Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 1, 970-990
https://doi.org/10.28919/jmcs/5054
ISSN: 1927-5307

SOLUTION OF SINGULARLY PERTURBED DIFFERENTIAL DIFFERENCE EQUATIONS USING LIOUVILLE GREEN TRANSFORMATION

M. ADILAXMI[1], D. KUMARA SWAMY[2,*]

[1]Department of Mathematics, K.L. University, Hyderabad, India

[2]Department of Mathematics, Indian Institute of Information Technology, Sonepat, India

Abstract: This paper envisages the use of Liouville Green Transformation to find the solution of singularly perturbed differential difference equations. First, using Taylor series, the given singularly perturbed differential difference equation is approximated by an asymptotically equivalent singularly perturbation problem. Then the Liouville Green Transformation is applied to get the solution. Several model examples are solved by taking various values for the delay parameter and perturbation parameter and compared the results with the exact solution.

Keywords: differential difference equations; singular perturbations; boundary layer.

2010 AMS Subject Classification: 65L11, 65Q10.

1. INTRODUCTION

The class of differential-difference equations which have characteristics of both classes, *i.e.* delay/advance and singularly perturbed behavior is known as singularly perturbed differential-difference equations. The expression "positive shift" and "negative shift" are also

---

[*]Corresponding author
E-mail address: diddi.k@gmail.com
Received September 24, 2020

used for “advance” and “delay” respectively. In general, an ordinary differential equation in which the highest order derivative is multiplied by a small positive parameter and containing at least one delay/advance is known as singularly perturbed differential-difference equation. Singularly perturbed differential-difference equations arise in the modelling of various practical phenomena in bioscience, engineering, control theory, specifically in variational problems, in describing the human pupil-light reflex, in a variety of models for physiological processes or diseases and first exit time problems in the modelling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. Stein [13] gave a differential-difference equation model incorporating stochastic effects due to neuron excitation. Lange and Miura [9, 10] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations. Detailed theory and analytical discussion on singularly perturbed differential difference equations is given in the books and high level monographs: Ref: [1-15].

In this paper we describe the use of Liouville Green Transformation to find the solution of singularly perturbed differential difference equations. First, using Taylor series, the given singularly perturbed differential difference equation is approximated by an asymptotically equivalent singularly perturbation problem. Then the Liouville Green Transformation is applied to get the solution. Several model examples are solved.

2. DESCRIPTION OF THE METHOD

Consider singularly perturbed differential equation with small delay as well as advance parameters of the form:

$$\varepsilon y''(x) + a(x)y'(x) + \alpha(x)y(x-\delta) + \omega(x)y(x) + \beta(x)y(x+\eta) = 0 \qquad (1)$$

for $0 < x < 1$ and subject to the conditions

$$y(x) = \phi(x), \quad -\delta \le x \le 0 \qquad (2)$$

$$y(x) = \gamma(x), \quad 1 \le x \le 1+\eta \qquad (3)$$

where $a(x)$, $\alpha(x)$, $\beta(x)$, $\omega(x)$, $\phi(x)$, and $\gamma(x)$ are bounded and continuously differentiable functions on (0,1), $0<\varepsilon<<1$ is the singular perturbation parameter; and $0<\delta=o(\varepsilon)$, $0<\eta=o(\varepsilon)$ are the delay and the advance parameters respectively. We assume that $\alpha(x)+\beta(x)+\omega(x)\le 0$, $a(x)-\delta\alpha(x)+\eta\beta(x)\ge M>0$ throughout the interval [0, 1], where M is some positive constant. Under these assumptions, Eq. (1) has a unique solution $y(x)$ which in general, exhibits a boundary layer of on the left-end of the underlying interval.

By using Taylor series expansion in the neighborhood of the point *x*, we have

$$y(x-\delta)\approx y(x)-\delta y'(x)$$

$$y\left(x+\eta\right)\approx y(x)+\eta y'(x)$$

Using these in Eq. (1), we obtain an asymptotically equivalent singularly perturbed first order differential-difference equation of the form:

$$\varepsilon y''(x)+f(x)y'(x)+g(x)y(x)=0,x\in[0,1] \tag{4}$$

where
$$f(x)=\delta\alpha(x)-a(x)-\eta\beta(x),$$
$$g(x)=-\alpha(x)-\beta(x)-w(x)$$

Transition from Eq. (1) to Eq. (4) is admitted, because of the condition that $0<\varepsilon<<1$ is sufficiently small. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in Elsgolt's and Norkin [5]. Thus, the solution of Eq. (4) will provide a good approximation to the solution of Eq. (1).

3. LIOUVILLE GREEN TRANSFORMATION

Rewrite the equation (4) as below:

$$-\varepsilon y''(x)+f(x)y'(x)+g(x)y(x)=0\ ,\quad x\in[0,1] \tag{5}$$

The Liouville –Green transforms $z,\varphi(x),v(z)$

$$z = \varphi(x) = \frac{1}{\varepsilon}\int f(x)dx \tag{6}$$

$$\phi(x) = \varphi'(x) = \frac{1}{\varepsilon} f(x) \tag{7}$$

$$v(z) = \phi(x)y(x) \tag{8}$$

According (8), we have

$$\frac{dy}{dx} = \frac{1}{\phi(x)}\frac{dv}{dz}z'(x) - \frac{\phi'(x)}{\phi^2(x)}v(z) = \frac{\phi'(x)}{\phi(x)}\frac{dv}{dz} - \frac{\phi'(x)}{\phi^2(x)}v(z), \tag{9}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\phi(x)}\left(\left(\varphi^2(x)\frac{d^2v}{dz^2} + \left(\phi'' - \frac{2\varphi'(x)\phi'(x)}{\phi(x)}\right)\frac{dv}{dz}\right) - \left(\frac{\phi''(x)}{\phi(x)} - \frac{2\varphi'^2(x)}{\phi^2(x)}v\right)\right). \tag{10}$$

