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VARIATIONAL-LIKE INCLUSION INVOLVING INFINITE FAMILY OF SET-VALUED MAPPINGS GOVERNED BY RESOLVENT EQUATIONS

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Abstract. The purpose of this work is to investigate generalized $H(.,.,.)$-φ-η-cocoercive operator and use its application via resolvent equation approach to solve the variational-like inclusion involving infinite family of set-valued mappings in semi-inner product spaces. We aim to establish an equivalence between the set-valued variational-like inclusion problem and fixed point problem. A relationship also obtain between the set-valued variational-like inclusion problem and the resolvent equation problem. This equivalent formulation suggests an idea to construct an iterative algorithm to find a solution of the resolvent equation problem.

Keywords: generalized $H(.,.,.)$-φ-η-cocorecive; variational-like inclusions; iterative algorithm; semi-inner product space.

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1. Introduction

Variational Inequality theory is very important due to its large application in various problem e.g. partial differential equation and optimization problems, see [3]. Therefore it have been

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developed and generalized in numerous directions. Variational inclusions is a natural generalization of variational inequalities. Monotonicity have a very crucial role in the study of variational inclusions. Therefore researchers introduced and studied many types of monotonicity e.g. maximal monotone mapping, relaxed monotone mapping, H-monotone mapping, A-monotone mapping etc., and discussed the solvability of different variational inclusion problems with the help of underlying different monotone mappings, see [4, 5],[7]-[9],[19, 20],[22]-[24],[25, 26]. The resolvent operator technique which is the generalized form of projection technique, is very efficient tool to solve variational inclusions and their generalizations. The resolvent equation is also a very significant approach. The resolvent operator equations technique is utilized to expand significant and feasible numerical approaches to find a the solution of many variational inequalities (inclusions) and linked optimization problems, see [1, 2].

Many heuristics generalized the monotonicity such as (H,η)-monotone, (A,η)-monotone, (A,η)-maximal relaxed monotone etc. They introduced and studied different variational inclusions problems involving these monotone mapping in Hilbert spaces (Benach spaces), see [8, 19, 22, 25].

"Recently, Sahu et al. [23] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces, which is more general than the results studied in [24]. Moreover, they constructed an iterative algorithm for approximating the solution for the class of implicit variational inclusion problems involving A-monotone and H-monotone operators by using the generalized resolvent operator technique. It is remarked that they discussed the existence and convergence analysis by relaxing the condition of monotonicity on the set-valued map considered", [4].

Very recently Luo and Huang [20], introduced and studied (H,φ)-η-monotone mapping in Banach spaces which provides a unifying framework for various classes of monotone mapping. Most recently, Bhat and Zahoor [4, 5], introduced and studied (H,ϕ)-η-monotone mapping in semi-inner product space and discussed the convergence analysis of proposed iterative schemes for some classes of variational inclusion through generalized resolvent operator. For the applications point of view of discussed operators in variational inequalities and variational inclusion, see [7]-[9],[14]-[20],[22]-[26],[28, 30].

The considered work is motivated by the noble research works discussed above. First, we investigate the notion generalized $H(.,.,.)$-φ-η-cocoercive operator which is the generalization of $H(.,.,.)$-η-cocoercive operator [15, 16]. Then we consider the variational inclusion involving infinite family of set-valued mappings. First, we obtain a relation between the variational-like inclusion and fixed point problem and also obtain a equivalance between the variational-like inclusion amd the resolvent operator equation involving generalized $H(.,.,.)$-φ-η-cocoercive operator. These equivalant fixed point problem and the resolvent equation formulation suggest us an idea to develop an iterative algorithm. As an application of resolvent equation approach, we will solve the considered variational-like inclusion problem. The obtained results are quite similar to above discussed research work but we utilize distinguished notion and approach to solve variational inclusion problems in 2-uniformly smooth Banach space. Our work is the extension and refinement of the existing results, see [1, 2, 4, 5, 14, 18, 20, 30].

