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PEBBLING ON SOME BRAID GRAPHS

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Abstract. Given a distribution of pebbles on the vertices of a connected graph, a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles at an adjacent vertex. The pebbling number, $f(G)$ of a connected graph G, is the smallest positive integer such that from every placement of $f(G)$ pebbles, we can move a pebble to any specified vertex by a sequence of pebbling moves. In this paper, we find the pebbling number for some braid graphs.

Keywords: pebbling number; braid graphs.

2010 AMS Subject Classification: 57M15.

1. Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [7].

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Consider a connected graph with fixed number of pebbles distributed on its vertices. A *pebbling move* consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The *pebbling number of a vertex v* in a graph G is the smallest number $f(G,v)$ such that for every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the *pebbling number* of G is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G,v)$ over all vertices v.

The pebbling number is known for many simple graphs including paths, cycles, and trees, [2], [3], [4], [6], [8], [9] but it is not known for most graphs and is hard to compute for any given graph that does not fall into one of these classes. Therefore, it is an interesting question if there is information we can gain about the pebbling number of more complex graphs from the knowledge of the pebbling number of some graphs for which we know.

In this paper, we find the pebbling number for some braid grpahs.

2. Preliminaries

We now introduce some definitions and notations which will be useful for the subsequent sections. For graph theoretic terminologies we refer to [5].

Definition 2.1. *The Braid graph $B(n)$ is obtained from a pair of paths $P_n^{'}$ and $P_n^{''}$ by joining i^{th} vertex of path $P_n^{'}$ with $(i+1)^{th}$ vertex of the path $P_n^{''}$ and the i^{th} vertex of the path $P_n^{''}$ with $(i+2)^{th}$ vertex of the path $P_n^{'}$ for all $1 \leq i \leq n-2$.*

Let the vertices of the path $P_n^{'}$ be $u_1, u_2, ..., u_n$ and the vertices of the path $P_n^{''}$ be $v_1, v_2, ..., v_n$.

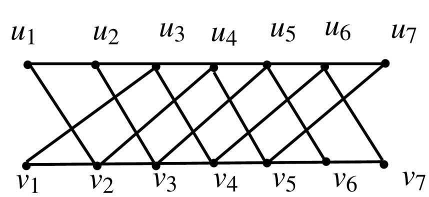

Figure 2.1. $B(7)$

Theorem 2.2. [2] *Let P_n be a path on n vertices. Then $f(P_n) = 2^{n-1}$.*

Theorem 2.3. [3] *Let $K_{1,n}$ be a star graph, where $n > 1$. Then $f(K_{1,n}) = n+2$.*

3. Main Results

Remark 3.1. A distribution of pebbles on the vertices of the graph G is a function $p : V(G) \to N \cup \{0\}$. Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the vertices of the set $A \subseteq V(G)$. Let v be a target vertex in the graph G. If $p(v) = 1$ or $p(u) \geq 2$, where $uv \in E(G)$, then we can move a pebble to v easily. So we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

Theorem 3.2. *For the Braid graph $B(3)$, $f(B(3)) = 6$.*

Proof. Placing 3 pebbles on the vertex v_3 and placing each pebble on both vertices v_1 and u_3, we cannot reach the vertex u_1. Thus $f(B(3)) \geq 6$.

Let D be any distribution of 6 pebbles on the vertices of the graph $B(3)$.

Case 1: Let u_1 be the target vertex. Then clearly, $p(u_1) = 0$ and $p(u_2) \leq 1$, $p(v_2) \leq 1$.

Subcase 1.1: Assume $p(u_2) = 1$ and $p(v_2) = 1$.

Then there will 4 pebbles distributed on the vertices u_3, v_3 and v_1. Thus we are done as there will be at least two pebbles in any one of the vertices v_1, u_3 or v_3. If u_3 or v_3 contains two pebbles then move a pebble to u_2 and hence we are done. Otherwise v_1 contains two pebbles. Moving a pebble from v_1 to v_2 we can reach the target.

Subcase 1.2: Assume $p(u_2) = 1$ and $p(v_2) = 0$.