From (5), (9) and (10), we obtain

$$-\frac{\varepsilon\varphi'^2}{\phi}\frac{d^2v}{dz^2} + \left(\frac{2\varepsilon\varphi'\phi'}{\phi^2} - \frac{\varepsilon\varphi''(x)}{\phi(x)} + f(x)\frac{\varphi'(x)}{\phi(x)}\right)\frac{dv}{dz} + \left(\frac{\varepsilon\phi''(x)}{\phi^2(x)} - \frac{2\varepsilon\varphi'^2(x)}{\phi^3(x)} - f(x)\frac{\phi'(x)}{\phi^2} + \frac{g(x)}{\phi}\right)v(z) = 0,$$

i.e.

$$\frac{d^2v}{dz^2} + \frac{1}{\varphi'^2}\left(\varphi''(x) - \frac{2\varphi'\phi'}{\phi} - f(x)\frac{\varphi'(x)}{\varepsilon}\right)\frac{dv}{dz} - \frac{1}{\varphi'^2}\left(\frac{\phi''(x)}{\phi(x)} - \frac{2\phi'^2}{\phi^2} - f(x)\frac{\phi'(x)}{\varepsilon\phi(x)} + \frac{g(x)}{\varepsilon}\right)v(z) = 0,$$

From (6), we have

$$\frac{d^2v}{dz^2} - \left(\varepsilon\frac{f'(x)}{f^2(x)} + 1\right)\frac{dv}{dz} - \frac{1}{f^2(x)}\left(\varepsilon^2\frac{f''(x)}{f(x)} - 2\varepsilon^2\frac{f'^2(x)}{f^2(x)} - \varepsilon f'(x) + \varepsilon g(x)\right)v(z) = 0,$$

i.e.

$$\frac{d^2v}{dz^2} - \frac{dv}{dz} = \varepsilon\frac{f'(x)}{f^2(x)}\frac{dv}{dz} + \varepsilon\frac{1}{f^2(x)}\left(\varepsilon\frac{f''(x)}{f(x)} - 2\varepsilon\frac{f'^2(x)}{f^2(x)} - f'(x) + g(x)\right)v(z) = \varepsilon M(x)\frac{dv}{dz} + \varepsilon N(x)v(z), \tag{11}$$

where $M(x) = \frac{f'(x)}{f^2(x)}, N(x,\varepsilon) = \frac{1}{f^2(x)}\left(\varepsilon\frac{f''(x)}{f(x)} - 2\varepsilon\frac{f'^2(x)}{f^2(x)} - f'(x) + g(x)\right).$

Since Ɛ is a small parameter $(0 < \varepsilon << 1)$, $\varepsilon M(x)$ and $\varepsilon N(x,\varepsilon)$ are sufficiently small on [0, 1].

So, as $\varepsilon \to 0$,the right hand side of equation (11) vanishes.

Therefore, we have

$$\frac{d^2v}{dz^2} - \frac{dv}{dz} \approx 0. \tag{12}$$

Therefore, the approximate solutions $v(z)$ of (12) are

$$v(z) \approx c_1 + c_2 e^z \tag{13}$$

where c_1, c_2 are two arbitrary constants. From (6)-(8), one has the asymptotic solutions of differential equations

$$y(x) = \frac{v(z)}{\phi(x)} = \varepsilon \frac{v(z)}{f(x)} \approx \frac{\varepsilon}{f(x)} \left(c_1 + c_2 e^{\frac{1}{\varepsilon}\int_0^x f(s)ds} \right) \tag{14}$$

where c_1,c_2 are arbitrary constants to be determined using the given boundary conditions.

4. NUMERICAL EXAMPLES

To validate the applicability of the method, we have applied it to five singularly perturbed differential difference equations (three left-end and two right-end boundary layer problems) of the type given by equations (1)- (3), and the exact solution is:

$$\varepsilon y''(x) + a(x)y'(x) + \alpha(x)y(x-\delta) + \omega(x)y(x) + \beta(x)y(x+\eta) = f(x)$$

under the boundary conditions $\begin{array}{l} y(x) = \phi(x), \quad -\delta \le x \le 0, \\ \mathrm{y(x)} = \gamma(x), \quad 1 \le x \le 1+\eta, \end{array}$

with constant coefficients is given by

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \frac{f}{c} \tag{15}$$

where $c = \alpha + \beta + \omega$, $c_1 = \dfrac{-f + \gamma c + e^{m_2}(f - \phi c)}{c(e^{m_1} - e^{m_2})}$, $c_2 = \dfrac{f - \gamma c + e^{m_1}(-f + \phi c)}{c(e^{m_1} - e^{m_2})}$

$$m_1 = \frac{-(a - \alpha\delta + \beta\eta) + \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4c\varepsilon}}{2\varepsilon}$$

$$m_2 = \frac{-(a - \alpha\delta + \beta\eta) - \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4c\varepsilon}}{2\varepsilon}$$

Example 1: Consider the model boundary value problem of the type given by (1) - (3) having the boundary layer at the left end

$$\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x+\eta) = 0$$

with boundaray conditions are $\varphi(x) = 1, \quad \gamma(x) = 0.$

The exact solution of the problem is given by Eq. (15). The numerical results are given in Table 1 for $\varepsilon = 10^{-2}, h = 10^{-2}, \eta = 0.005$.