Definition 1.1. [21, 23] Let us consider the vector space Y over the field F of real or complex numbers. A functional $[.,.]: Y \times Y \to F$ is called a semi inner product if

(i) $[u^1+u^2,v^1] = [u^1,v^1]+[u^2,v^1],\ \forall u^1,u^2,v^1 \in Y$

(ii) $[\alpha u^1,v^1] = \alpha[u^1,v^1],\ \forall \alpha \in F,\ u^1,v^1 \in Y$

(iii) $[u^1,u^1] \geq 0,\ for\ u^1 \neq 0$

(iv) $|[u^1,v^1]|^2 \leq [u^1,u^1][v^1,v^1],\ \forall u^1,v^1 \in Y$

The pair $(Y,[.,.])$ is called a semi-inner product space.

"We observed that $\|u^1\| = [u^1,u^1]^{1/2}$ is a norm and we can say a semi-inner product space is a normed linear space with the norm. Every normed linear space can be made into a semi-inner product space in infinitely many different ways. Giles [10] had shown that if the underlying space Y is a uniformly convex smooth Banach space then it is possible to define a semi-inner product uniquely" [4].

Remark 1.2. "This unique semi-inner product has the following nice properties:

(i) $[u^1,v^1] = 0$ iff v^1 is orthogonal to u^1, that is iff $\|v^1\| \leq \|v^1+\alpha u^1\|$, for all scalars α.

(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on Y then there is a unique vector $v^1 \in Y$ such that $f(u^1) = [u^1,v^1]$, for all $u^1 \in Y$.

(iii) The semi-inner product is continuous, that is for each $u^1, v^1 \in Y$, we have $Re[v^1, u^1 + \alpha v^1] \to Re[v^1, u^1]$ as $\alpha \to 0$", [4].

Definition 1.3. [23] The real sequence space $l^p\ for\ 1 < p < 1$ is a semi-inner product space with the semi-inner product defined by

$$[v,w] = \frac{1}{\|w\|_p^{p-2}} \sum_j v_j w_j |w|^{p-2},\ v,w \in l^p.$$

Definition 1.4. [10, 23] The real Banach space $L^p(Y,\mu)$ for $1 < p < 1$ is a semi-inner product space with the semi-inner product defined by

$$[g,h] = \frac{1}{\|h\|_p^{p-2}} \int_Y g(u)|h(u)|^{p-1} sgn(h(u))d\mu,\ v,w \in L^p.$$

Definition 1.5. [23, 27] The Y be a Banach space, then

(i) modulus of smoothness of Y defined as

$$\rho_Y(s) \ = \ \sup\left\{\frac{\|u^1+v^1\|+\|u^1-v^1\|}{2} - 1 :\ \|u^1\| \leq 1,\ \|v^1\| \leq s\right\}.$$

(ii) be uniformly smooth if $\lim_{s\to 0} \rho_Y(s)/s \ = \ 0$

(iii) Y be p-uniformly smooth for $p > 1$, if there exists $c > 0$ such that $\rho_Y(s) \leq cs^p$.

(iv) Y be 2-uniformly smooth if there exists $c > 0$ such that $\rho_Y(s) \leq cs^2$.

Lemma 1.6. [23, 27] Let $p > 1$ be a real number and Y be a smooth Banach space. Then the following statements are equivalent:

(i) Y is 2-uniformly smooth.

(ii) There is a constant $k > 0$ such that for every $v^1, w^1 \in Y$, the following inequality holds

$$\|v^1+w^1\|^2 \leq \|v^1\|^2 + 2\langle w^1, f_{v^1}\rangle + k\|w^1\|^2, \tag{1.1}$$

where $f_{v^1} \in J(v^1)$ and $J(v^1) = \{v^{1*} \in Y^* : \langle v^1, v^{1*}\rangle = \|v^1\|^2$ and $\|v^{1*}\| = \|v^1\|\}$ is the normalized duality mapping.

Remark 1.7. "Every normed linear space Y is a semi-inner product space (see [21]). Infact, by Hahn-Banach theorem, for each $v^1 \in Y$, there exists at least one functional $f_{v^1} \in Y^*$ such that

$\langle v^1, f_{v^1}\rangle = \|v^1\|^2$. Given any such mapping $f : Y \to Y^*$, we can verify that $[w^1, v^1] = \langle w^1, f_{v^1}\rangle$ defines a semi-inner product. Hence we can write the inequality (2.1) as

$$\|v^1 + w^1\|^2 \leq \|v^1\|^2 + 2[w^1, f_{v^1}] + s\|w^1\|^2. \tag{1.2}$$

The constant s is known as constant of smoothness of Y, is chosen with best possible minimum value", [23].