Then there will be 5 pebbles distributed on the vertices v_1, v_3 and u_3. Thus any one of these vertices will have at least two pebbles. If u_3 or v_3 have two pebbles then moving a pebble to u_2, we are done. Otherwise assume $p(u_3) \leq 1$ or $p(v_3) \leq 1$. Then atleast three pebbles will be placed on v_1. If u_3 is occupied then we reach the target by using the path $P : v_1, u_3, u_2, u_1$. Otherwise v_1 itself contains four pebbles and thus we can reach the target.

Subcase 1.3: Assume $p(u_2) = 0$ and $p(v_2) = 1$

Then there will be five pebbles distribtued on the vertices of v_1, u_3 and v_3. Thus any one of these vertices may have at least two pebbles. If v_1 or v_3 contains at least two pebbles then we reach the target by moving a pebble to v_2 and then to u_1. Otherwise assume $p(v_1) \leq 1$ and $p(v_3) \leq 1$. Then there are at least three remaining pebbles will be in u_3. If v_1 is occupied then we reach the target by moving a pebble to u_3 and we hence we are done. Otherwise u_3 contains four pebbles they by using the path $P: u_3, u_2, u_1$ can be reached.

Subcase 1.4: Assume $p(u_2) = 0$ and $p(v_2) = 0$

Then the six pebbles will be placed on the vertices v_1, u_3, v_3. If $p(u_3) \geq 4$ or $p(v_3) \geq 4$ or $p(v_1) \geq 4$ then we are done. Therefore assume that $p(u_3) \leq 3, p(v_3) \leq 3, p(v_1) \leq 3$. If $p(u_3) \geq 2$ and $p(v_3) \geq 2$ then we can reach the target by moving a pebble from u_3 and v_3 to u_2 and then to u_1. Hence assume that $p(u_3) \leq 1$ and $p(v_3) \leq 1$. Then there will be remaining atleast four pebbles on v_1. Thus we are done by using the path $P: v_1, v_2, u_1$. If v_3 is the target vertex, by symmetry, we are done.

Case 2: Let v_1 be the target vertex. Clearly, $p(v_1) = 0$, $p(v_2) \leq 1$ and $p(u_3) \leq 1$.

Subcase 2.1: Assume that $p(u_3) = 1$ and $p(v_2) = 1$.

Then there will be at least four pebbles placed on the vertices of u_1, u_2 and v_3. Thus one of those vertex contains at least two pebbles. On moving a pebble from eihter u_1 or v_3 which contains two pebbles to v_2, we are done. Otherwise moving a pebble from u_2 to u_3 and hence to v_1, we reach our target.

Subcase 2.2: Assume $p(u_3) = 1$ and $p(v_2) = 0$.

Then among the remaining five pebbles placed on the vertices u_1, u_2 and v_3, any one vertex contains at least two pebbles. If u_2 contains at least two pebbles then moving a pebble to u_3 and hence to v_1, we are done. Thus assume that $p(u_2) \leq 1$. If u_2 is occupied then the remaining four pebbles are placed on u_1 and v_3. If $p(u_1) \leq 1$ then we can reach the target by moving a pebble from the path $P: v_3, u_2, u_3, v_1$. If $2 \leq p(u_1) \leq 3$, then we are done by using the path $P: u_1, u_2, u_3, v_1$. Otherwise we can reach the target by moving pebbles from u_1 to v_2 and then to v_1. If u_2 is unoccupied then we can easily move two pebbles from either u_1 or v_3 and hence we are done.

Subcase 2.3: Assume $p(u_3) = 0$ and $p(v_2) = 1$.

Then among the five pebbles placed on u_1, u_2 and v_3 at least one vertex contains at least two pebbles. If u_1 or v_3 contains at least two pebbles then moving a pebble to v_2 and then to v_1, we are done. Thus assume $p(u_1) \leq 1$ and $p(v_3) \leq 1$. Then u_2 contains at least three pebbles. If u_1

is occupied we can reach the target using the path $P : u_2, u_1, v_2, v_1$. Suppose u_1 is unoccupied then u_2 contains exactly four pebbles and thus we are done using the path $P : u_2, u_3, v_1$.