Example 2: Consider the model boundary value problem of the type given by (1)-(3) having the boundary layer at the left end

$$\varepsilon y''(x) + y'(x) + 2y(x-\delta) - 3y(x) = 0$$

with boundaray conditions are $\varphi(x) = 1, \quad \gamma(x) = 0.$

The exact solution of the problem is given by Eq. (15). The numerical results are given in Table 2 for $\varepsilon = 10^{-2}, h = 10^{-2}, \delta = 0.005$.

Example 3: Consider the model boundary value problem of the type given by (1) -(3) having the boundary layer at the left end

$$\varepsilon y''(x) + y'(x) - 2y(x-\delta) - 5y(x) + y(x+\eta) = 0$$

with boundaray conditions are $\varphi(x) = 1, \quad \gamma(x) = 0.$

The exact solution of the problem is given by Eq. (15). The numerical results are given in Tables 3, 4, 5 for different values of ε, δ and η.

Example 4 Consider the model boundary value problem of the type given by (1) -(3) having the boundary layer at the right end

$$\varepsilon y''(x) - y'(x) - 2y(x-\delta) + y(x) = 0$$

with boundaray conditions are $\varphi(x) = 1, \quad \gamma(x) = -1.$

The exact solution of the problem is given by Eq. (15). The numerical results are given in Table 6, 7 for $\varepsilon = 10^{-1}, h = 10^{-3}$ for different values of δ.

Example 5: Consider the model boundary value problem of the type given by (1) -(3) having the boundary layer at the right end

$$\varepsilon y''(x)\text{-}y'(x)\text{-}2y(x\text{-}\delta)+y(x)\text{-}2y(x+\eta)=0$$

with boundaray conditions are $\varphi(x)=1,\quad \gamma(x)=-1.$

The exact solution of the problem is given by Eq. (15). The numerical results are given in Tables 8, 9 for different values of ε, δ and η.

5. DISCUSSION AND CONCLUSION

In this paper we have described the use of Liouville Green Transformation to find the solution of singularly perturbed differential difference equations. First, using Taylor series, the given singularly perturbed differential difference equation is approximated by an asymptotically equivalent singularly perturbation problem. Then the Liouville Green Transformation is applied to get the solution. To validate the efficiency of the method, five examples (three with left-end and two with right-end boundary layers) have been solved for different values of δ, η and ε. Even though the numerical results are computed at all points of the mesh size h, only few values have been reported. The numerical solutions are presented in tables and compared with the exact solutions. It is observed that the proposed method approximates the exact solution very well (Tables 1-9). Further, to understand the effect of the delay and advance parameters on the solution of the problem, the numerical solutions have also been sketched in graphs. The present method does not depend on asymptotic expansion as well as on the matching of the coefficients. Therefore, the method is simple, easy and efficient technique for solving boundary value problems involving singularly perturbed differential-difference equations.

Table 1: Results in the solution of Example 1 with $\varepsilon = 10^{-2}, h = 10^{-2}, \eta = 0.005$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.01	0.36066497	0.36421897	0.00355400
0.02	0.13007922	0.13265546	0.00257623
0.03	0.04691502	0.04831563	0.00140061
0.04	0.01692060	0.01759747	0.00067686
0.05	0.00610266	0.00640933	0.00030666
0.06	0.00220101	0.00233440	0.00013338
0.07	0.00079383	0.00085023	5.6402e-05
0.08	0.00028630	0.00030967	2.3364e-05
0.09	0.00010326	0.00011278	9.5272e-06
0.10	3.7242e-05	4.1079e-05	3.8369e-06
0.90	1.3782e-40	3.3314e-40	1.9531e-40
0.91	4.9707e-41	1.2133e-40	7.1629e-41
0.92	1.7924e-41	4.4193e-41	2.6268e-41
0.93	6.4617e-42	1.6096e-41	9.6343e-42
0.94	2.3274e-42	5.8624e-42	3.5350e-42
0.95	8.3627e-43	2.1352e-42	1.2989e-42
0.96	2.9844e-43	7.7769e-43	4.7925e-43
0.97	1.0443e-43	2.8324e-43	1.7881e-43
0.98	3.4429e-44	1.0316e-43	6.8735e-44
0.99	9.1496e-45	3.7574e-44	2.8424e-44
1	0	0	0

Table 2: Results in the solution of Example 2 with

$$\varepsilon = 10^{-2}, h = 10^{-2}, \delta = 0.005$$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.01	0.36787944	0.37157669	0.00369724
0.02	0.13533528	0.13806923	0.00273395
0.03	0.04978706	0.05130331	0.00151624
0.04	0.01831563	0.01906311	0.00074747
0.05	0.00673794	0.00708340	0.00034546
0.06	0.00247875	0.00263202	0.00015327
0.07	0.00091188	0.00097800	6.6118e-05
0.08	0.00033546	0.00036340	2.7939e-05
0.09	0.00012340	0.00013503	1.1622e-05
0.10	4.5399e-05	5.0174e-05	4.7747e-06
0.901	8.1936e-40	2.0154e-39	1.1960e-39
0.91	3.0140e-40	7.4887e-40	4.4746e-40
0.92	1.1085e-40	2.7826e-40	1.6740e-40
0.93	4.0760e-41	1.0339e-40	6.2635e-41
0.94	1.4972e-41	3.8419e-41	2.3447e-41
0.95	5.4856e-42	1.4275e-41	8.7902e-42
0.96	1.9953e-42	5.3045e-42	3.3092e-42
0.97	7.1109e-43	1.9710e-42	1.2599e-42
0.98	2.3841e-43	7.3240e-43	4.9398e-43
0.99	6.4291e-44	2.7214e-43	2.0785e-43
1	0	0	0