Example 1.8. "The function space L^p is 2-uniformly smooth for $p \geq 2$ and it is p-uniformly smooth for $1 < p < 2$. If $2 \leq p < \infty$, then we have for all $v^1, w^1 \in L^p$,

$$\|v^1 + w^1\|^2 \leq \|v^1\|^2 + 2[w^1, f_{v^1}] + (p-1)\|w^1\|^2.$$

where the constant of smoothness is $p-1$", [23].

2. Preliminaries

Let Y be a 2-uniformly smooth Banach space. Its norm and topological dual space is given by $\|.\|$ and Y^*, respectively. The semi-inner product $[.,.]$ signify the dual pair among Y and Y^*.

Definition 2.1. [20, 23] Let Y be real 2-uniformly smooth Banach space. Let single-valued mapping $Q : Y \to Y$ and mapping $\eta : Y \times Y \to Y$, then

(i) Q is (r, η)-strongly monotone if there $\exists$ constant $r > 0$ such that

$$[Q(u) - Q(u'), \eta(u, u')] \ \geq\ r\,\|u - u'\|^2,\ \forall u,\ u' \in Y;$$

(ii) Q is (s, η)-cocoercive if there $\exists$ constant $s > 0$ such that

$$[Q(u) - Q(u'), \eta(u, u')] \ \geq\ s\,\|Q(u) - Q(u')\|^2,\ \forall u,\ u' \in Y;$$

(iii) Q is (s', η)-relaxed cocoercive if there $\exists$ constant $s > 0$ such that

$$[Q(u) - Q(u'), \eta(u, u')] \ \geq\ -s'\,\|Q(u) - Q(u')\|^2,\ \forall u,\ u' \in Y;$$

(iv) Q is α-expansive if there $\exists$ constant $\alpha > 0$

$$\|Q(u) - Q(u')\| \ \geq\ \alpha\,\|u - u'\|,\ \forall u,\ u' \in Y;$$

(v) η is be τ-Lipschitz continuous if there $\exists$ constant $\tau > 0$ such that

$$\|\eta(u,u')\| \leq \tau \|u-u'\|, \forall u, u' \in Y.$$

Definition 2.2. [15, 16] Let us consider the single-valued mappings $Q,R,S : Y \to Y$, mapping $\eta : Y \times Y \to Y$, $H : Y \times Y \times Y \to Y$, then

(i) $H(Q,.,.)$ is (μ,η)-cocoercive in regards R if there $\exists$ constant $\mu > 0$ such that

$$[H(Qu,x,x) - H(Qu',x,x),\ \eta(u,u')] \geq \mu \|Qu - Qu'\|^2, \forall x, u, u' \in Y;$$

(ii) $H(.,R,.)$ is (γ,η)-relaxed cocoercive in regards R if there $\exists$ constant $\gamma > 0$ such that

$$[H(x,Ru,x) - H(x,Ru',x), \eta(u,u')] \geq -\gamma \|Ru - Ru'\|^2, \forall x, u, u' \in Y;$$

(iii) $H(.,.,S)$ is (δ,η)-strongly monotone in regards S if there $\exists$ constant $\delta > 0$ such that

$$[H(x,x,Su) - H(x,x,Su'), \eta(u,u')] \geq \delta \|u-u'\|^2, \forall x, u, u' \in Y;$$

(iv) $H(Q,.,.)$ is κ_1-Lipschitz continuous in regards Q if there $\exists$ constant κ_1 such that

$$\|H(Qu,x,x) - H(Qu',x,x)\| \leq \kappa_1 \|u-u'\|, \forall x, u,u' \in Y.$$

Similarly we can define the Lipschitz continuity for $H(.,.,.)$ in regards second and third component.

"Let $M : Y \multimap Y$ be a set-valued mapping, then graph of M is given by graph$(M) = \{(v,w) : w \in M(v)\}$. The domain of M is given by

$$\text{Dom}(M) = \{v \in Y : \exists w \in Y : (v,w) \in M\}.$$

The Range of (M) is given by

$$\text{Range}(M) = \{w \in Y : \exists V \in Y : (v,w) \in M\}.$$

The inverse of (M) is given by

$$M^{-1} = \{(w,v) : (v,w) \in M\}.$$

For any two set-valued mappings N and M, and any real number β, we define

$$N + M = \{(v,w+w') : (v,w) \in N, (v,w') \in M\},$$

$$\beta M = \{(v, \beta w) : (v, w,) \in M\}.$$

For a mapping A and a set-valued map $M : Y \multimap Y$, we define $A + M = \{(v, w + w') : Av = w, (v, w') \in M\}$", [4].