Subcase 2.4: Assume $p(v_2) = 0$ and $p(u_3) = 0$.

Then all the six pebbles will be placed on the vertices u_1, u_2 and v_3. If any of these vertices contains at least four pebbles then we are done as the distance from these vertices to the target is two. Thus assume $p(u_1) \leq 3$, $p(u_2) \leq 3$ and $p(v_3) \leq 3$. If $p(u_1) \geq 2$ and $p(v_3) \geq 2$ then moving a pebble to v_2 from both u_1 and v_3 we are done. Thus assume $p(u_1) \leq 1$ and $p(v_3) \leq 1$. Thus u_2 contains at least four pebbles. By using the path $P : u_2, u_3, v_1$ we can reach the target. If u_3 is the target vertex, by symmetry, we are done.

Case 3: Let u_2 be the target vertex. Clearly, $p(u_2) = 0, p(u_1) \leq 1, p(u_3) \leq 1$ and $p(v_3) \leq 1$.

Then there are at least three remaining pebbles are distributed on the verices v_1 and v_2. Suppose v_2 contains at least two pebbles and if u_1 or v_3 is occupied then we are done by moving a pebble from v_2 to u_1 or v_3. Otherwise assume that $p(u_1) = 0 = p(v_3)$. Suppose $p(v_2) \geq 4$ or $p(v_1) \geq 4$ we are done. Thus assume $p(v_1) \leq 3$ and $p(v_2) \leq 3$. Now using the pebbles in the spanning path $P : v_2, v_1, u_3, u_2$ we can reach the target vertex. By symmetry we can reach the vertex v_2. □

Theorem 3.3. *For the Braid graph $B(4)$, $f(B(4)) = 10$.*

Proof. Placing 7 pebbles on the vertex v_4 and each pebble on the vertices u_4 and v_1, we cannot reach the vertex u_1. Thus $f(B(4)) \geq 10$.

Now we prove the sufficient part. Let D be any distribution of 10 pebbles on the vertices of the graph $B(4)$.

Case 1: Let u_1 be the target vertex.

Clearly, $p(u_1) = 0$, $p(v_2) \leq 1$ and $p(u_2) \leq 1$.If $p(u_4) \geq 4$ or $p(v_4) \geq 8$, we can reach the target as $d(u_1, u_4) = 2$ and $d(u_1, v_4) = 3$. Thus assume that $p(u_4) \leq 3$ and $p(v_4) \leq 7$. Let $G_1 = G - < \{u_4, v_4\} >$. If G_1 contains at least six pebbles then we can reach the target since G_1 is isomorphic to $B(3)$ and $f(B(3)) = 6$. Thus we assume that $p(G_1) \leq 5$.

Subcase 1.1: $p(G_1) = 5$.

Then $p(v_4) \geq 2$. Thus we can move a pebble from v_4 to G_1 and hence $p(G_1) = 6$. Since G_1 is isomorphic to $B(3)$ and $f(B(3)) = 6$, we are done.

Subcase 1.2: $p(G_1) = 4$

Then $3 \leq p(v_4) \leq 6$. If $p(v_4) \geq 4$, we can move at least two pebbles from v_4 to G_1 and thus we are done as $p(G_1) = 6$. If $p(v_4) = 3$ then one pebble can be moved from v_4 and the another from u_4. Thus $p(G_1) = 6$ and hence we can reach the target.

Subcase 1.3: $p(G_1) = 3$

Then $4 \leq p(v_4) \leq 7$. If $p(v_4) \geq 6$, then three pebbles can be moved to G_1 and hence we are reached. If $4 \leq p(v_4) \leq 5$, then two pebbles can be moved from v_4 and another pebble can be moved from u_4 to G_1. Thus $p(G_1) = 6$ and hence we are done.

Subcase 1.4: $p(G_1) = 2$

Then $5 \leq p(v_4) \leq 8$. If $2 \leq p(u_4) \leq 3$ then moving a pebble from u_4 and another from v_4 to V_2 we can reach the target. If $p(u_4) = 1$ and v_2 is occupied then we can reach the target by moving the second pebble from v_4. Suppose $p(v_1) \leq 1$ or v_2 is unoccupied then any other vertices u_2, u_3 or v_3 will be occupied and hence we can reach the target by using the path through that vertex.