Table 3: Results in the solution of Example 3 with

$$\varepsilon = 10^{-2}, h = 10^{-2}, \delta = 0.001 \text{ and } \eta = 0.001$$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.01	0.34658614	0.36677745	0.02019130
0.02	0.12012195	0.13452570	0.01440374
0.03	0.04163260	0.04934099	0.00770838
0.04	0.01442928	0.01809716	0.00366787
0.05	0.00500099	0.00663763	0.00163664
0.06	0.00173327	0.00243453	0.00070125
0.07	0.00060072	0.00089293	0.00029220
0.08	0.00020820	0.00032750	0.00011930
0.09	7.2160e-05	0.00012012	4.7961e-05
0.10	2.5009e-05	4.4058e-05	1.9048e-05
0.90	3.8282e-42	6.2551e-40	6.2168e-40
0.91	1.3267e-42	2.2942e-40	2.2809e-40
0.92	4.5980e-43	8.4147e-41	8.3687e-41
0.93	1.5931e-43	3.0863e-41	3.0704e-41
0.94	5.5171e-44	1.1320e-41	1.1264e-41
0.95	1.9073e-44	4.1519e-42	4.1328e-42
0.96	6.5591e-45	1.5228e-42	1.5162e-42
0.97	2.2189e-45	5.5854e-43	5.5632e-43
0.98	7.1157e-46	2.0486e-43	2.0414e-43
0.99	1.8577e-46	7.5138e-44	7.4952e-44
1	0	0	0

Table 4: Results in the solution of Example 3 with

$$\varepsilon = 10^{-2}, h = 10^{-2}, \delta = 0.001 \text{ and } \eta = 0.005$$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.01	0.34527238	0.36531327	0.02004089
0.02	0.11921302	0.13345379	0.01424076
0.03	0.04116096	0.04875244	0.00759147
0.04	0.01421174	0.01780991	0.00359816
0.05	0.00490692	0.00650619	0.00159927
0.06	0.00169422	0.00237680	0.00068257
0.07	0.00058496	0.00086827	0.00028330
0.08	0.00020197	0.00031719	0.00011521
0.09	6.9735e-05	0.00011587	4.6138e-05
0.10	2.4077e-05	4.2330e-05	1.8252e-05
0.901	2.7199e-42	4.3640e-40	4.3368e-40
0.91	9.3909e-43	1.5942e-40	1.5848e-40
0.92	3.2421e-43	5.8240e-41	5.7915e-41
0.93	1.1191e-43	2.1275e-41	2.1163e-41
0.94	3.8609e-44	7.7723e-42	7.7337e-42
0.95	1.3297e-44	2.8393e-42	2.8260e-42
0.96	4.5560e-45	1.0372e-42	1.0326e-42
0.97	1.5358e-45	3.7892e-43	3.7738e-43
0.98	4.9086e-46	1.3842e-43	1.3793e-43
0.99	1.2778e-46	5.0568e-44	5.0440e-44
1	0	0	0

Table 5: Results in the solution of Example 3 with

$$\varepsilon = 10^{-2}, h = 10^{-2}, \delta = 0.005 \text{ and } \eta = 0.001$$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.01	0.34396313	0.36385494	0.01989180
0.02	0.11831063	0.13239041	0.01407978
0.03	0.04069449	0.04817090	0.00747641
0.04	0.01399740	0.01752722	0.00352981
0.05	0.00481459	0.00637736	0.00156277
0.06	0.00165604	0.00232043	0.00066439
0.07	0.00056961	0.00084430	0.00027468
0.08	0.00019592	0.00030720	0.00011127
0.09	6.7391e-05	0.00011177	4.4385e-05
0.10	2.3180e-05	4.0670e-05	1.7490e-05
0.90	1.9322e-42	3.0447e-40	3.0253e-40
0.91	6.6460e-43	1.1078e-40	1.1011e-40
0.92	2.2858e-43	4.0308e-41	4.0080e-41
0.93	7.8603e-44	1.4666e-41	1.4588e-41
0.94	2.7014e-44	5.3365e-42	5.3095e-42
0.95	9.2693e-45	1.9417e-42	1.9324e-42
0.96	3.1641e-45	7.0650e-43	7.0334e-43
0.97	1.0628e-45	2.5706e-43	2.5600e-43
0.98	3.3855e-46	9.3534e-44	9.3195e-44
0.99	8.7877e-47	3.4032e-44	3.3945e-44
1	0	0	0