Definition 2.3. [20, 23] A set-valued mapping $M : Y \multimap Y$ is said to be (m, η)-relaxed monotone if $\exists$ a constant $m > 0$ such that

$$[v^* - w^*, \eta(v, w)] \ \geq \ -m \, \|v - w\|^2, \ \forall v, w \in Y, \ v^* \in M(v), \ w^* \in M(w).$$

Definition 2.4. Let $G : Y^\infty = Y \times Y \times Y ... \to Y$ be a mapping. Then G is α_i-Lipschitz continuous in regards i^{th} component if $\exists$ a constant $\alpha_i > 0$ such that

$$\|G(., ., v_i, ...) - G(., ., ., w_i, ...)\| \ \leq \ \alpha_i \, \|v_i - w_i\|, \ \forall v_i, w_i \in Y.$$

Definition 2.5. The Hausdorff metric $D(., .)$ on $CB(Y)$, is defined by

$$D(A, B) = \max \left\{ \sup_{u \in A} \inf_{v \in B} d(u, v), \sup_{v \in B} \inf_{u \in A} d(u, v) \right\}, \ A, B \in CB(Y),$$

where $d(., .)$ is the induced metric on Y and $CB(Y)$ denotes the family of all nonempty closed and bounded subsets of X.

Definition 2.6. [6]A multi-valued mapping $S : Y \to CB(Y)$ is called D-Lipschitz continuous with constant $\lambda_S > 0$, if

$$D(Sv, Sw) \leq \ \lambda_S \, \|v - w\|, \ \forall v, \ w \in Y.$$

3. GENERALIZED $H(., ., .)$-φ-η-COCOERCIVE OPERATOR

First, we give some definitions and important theorems associates with generalized $H(., ., .)$-φ-η-cocoercive operator.

Let Y be 2-uniformly smooth Banach space. Assume that $\eta, H : Y \times Y \times Y \to Y$ be the mappings and $\varphi, Q, R, S : Y \to Y$ be the single-valued mappings and $M : Y \multimap Y$ be a multi-valued mapping.

Definition 3.1. Let $H(., ., .)$ is (μ, η)-cocoercive in regards Q with non-negative constant μ, (γ, η)-relaxed cocoercive in regards R with non-negative constant γ and (δ, η)-strongly monotone in regards S with non-negative constant δ, then M is called generalized $H(., ., .)$-φ-η-*cocoercive* in regards Q, R and S if

(i) φoM is (m,η)-relaxed monotone;

(ii) $(H(.,.,.)+\lambda\varphi oM)(Y) = Y$, $\lambda>0$.

Let us consider the following

Assumption $\mathbf{M_1}$: Let H is (μ,η)-cocoercive in regards Q with non-negative constant μ, (γ,η)-relaxed cocoercive in regards R with non-negative constant γ and (δ,η)-strongly monotone in regards S with non-negative constant δ with $\mu>\gamma$.

Assumption $\mathbf{M_2}$: Let Q is α-expansive and R is β-Lipschitz continuous with $\alpha>\beta$.

Assumption $\mathbf{M_3}$: Let η is τ-Lipschitz continuous.

Assumption $\mathbf{M_4}$: Let M is generalized $H(.,.,.)$-φ-η-cocoercive operator in regards Q, R and S.

