, Sm_{2j}, t)$$

$$+ T_p(m_{2j+1}, Tm_{2j+1}, t)] \left[\frac{T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, Tm_{2j+1}, t)}{T_p(m_{2j}, Tm_{2j+1}, t)} \right]$$

$$+ \alpha_3 [T_p(m_{2j}, Tm_{2j+1}, t) + T_p(m_{2j+1}, Sm_{2j}, t)]$$

$$\cdot \left[\frac{\{T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, Tm_{2j+1}, t) + T_p(m_{2j}, Tm_{2j+1}, t)\}^2}{\{T_p(m_{2j}, Tm_{2j+1}, t)\}^2} \right]$$

$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t) + \alpha_2 [T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)]$$

$$\cdot \left[\frac{T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)}{T_p(m_{2j}, m_{2j+2}, t)} \right]$$

$$+ \alpha_3 [T_p(m_{2j}, m_{2j+2}, t) + T_p(m_{2j+1}, m_{2j+1}, t)]$$

$$\cdot \left[\frac{\{T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t) + T_p(m_{2j}, m_{2j+2}, t)\}^2}{\{T_p(m_{2j}, m_{2j+2}, t)\}^2} \right]$$

$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t) + \alpha_2 [T_p(m_{2j}, m_{2m+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)]$$

$$+ \alpha_3 [T_p(m_{2j}, m_{2j+2}, t) + T_p(m_{2j+1}, m_{2j+1}, t)][\{2\}^2]$$

$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t) + \alpha_2 [T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t)]$$

$$+ 4\alpha_3 [T_p(m_{2j}, m_{2j+1}, t) + T_p(m_{2j+1}, m_{2j+2}, t) + T_p(m_{2j}, m_{2j+1}, t)$$

$$+ T_p(m_{2j+1}, m_{2j+2}, t)]$$

$$\leq \alpha_1 T_p(m_{2j}, m_{2j+1}, t) + \alpha_2 T_p(m_{2j}, m_{2j+1}, t) + 12\alpha_3 T_p(m_{2j}, m_{2j+1}, t)$$

$$\leq \frac{\alpha_1 + \alpha_2 + 12\alpha_3}{1 - \alpha_2 - 4\alpha_3} T_p(m_{2j}, m_{2j+1}, t)$$

$$T_p(m_{2j+1}, m_{2j+2}, t) \leq k(m_{2j}, m_{2j+1}, t),$$

where $k = \dfrac{\alpha_1 + \alpha_2 + 12\alpha_3}{1 - \alpha_2 - 4\alpha_3}$; $0 < k < 1$.
Continue in this way, we have

$$T_p(m_{2j+1}, m_{2j+2}, t) \leq k^{2j} T_p(m_{2j}, m_{2j+1}, t); \ \ 0 < k < 1$$

$k^{2j} \to 0$ as $j \to \infty$. Using Lemma 2.5 sequence $\{m_j\}_{j \in N}$ is Cauchy sequence. Thus, $\exists\ \mu \in A$ s.t $\{m_j\}$ converges to μ. Further the subsequence $\{Sm_{2j}\} \to \mu$ and $\{Tm_{2j}\} \to \mu$. Since S, T :

$A \to A$ are continuous, we have

$$S\mu = \mu \text{ and } T\mu = \mu .$$

Then, μ is a fixed point of S and T.

$$\Rightarrow \quad S\mu = \mu = T\mu .$$

Now, for uniqueness, let μ and ρ be the two-fixed point of S and T, then by (3.11) we get

$$\begin{aligned}
T_p(\mu,\rho,t) &= T_p(S\mu,T\rho,t) \\
&\leq \alpha_1 T_p(\mu,\rho,t) + \alpha_2[T_p(\mu,S\mu,t) + T_p(\rho,\rho,t)\left[\frac{T_p(\mu,\rho,t)+T_p(\rho,T\rho,t)}{T_p(\mu,T\rho,t)}\right] \\
&+ \alpha_3[T_p(\mu,T\rho,t) + T_p(\rho,S\mu,t)\left[\frac{T_p(\mu,\rho,t)+T_p(\rho,T\rho,t)+T_p(\mu,T\rho,t)^2}{[T_p(\mu,T\rho,t)]^2}\right] \\
&\leq \alpha_1 T_p(\mu,\rho,t) + \alpha_2[T_p(\mu,\mu,t) + T_p(\rho,\rho,t)\left[\frac{T_p(\mu,\rho,t)+T_p(\rho,\rho,t)}{T_p(\mu,\rho,t)}\right] \\
&+ \alpha_3[T_p(\mu,\rho,t) + T_p(\rho,\mu,t)\left[\frac{T_p(\mu,\rho,t)+T_p(\rho,\rho,t)+T_p(\mu,\rho,t)^2}{[T_p(\mu,\rho,t)]^2}\right]
\end{aligned}$$

Hence $T_p(\mu,\mu,t) = 0$ and $T_p(\rho,\rho,t) = 0$

$$\begin{aligned}
&\leq \alpha_1 T_p(\mu,\rho,t) + \alpha_3[T_p(\mu,\rho,t) + T_p(\rho,\mu,t)\left[\frac{T_p(\mu,\rho,t)+T_p(\mu,\rho,t)^2}{[T_p(\mu,\rho,t)]^2}\right] \\
&\leq \alpha_1 T_p(\mu,\rho,t) + 4\alpha_3[T_p(\mu,\rho,t) + \alpha_3 T_p(\rho,\mu,t)] \\
&\leq (\alpha_1 + 4\alpha_3)T_p(\mu,\rho,t) + 4\alpha_3 T_p(\rho,\mu,t)
\end{aligned}$$

$$T_p(\mu,\rho,t) \leq (\alpha_1 + 4\alpha_3)T_p(\mu,\rho,t) + 4\alpha_3 T_p(\rho,\mu,t). \tag{3.12}$$

Similarly,

$$T_p(\rho,\mu,t) \leq (\alpha_1 + 4\alpha_3)T_p(\rho,\mu,t) + 4\alpha_3 T_p(\mu,\rho,t). \tag{3.13}$$

Subtract above two equations, we have

$$\begin{aligned}
|T_p(\mu,\rho,t) - T_p(\rho,\mu,t)| &\leq |\alpha_1 + 4\alpha_3 - 4\alpha_3|\,|T_p(\mu,\rho,t) - T_p(\rho,\mu,t)| \\
&\leq |\alpha_1|\,|T_p(\mu,\rho,t) - T_p(\rho,\mu,t) \qquad (3.14)
\end{aligned}$$

Clearly, $|\alpha_1| < 1$.

So, above inequality holds.

$$\Rightarrow \quad T_p(\mu,\rho,t) - T_p(\rho,\mu,t) = 0. \tag{3.15}$$

From (3.12), (3.13) and (3.15), we have

$$T_p(\mu,\rho,t) = 0 \text{ and } T_p(\rho,\mu,t) = 0$$

$$\Rightarrow \quad \mu = \rho.$$

This completes the proof. T and S have unique common fixed point. □

4. CONCLUSION

Here, some common fixed point theorems are proved on parametric metric space by using the various expansive contractions condition. We get the unique fixed point for single mapping in complete parametric metric space as well as common unique fixed point.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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