Subcase 1.5: $p(G_1) \leq 1$

If $p(u_4) \leq 1$ then $p(v_4) \geq 8$ and hence we are done. Thus assume $2 \leq p(u_4) \leq 3$. Moving a pebble from u_4 and another pebble from v_4 to v_2, we can reach the target. If v_4 is the target vertex, by symmetry we are done.

Case 2: Let v_1 be the target vertex.

Clearly, $p(v_1) = 0$, $p(v_2) \leq 1$ and $p(u_3) \leq 1$. First let us assume that $p(v_2) = 1$. If $p(u_1) \geq 2$ then we can move a pebble and hence we reach the target. Thus asuume that $p(u_1) \leq 1$. Let $G_1 = G - < \{u_1, v_1\} >$. Thus 9 pebbles will be placed on the vertices of G_1 and since G_1 is isomorphic to $B(3)$ and $f(B(3)) = 6$, using 6 pebbles we can move another pebble to v_2 and hence we are done. Thus assume $p(v_2) = 0$. If $p(u_1) \geq 4$, we are done. If $2 \leq p(u_1) \leq 3$, then we can move a pebble from u_1 to v_2 and since G_1 contains at least 7 pebbles, we can move another pebble to v_2 and hence we can reach the target. Thus assume $p(u_1) \leq 1$. If $p(v_3) \geq 4$, we can reach the target. Hence assume $p(v_3) \leq 3$. Thus the remaining six pebbles will be distributed on the vertices u_2, u_3, u_4 and v_4. We can see that $< \{u_2, u_3, u_4, v_1, v_4\} >$ is a spanning subgraph of $K_{1,4}$ and since $f(K_{1,4}) = 6$ we can pebble the target. By symmetry if u_4 is the target we are done.

Case 3: Let u_2 be the target vertex.

Assume $p(u_2) = 0, p(u_1) \leq 1, p(u_3) \leq 1$ and $p(v_3) \leq 1$. If $p(v_1) \geq 4$, or $p(u_1) \geq 2$then we are done. Therefore assume that $p(v_1) \leq 3$ and $p(u_1) \leq 1$. Then there are at least 6 pebbles are distributed on the vertices of the graph $G_1 = G - \{u_1, v_1\}$. Since $f(G_1)$ is isomorphic to $B(3)$ and $f(B(3) = 6$ we are done. By symmetry if v_3 is the target, we are done.

Case 4: Let v_2 be the target vertex.

Assume $p(v_2) = 0$, $p(u_1) \leq 1$, $p(u_4) \leq 1$, $p(v_1) \leq 1$ and $p(v_3) \leq 1$. Since $p(G_1) = p(G- < \{u_1, v_1\} >)$ have at least eight pebbles and $p(G_1) \geq f(B(3))$ we can reach our target. By symmetry if u_3 is the target, we are done. □

We now consider the braid graphs obtained by the paths of length $3m+1$.

Theorem 3.4. *For the Braid graph $B(3m+1)$, $f(B(3m+1)) = 2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} + 2$.*

Proof. Placing $2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} - 1$ pebbles on the vertex v_{3m+1} and each pebble on the vertices v_1 and u_{3m+1}, we cannot reach the vertex u_1. Thus $f(B(3m+1)) \geq 2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} + 2$.

Let D be any distribution of $2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} + 2$ pebbles on the vertices of the graph $G = B(3m+1)$. We now prove the sufficient part by induction on m.

Let $G_1 = G- < \{u_{3m-1}, u_{3m}, u_{3m+1}, v_{3m-1}, v_{3m}, v_{3m+1}\} >$ and $p_1 = p(G_1)$ and let $G_2 =< \{u_{3m-1}, u_{3m}, u_{3m+1}, v_{3m-1}, v_{3m}, v_{3m+1}\} >$ and $p_2 = p(G_2)$.

Case 1: Let u_1 be the target vertex.