Theorem 3.2. *Let assumptions M_1, M_2 and M_4 hold good with $\ell=\mu\alpha^2-\gamma\beta^2+\delta>m$, then $(H(Q,R,S)+\lambda\varphi oM)^{-1}$ is single-valued.*

Proof. Let $y,z\in(H(Q,R,S)+\lambda\varphi oM)^{-1}(x)$ for any given $x\in Y$. It is obvious that

$$\begin{cases} -H(Qy,Ry,Sy)+x\in\ \lambda\varphi oM(y),\\ -H(Qz,Rz,Sz)+x\in\ \lambda\varphi oM(z).\end{cases}$$

Since φoM is (m,η)-relaxed monotone in the first argument, we have

$$\begin{aligned} -m\lambda\|y-z\|^2 \ \le\ & [-H(Qy,Ry,Sy)+x-(-H(Qz,Rz,Sz)+x),\ \eta(y,z)]\\ =\ & [H(Qy,Ry,Sy)-H(Qz,Rz,Sz),\ \eta(y,z)]\\ =\ & -[H(Qy,Ry,Sy)-H(Qz,Ry,Sy),\ \eta(y,z)]\\ & -[H(Qz,Ry,Sy)-H(Qz,Rz,Sy),\ \eta(y,z)]\\ & -[H(Qz,Rz,Sy)-H(Qz,Rz,Sz),\ \eta(y,z)].\end{aligned}$$

Since assumption M_1 holds, we have

$$-m\lambda\|y-z\|^2\le-\mu\|Qy-Qz\|^2+\gamma\|Ry-Rz\|^2-\delta\|y-z\|^2.$$

Since assumption M_2 holds, we have

$$-m\lambda\|y-z\|^2 \leq -\mu\alpha^2\|y-z\|^2+\gamma\beta^2\|y-z\|^2-\delta\|y-z\|^2$$
$$= -(\mu\alpha^2-\gamma+\delta)\ \|y-z\|^2$$
$$0 \leq -(\ell-m\lambda)\ \|y-z\|^2\ \leq\ 0, \text{where } \ell\ =\ \mu\alpha^2-\gamma\beta^2+\delta.$$

Since $\mu>\gamma,\ \alpha>\beta,\delta>0$, it follows that $\|y-z\|\ \leq\ 0$. We get $y=z$, therefore $(H(Q,R,S)+\lambda\varphi oM)^{-1}$ is single-valued.

Definition 3.3. Let assumptions M_1, M_2 and M_4 hold good with $\ell=\mu\alpha^2-\gamma\beta^2+\delta>m\lambda$ then the *resolvent operator* $R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}: Y\to\ Y$ is given as

$$R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(u)\ =\ (H(Q,R,S)+\lambda\varphi oM)^{-1}(u),\ \forall\, u\in Y. \tag{3.1}$$

Theorem 3.4. *Let assumptions M_1-M_4 hold good with $\ell=\mu\alpha^2-\gamma\beta^2+\delta>m\lambda$ and η is τ-Lipschitz then $R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}: Y\to\ Y$ is $\frac{\tau}{\ell-m\lambda}$-Lipschitz continuous, that is,*

$$\|R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y)-R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)\|\leq\frac{\tau}{\ell-m\lambda}\|y-z\|,\ \forall\, y,z\in Y.$$

Proof. Let any given points $y,z\in Y$. From (3.3), we have

$$R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y)\ =\ (H(Q,R,S)+\lambda\varphi oM)^{-1}(y),$$
$$R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)\ =\ (H(Q,R,S)+\lambda\varphi oM)^{-1}(z).$$

Let $u_0\ =\ R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y)$ and $u_1\ =\ R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)$.

$$\begin{cases} \lambda^{-1}\Big(y-H\Big(Q(u_0),R(u_0),S(u_0)\Big)\Big)\in\ \varphi oM(u_0)\\ \lambda^{-1}\Big(z-H\Big(Q(u_1),R(u_1),S(u_1)\Big)\Big)\in\ \varphi oM(u_1).\end{cases}$$

Since φoM is (m,η)-relaxed monotone in the first arguments, we have

$$[(y-H(Q(u_0),R(u_0),S(u_0)))-(z-H(Q(u_1),R(u_1),S(u_1))),\eta(u_0,u_1)]\ \geq\ -m\lambda\ \|u_0-u_1\|^2,$$

which implies

$$[y-z,\eta(u_0,u_1)]\geq[H(Q(u_0),R(u_0),S(u_0))-H(Q(u_1),R(u_1),S(u_1)),\eta(u_0,u_1)]-m\lambda\|u_0-u_1\|^2.$$

Now, we have

$$\|y-z\|\ \|\eta(u_0,u_1)\| \geq\ [y-z,\ \eta(u_0,u_1)]$$
$$\geq\ [H(Q(u_0),R(u_0),S(u_0))-H(Q(u_1),R(u_1),S(u_1)),\eta(u_0,u_1)]-m\lambda\ \|u_0-u_1\|^2$$

Since assumptions M_1,M_2,M_3 hold and η is τ-Lipschitz continuous, we have Hence,

$$\|y-z\|\ \tau\|u_0-u_1\|\ \geq\ (\ell-m\lambda)\ \|u_0-u_1\|^2$$
$$\text{or } \|R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(y)-R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(z)\| \leq \frac{\tau}{\ell-m\lambda}\ \|y-z\|,\ \forall\, y,z\in Y.$$

Hence, we get the required result.