Table 6: Results in the solution of Example 4 with

$$\varepsilon = 10^{-1}, h = 10^{-3}, \delta = 0.05$$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.001	0.99915593	0.99999963	0.00084369
0.002	0.99831258	0.99999925	0.00168667
0.003	0.99746994	0.99999887	0.00252893
0.004	0.99662800	0.99999849	0.00337048
0.005	0.99578678	0.99999811	0.00421132
0.006	0.99494626	0.99999772	0.00505145
0.007	0.99410646	0.99999732	0.00589086
0.008	0.99326735	0.99999692	0.00672956
0.009	0.99242896	0.99999652	0.00756755
0.010	0.99159127	0.99999611	0.00840483
0.990	-0.83664276	-0.79166477	0.04497797
0.991	-0.85214214	-0.81148226	0.04065988
0.992	-0.86782153	-0.83151893	0.03630259
0.993	-0.88368307	-0.85177723	0.03190583
0.994	-0.89972893	-0.87225959	0.02746933
0.995	-0.91596131	-0.89296850	0.02299281
0.996	-0.93238245	-0.91390647	0.01847598
0.997	-0.94899460	-0.93507603	0.01391856
0.998	-0.96580003	-0.95647974	0.009320289
0.999	-0.98280105	-0.97812019	0.004680861
1	-1	-1	0

Table 7: Results in the solution of Example 4 with

$$\varepsilon = 10^{-1}, h = 10^{-3}, \delta = 0.005$$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.001	0.99909173	0.99999916	0.00090742
0.002	0.99818429	0.99999832	0.00181402
0.003	0.99727767	0.99999747	0.00271979
0.004	0.99637187	0.99999661	0.00362473
0.005	0.99546689	0.99999574	0.00452885
0.006	0.99456272	0.99999486	0.00543213
0.007	0.99365938	0.99999398	0.00633459
0.008	0.99275685	0.99999308	0.00723623
0.009	0.99185514	0.99999218	0.00813703
0.010	0.99095424	0.99999126	0.00903702
0.990	-0.85005191	-0.80785817	0.04219374
0.991	-0.86433434	-0.82621089	0.03812345
0.992	-0.87877044	-0.84474991	0.03402052
0.993	-0.89336192	-0.86347713	0.02988478
0.994	-0.90811049	-0.88239445	0.02571603
0.995	-0.92301791	-0.90150381	0.02151410
0.996	-0.93808593	-0.92080714	0.01727878
0.997	-0.95331634	-0.94030643	0.01300990
0.998	-0.96871093	-0.96000366	0.00870727
0.999	-0.98427154	-0.97990084	0.00437070
1	-1	-1	0

Table 8: Results in the solution of Example 5 with

$\varepsilon = 10^{-2}, h = 10^{-3}, \delta = 0.001$ and $\eta = 0.005$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.001	0.99759959	0.999999150	0.00239955
0.002	0.99520495	0.999998292	0.00479333
0.003	0.99281606	0.999997426	0.00718136
0.004	0.99043290	0.999996551	0.00956364
0.005	0.98805546	0.999995667	0.01194020
0.006	0.98568372	0.999994773	0.01431104
0.007	0.98331768	0.999993871	0.01667618
0.008	0.98095732	0.999992960	0.01903563
0.009	0.97860262	0.999992040	0.02138941
0.010	0.97625357	0.999991110	0.02373753
0.990	-0.86983351	-0.80821963	0.06161387
0.991	-0.88214574	-0.82653951	0.05560622
0.992	-0.89460930	-0.84504498	0.04956431
0.993	-0.90722611	-0.86373794	0.04348817
0.994	-0.91999809	-0.88262027	0.03737782
0.995	-0.93292720	-0.90169389	0.03123331
0.996	-0.94601543	-0.92096075	0.02505467
0.997	-0.95926478	-0.94042281	0.01884197
0.998	-0.97267728	-0.96008203	0.01259525
0.999	-0.98625499	-0.97994042	0.00631457
1	-1	-1	0

Table 9: Results in the solution of Example 6 with

$\varepsilon = 10^{-2}, h = 10^{-3}, \delta = 0.005$ and $\eta = 0.001$

x	Exact solution	Asymptotic solution	Error
0	1	1	0
0.001	0.99757358	0.99999901	0.00242543
0.002	0.99515306	0.99999802	0.00484496
0.003	0.99273841	0.99999702	0.00725861
0.004	0.99032961	0.99999601	0.00966640
0.005	0.98792666	0.99999499	0.01206833
0.006	0.98552954	0.99999396	0.01446442
0.007	0.98313823	0.99999292	0.01685468
0.008	0.98075273	0.99999187	0.01923914
0.009	0.97837301	0.99999081	0.02161779
0.010	0.97599906	0.99998974	0.02399067
0.990	-0.87142677	-0.81111386	0.06031290
0.991	-0.88359565	-0.82917050	0.05442514
0.992	-0.89591248	-0.84740715	0.04850532
0.993	-0.90837911	-0.86582561	0.04255350
0.994	-0.92099742	-0.88442769	0.03656972
0.995	-0.93376929	-0.90321522	0.03055406
0.996	-0.94669664	-0.92219005	0.02450658
0.997	-0.95978141	-0.94135405	0.01842735
0.998	-0.97302556	-0.96070910	0.01231646
0.999	-0.98643108	-0.98025711	0.00617397
1	-1	-1	0