Suppose $p_1 = 0$ then $2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} + 2$ pebbles will be distributed on the vertices of the graph G_2. Since G_2 is isomorphic to $B(3)$ and $f(B(3)) = 6$, using 6 pebbles in G_2 we can move a pebble to u_{3m-1}. Also the distance between the u_{3m-1} to any vertex in G_2 is at most two, using at a cost of at most 4 pebbles we can move a pebble to u_{3m-1}. Further since,

$$\frac{2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} + 2 - 6}{4} \geq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} - 1$$

we can move $2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} - 1$ additional pebbles to u_{3m-1}. Also the distance from u_{3m-1} to the target is $\left\lceil \frac{2(3m-2)}{3} \right\rceil$ we are done.

Also, G_1 is isomorphic to $B(3(m-1)+1)$ if $p_1 \geq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} + 2$ then by induction we can reach the target. Hence assume that $1 \leq p_1 \leq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} + 1$.

Since G_2 is isomorphic to $B(3)$, $f(B(3)) = 6$ and the distance from either u_{3m-2} or v_{3m-3} to any vertex in G_2 is at most three, we can move a pebble at a cost of at most 8 pebbles. Thus we can move at least

$$\frac{2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil}+2-2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil}+1-6}{8}+1$$

pebbles from G_2 to either u_{3m-2} or v_{3m-3}. Since,

$$\frac{2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil}+2-2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil}+1-6}{8}+1 \geq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil}-1$$

using these pebbles, we can reach either u_2 or v_2. If $p(v_1) \geq 2$, then a pebble from v_1 to v_2 and hence we are done. Thus assume $p(v_1) \leq 1$. After moving as many pebbles as possible from G_2 to either v_{3m-3} or u_{3m-2}. Now we can consider the following paths $P_A : u_1, u_2, v_3, u_5, v_6, \ldots, v_{3m-3}$ and $P_B : u_1, v_2, u_4, v_5, u_7, \ldots u_{3m-2}$ of lengths $\left\lceil \frac{2(3m-2)}{3} \right\rceil - 1$. Without loss of generality let us assume that $p(P_A) \geq p(P_B)$. Suppose that $p(P_A) \geq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil-1}$ then we are done. Otherwise $p(P_A) \leq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil-1} - 1$ and $p(P_B) \leq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil-1} - 1$. Now the remaining pebbles will be distributed on the $u_3, v_4, u_6, v_7, \ldots, u_{3m-3}, v_{3m-2}$. The pebbles remain in these vertices creates a spanning path to the target, otherwise by moving as many pebbles as possible from these vertices and from the P_B to the neighbouring vertices that is on the path P_A, $p(P_A) \geq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil-1}$. Thus we can easily reach the target. By symmetry, if v_{3m+1} is the target, we are done.

Case 2: Let v_1 be the target vertex.

Suppose $p_1 = 0$ then $2^{\left\lceil \frac{2(3m+1)}{3} \right\rceil} + 2$ pebbles will be distributed on the vertices of the graph G_2. Since G_2 is isomorphic to $B(3)$ and $f(B(3) = 6$, using 6 pebbles in G_2 we can move a pebble to u_{3m-1}. Also the distance between the u_{3m-1} to any vertex in G_2 is at most two, using at a cost of at most 4 pebbles we can move a pebble to u_{3m-1}.

And we can move $2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} - 1$ additional pebbles to u_{3m-1}. Also the distance from u_{3m-1} to the target is $\left\lceil \frac{2(3m-2)}{3} \right\rceil$ we are done. Also since G_1 is isomorphic to $B(3(m-1)+1)$ if $p_1 \geq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} + 2$ then by induction we can reach the target. Hence assume that $1 \leq p_1 \leq 2^{\left\lceil \frac{2(3m-2)}{3} \right\rceil} + 1$.