4. Application

Now we make an attempt to show that generalized $H(.,.,.)$-φ-η-cocorecive operator under acceptable assumptions can be used as a powerful tool to solve variational inclusion problems.

Let Y be 2-uniformly smooth Banach space. Let $V,W_i:Y\to CB(Y),\ i=1,2,...\infty$ be the infinite family of multi-valued mappings and $Q,R,S,h,k,\varphi:Y\to Y$ be the single-valued mappings Let $\eta:Y\times Y\to Y$, $H:Y\times Y\times Y\to Y$ and $G:Y^\infty=Y\times Y\times Y...\to Y$ be the mappings. Suppose that multi-valued mapping $M:Y\multimap Y$ be a generalized $H(.,.,.)$-φ-η-cocoercive operator in regards Q, R and S. We consider the following variational like inclusion problem involving infinite family of set-valued mappings to find $v\in Y$, $a\in V(v)$ and $v_i\in W_i(v)$, $i=1,2,...\infty$ such that

$$0\in\ G(v_1,v_2,v_3,...)+k(a)+M(h(v)-k(a)). \tag{4.1}$$

Varaitionl inclusion problem type of (4.1), studied by Ahmad and Dilshad [1] and Wang [29] in the setting of real Banach space, .

Lemma 4.1. *Let us consider the mapping $\varphi:Y\to Y$ such that $\varphi(v_1+v_2)=\varphi(v_1)+\varphi(v_2)$ and $Ker(\varphi)=\{0\}$, where $Ker(\varphi)=\{v_1\in Y:\ \varphi(v_1)=0\}$. If $(v,a,(v_1,v_2,...))$, where $v\in Y$, $a\in V(v)$ and $v_i\in W_i(v)$, $i=1,2,...\infty$ is a solution of problem (4.1) if and only if $(v,a,(v_1,v_2,...))$ satisfies the following relation:*

$$h(v)\ =\ k(a)+R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}\ [H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a)))$$

$$-\lambda\{\varphi o G(v_1,v_2,v_3,...)+k(a)\}]. \tag{4.2}$$

The resolvent equation corresponding to generalized set-valued variational-like inclusion problem (4.1).

$$\varphi o\ G(v_1,v_2,v_3,....)+k(a)+\lambda^{-1}J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)=0. \tag{4.3}$$

where $\lambda>0$,

$$J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)=\Big[I-H(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)),R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)),S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)))\Big],$$

I is the identity mapping and

$$H(Q,R,S)\Big[R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)\Big]=H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)),R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)),S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q))\Big).$$

Now, we show that the problem (4.1) is equivalent to the resolvent equation problem (4.3).

Lemma 4.2. *If $(v,a,(v_1,v_2,...))$ with $v\in Y$, $a\in V(v)$ and $v_i\in W_i(v)$, $i=1,2,...\infty$ is a solution of problem (4.1) if and only if the resolvent equation problem (4.3) has a solution $(q,v,a,(v_1,v_2,...))$ with $v,q\in Y$, $a\in V(v)$ and $v_i\in W_i(v)$, $i=1,2,3,....$, where*

$$h(v)\ =\ R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q), \tag{4.4}$$

and $q=H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a)))-\lambda\{\varphi o G(v_1,v_2,v_3,...)+k(a)\}$.