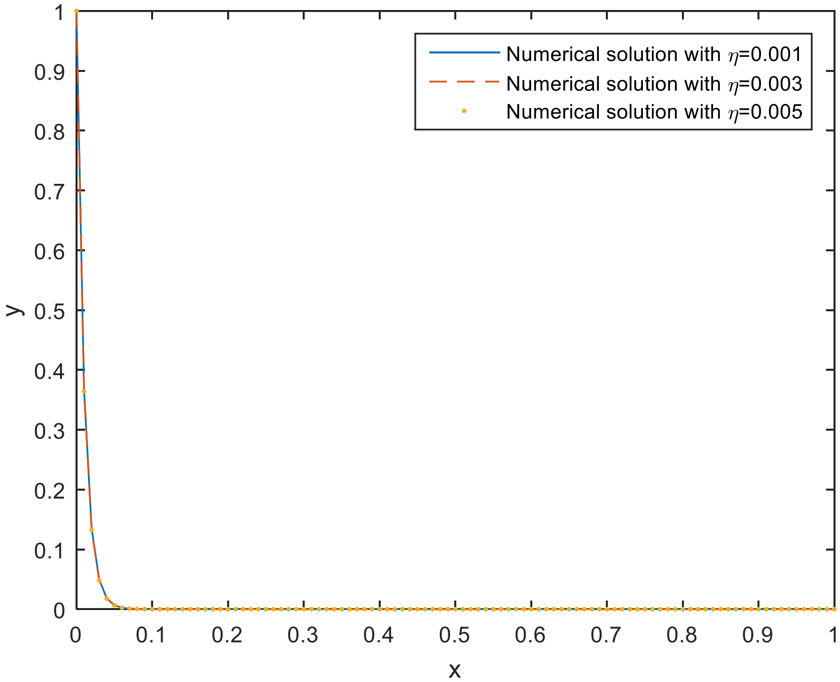

Fig1. Example 1 with $\varepsilon = 10^{-2}$ for different values of η

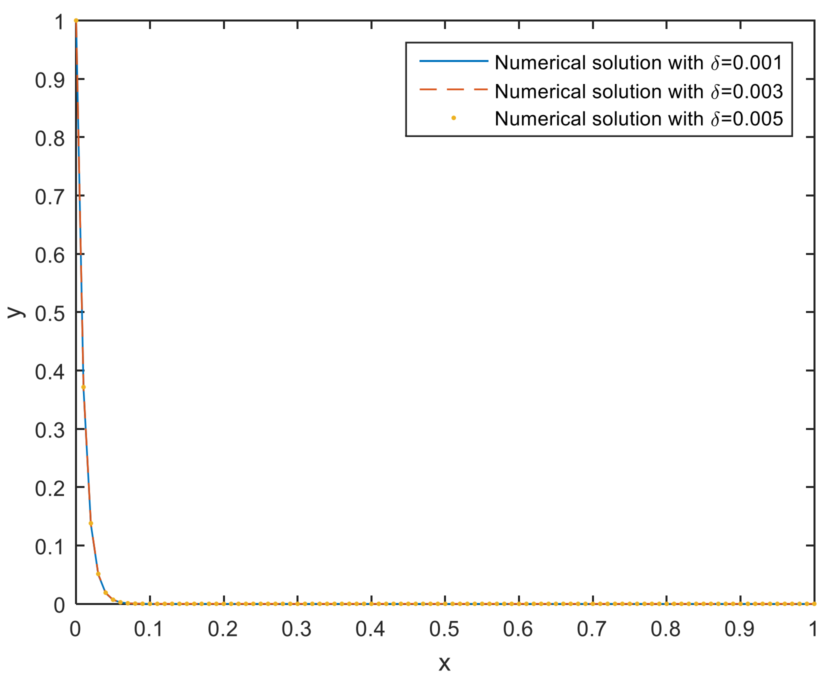

Fig2. Example 2 with $\varepsilon = 10^{-2}$ for different values of δ

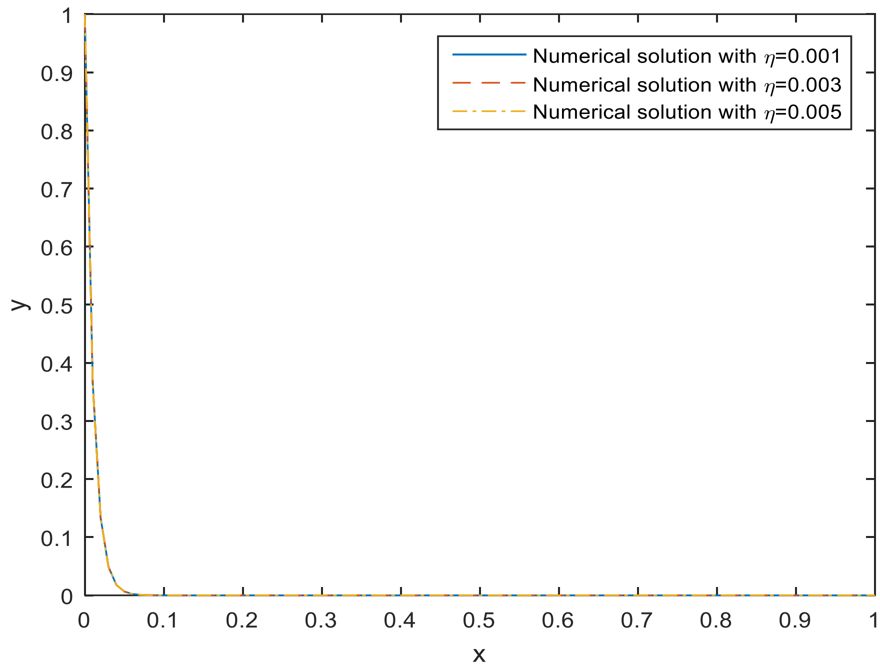

Fig3. Example 3 with $\varepsilon = 10^{-2}, \delta = 0.003$ for different values of η