Since G_2 is isomorphic to $B(3)$, $f(B(3)) = 6$ and the distance from either u_{3m-2} or v_{3m-2} to any vertex in G_2 is at most three, we can move a pebble at a cost of at most 8 pebbles. Thus we can move at least

$$\frac{2^{\left\lceil\frac{2(3m+1)}{3}\right\rceil+2}-2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil+1}-6}{8}+1$$

pebbles from G_2 to either u_{3m-2} or v_{3m-3}. Since,

$$\frac{2^{\left\lceil\frac{2(3m+1)}{3}\right\rceil+2}-2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil+1}-6}{8}+1 \geq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil}-1$$

using these pebbles, we can reach either u_2 or v_2. Suppose $p(u_1) \geq 2$. After moving pebbles from G_2 we can reach v_2 and an another pebble can be moved from u_1 and hence we can reach the target. Suppose $p(u_2) \geq 2$. After moving pebbles from G_2 we can reach u_3 and the second pebble can be moved from u_2 and hence we can reach the target. Therefore assume that $p(u_1) \leq 1$ and $p(u_2) \leq 1$. Consider the paths $P_A : v_1, v_2, u_4, v_5, \ldots, u_{3m-2}$ and $P_B : v_1, u_3, v_4, \ldots u_{3m-2}$ of lengths $\left\lceil\frac{2(3m-2)}{3}\right\rceil - 1$. Without loss of generality, let us assume that $p(P_A) \geq p(P_B)$. After moving as many pebbles as possible from G_2 to u_{3m-2} suppose $p(P_A) \geq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil-1}$. Then we can reach the target. Suppose $p(P_A) \leq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil-1} - 1$ and $p(P_B) \leq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil-1} - 1$. Then there exists a spanning path with the vertices $v_{3(m-1)}, u_{3(m-1)-1}, \ldots, v_3, v_2, v_1$ consisting of the remaining pebbles and thus we can reach the target. Otherwise by moving as many pebbles as possible from these vertices and from the P_B to the neighbouring vertices that is on the path P_A, $p(P_A) \geq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil-1}$. Thus we can easily reach the target. By symmetry, if u_{3m+1} is the target then we are done.

Case 3: Let x be any target vertex otherthan $G_1 - \{u_1, v_1\}$.

Suppose $p_1 \geq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil} + 2$ then we can reach the target by induction. Thus assume $p_1 \leq 2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil} + 1$. Therefore as discussed in the earlier cases, we can move $2^{\left\lceil\frac{2(3m-2)}{3}\right\rceil} - 1$ pebbles from G_2 to u_{3m-2} or $v_{3(m-1)}$ and hence we can reach any vertex in $G_1 - \{u_1, v_1\}$. By symmetricity, we can reach any vertex in the graph $B(3m+1)$. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] F.R.K. Chung, Pebbling in hypercubes, SIAM J. Discrete Math. 2(4) (1989), 467-472.

[2] J. A. Foster, H. S. Snevily, The 2-pebbling property and a conjecture of Graham's, Graphs Comb. 16 (2000), 231-244.

[3] R. Feng, J. Y. Kim, Graham's pebbling conjecture of production complete bipartite graph, Sci. China Ser. A, 44(2001), 817-822.

[4] R. Feng, J. Y. Kim, Pebbling numbers of some graphs, Sci. China Ser. A, 45(2002), 470-478.

[5] G. Chartrand, L. Lesniak, P. Zhang, Graphs & digraphs, Fourth edition, CRC Press, Boca Raton, 2005.

[6] D.S. Herscovici and A.W. Higgins, The Pebbling Number of $C_5 \times C_5$, Discrete Math. 187(1–3) (1998), 123–135.

[7] G.H. Hurlbert, A survey of graph pebbling, Congr. Numerantium, 139 (1999), 41–64.

[8] A. Lourdusamy and A. P. Tharani, On $t-$pebbling graphs, Utilitas Math. 87(2012), 331-342.

[9] A. Lourdusamy, A.P. Tharani, t-pebbling conjecture on the products of complete r-partite graphs, Ars Combin. 102 (2011), 201-212.

[10] N. B. Rathod, K.K. Kanani, $k-$cordial labeling of Triangular Belt, Alternate Triangular Belt, Braid Graph and $Z-P_n$, Int. J. Math. Appl. 5(4) (2017), 655-662.

[11] S. S. Wang, Pebbling and Graham's Conjecture, Discrete Math. 226(2001), 431-438.