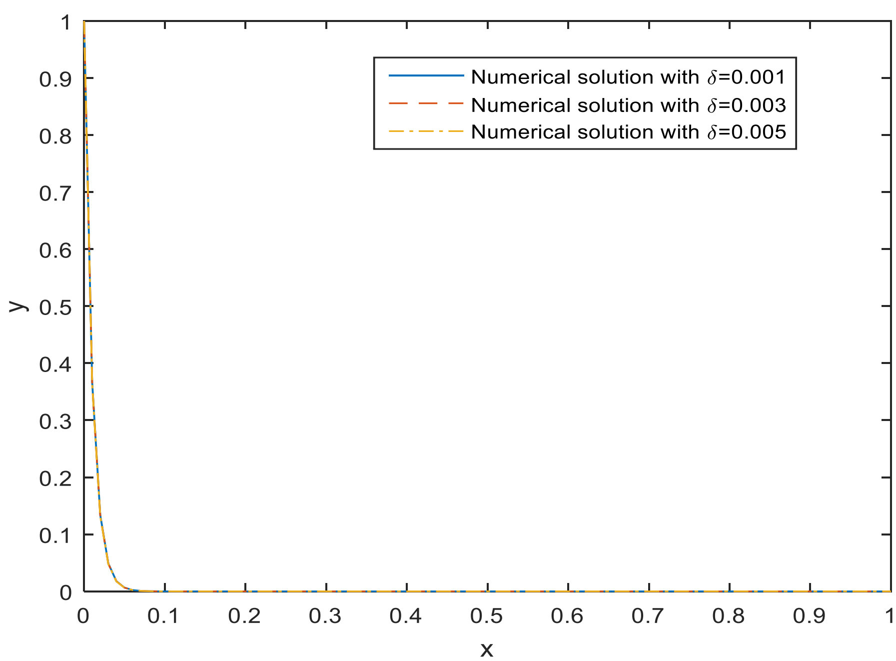

Fig4. Example 3 with $\varepsilon = 10^{-2}, \eta = 0.003$ for different values of δ

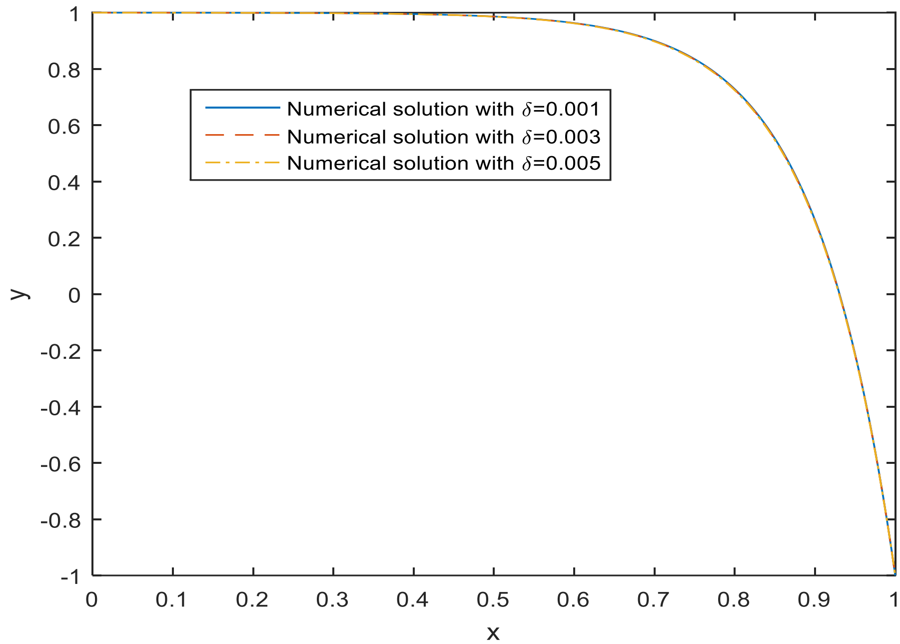

Fig5. Example 4 with $\varepsilon = 10^{-2}$ for different values of δ

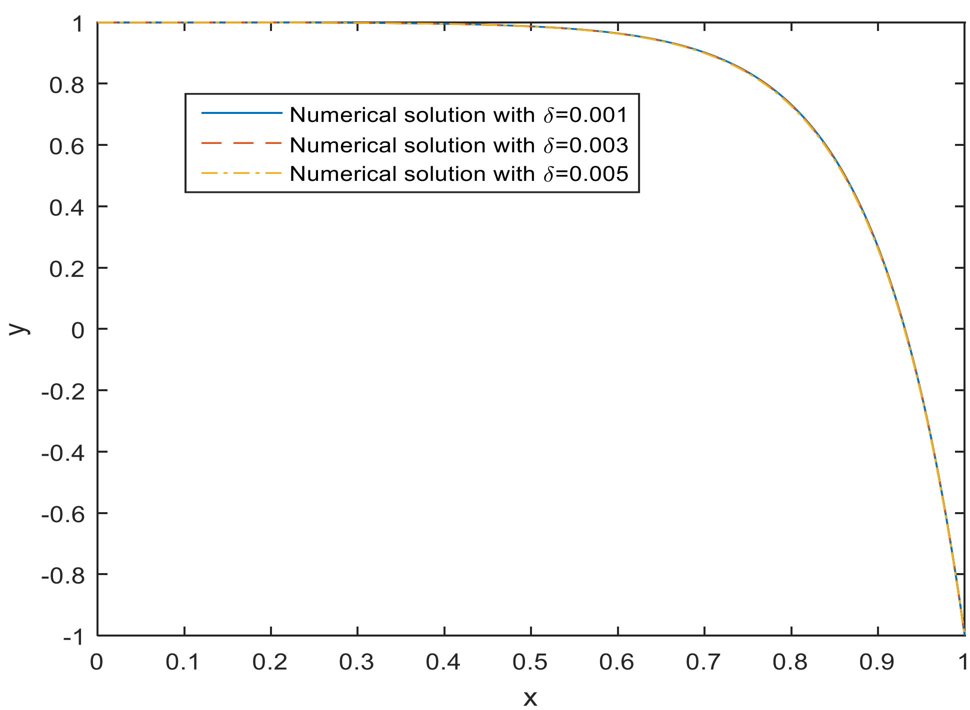

Fig6. Example 4 with $\varepsilon = 10^{-2}, \eta = 0.003$ for different values of δ

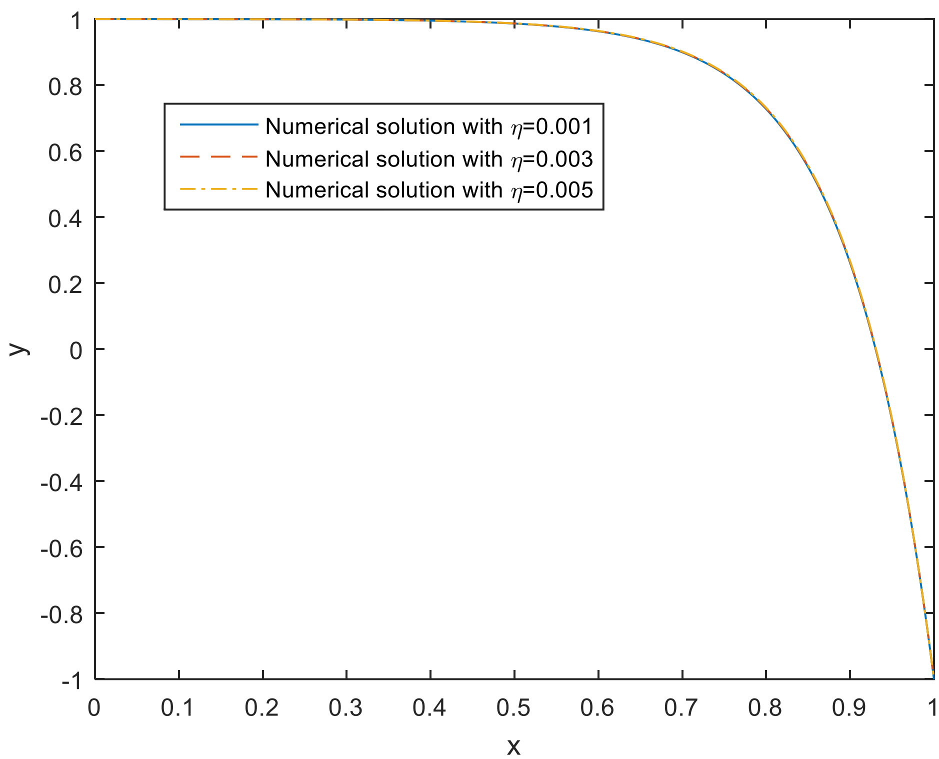

Fig7. Example 5 with $\varepsilon = 10^{-2}, \delta = 0.003$ for different values of η

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] R. Bellman, Perturbation Techniques in Mathematics, Physics and Engineering, Holt, Rinehart, Winston, New York, 1964.

[2] R.E. Bellman, K.L. Cooke, Differential-Difference Equations. Academy Press, New York, 1963.

[3] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.

[4] R.D. Driver, Ordinary and Delay Differential Equations. Springer-Verlag, New York, 1977.

[5] L.E. El'sgol'ts, S.B. Norkin, Introduction to the theory and applications of differential equations with deviating arguments. Mathematics in Science and Eng., 105, Academic Press, New York, 1973.

[6] J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[7] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary value problems for differential difference equations, V. Small shifts with layer behaviour, SIAM J. Appl. Math. 4 (1994), 249-272.

[8] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary-value problems for differential-difference equations. vi. Small shifts with rapid oscillations, SIAM J. Appl. Math. 54 (1994), 273–283.

[9] P.V. Kokotovic, H.K. Khalil, J. O'Reilly, Singular Perturbation Methods in Control: Analysis and Design, Academic Press, New York, 1986.

[10] J.J.H.Miller, E. O'Riordan, G.L Shishkin, Fitted Numerical methods for singular perturbation problems, World Scientific, River Edge, NJ, 1996.

[11] A.H. Nayfeh, Perturbation Methods, Wiley, New York, 1979.

[12] R.E. O'Malley, Introduction to Singular Perturbations, Academic Press, New York, 1974.

[13] R.B. Stein, A Theoretical Analysis of Neuronal Variability, Biophys. J. 5 (1965), 173–194.

[14] D. Kumara Swamy, K. Phaneendra, Y.N. Reddy, Accurate Numerical method for SP Differential-Difference Equations with mixed shifts, Khayyam J. Math. 4 (2018), 110-122.

[15] D. Kumara Swamy, K. Phaneendra, A. Benerji Babu, Y.N. Reddy, Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behaviour, Ain Shams Eng. J. 6 (2015), 391–398.