Proof: Let $(v,a,(v_1,v_2,...))$ be a solution of problem (4.1), and from Lemma 4.1 Using the fact that

$$\begin{aligned}
J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}&=\Big[I-H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}),R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}),S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi})\Big)\Big],\\
J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}(q)&=J^{H(.,.,.)-\eta}_{M,\lambda,\varphi}\Big[H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a)))\\
&\qquad -\lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big]\\
&=\Big[I-H\Big(Q(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}),R(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi}),S(R^{H(.,.,.)-\eta}_{M,\lambda,\varphi})\Big)\Big]\\
&\quad\Big[H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a)))-\lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \Big[H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) - \lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big] \\
&\quad -H\Big(Q(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}),R(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}),S(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta})\Big) \\
&\quad \Big[H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) - \lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big] \\
&= \Big[H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) - \lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big] \\
&\quad -H\Big(Q(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta})\Big(H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) \\
&\qquad -\lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big), \\
&\qquad R(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta})\Big(H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) \\
&\qquad -\lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big), \\
&\qquad S(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta})\Big(H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) \\
&\qquad -\lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big)\Big) \\
&= \Big[H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) - \lambda\{\varphi o G(v_1,v_2,...)+k(a)\}\Big] \\
&\qquad -H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) \\
&= -\lambda[\varphi o\, G(v_1,v_2,v_3,....)+k(a)]
\end{aligned}
$$

This implies that

$$
\varphi o\, G(v_1,v_2,v_3,....)+k(a)+\lambda^{-1}J_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q)=0. \tag{4.5}
$$

Conversely, let $(q,v,a,(v_1,v_2,...))$ is a solution of resolvent equation problem (4.3), then

$$
\begin{aligned}
&J_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q) = -\lambda[\varphi o\, G(v_1,v_2,v_3,....)+k(a)] \\
&\Big[I-H\Big(Q(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}),R(R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}),S(R_{M,\lambda\varphi}^{H(.,.,.)-\eta})\Big)\Big](q) = -\lambda[\varphi o\, G(v_1,v_2,v_3,....)+k(a)] \\
&q-H(Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) = -\lambda[\varphi o G(v_1,v_2,v_3,....)+k(a)].
\end{aligned}
$$

This implies that

$$
q = H((Q(h(v)-k(a)),R(h(v)-k(a)),S(h(v)-k(a))) - \lambda[\varphi o\, G(v_1,v_2,v_3,....)+k(a)].
$$

Hence $(v,a,(v_1,v_2,...))$ is a solution of variational inclusion problem (4.1).

Lemma 4.1 and Lemma 4.2 are very crucial from the numerical point of view. They permit us to suggest the following iterative scheme for finding the approximate solution of (4.3).

Algorithm 4.3. For any given $(q_0, v_0, a_0, (v_1^0, v_2^0, v_3^0, ...))$, we can choose $q_0,\ v_0 \in Y$, $a_0 \in V(v_0)$ and $v_i^0 \in W_i(v_0)$, $i = 1,2,3.....$ and $0 < \varepsilon < 1$ such that sequences $\{q_k\}$, $\{v_k\}$, $\{a_k\}$ and $\{v_i^k\}$ satisfy

$$\begin{cases} h(v_k) = k(a_k) + R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q_k), \\ a_k \in V(a_k),\ \|a_k - a_{k+1}\| \leq\ D(V(v_k), V(v_{k+1})) + \varepsilon^{k+1}\|v_k - v_{k+1}\|, \\ for\ each\ i,\ v_i^k \in W_i(v_k), \|v_i^k - v_i^{k+1}\|\ \leq D(W_i(v_k), W_i(v_{k+1})) + \varepsilon^{k+1}\|v_k - v_{k+1}\|, \\ q_{k+1} = H(Q(h(v_k) - k(a_k)), R(h(v_k) - k(a_k)), S(h(v_k) - k(a_k))) - \lambda\{\varphi oG(v_k, w_k) + k(a_k)\}, \end{cases}$$

where $\lambda > 0$, $k \geq 0$, and $D(.,.)$ is the Hausdorff metric on $\mathrm{CB}(Y)$.

Next, we find the convergence of the iterative algorithm for the resolvent equation problem (4.3) corresponding generalized set-valued variational inclusion problem (4.1). the unique solution (t, u, v, w) of the resolvent equation problem (4.3).

Theorem 4.4. *Let us consider the problem (4.1) with assumptions M_1-M_4 hold good and φ : $Y \to Y$ be a single-valued mapping with $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$ and $Ker(\varphi) = \{0\}$. Let multi-valued mappings $V,\ W_i : Y \to CB(Y)$, $i = 1,2,...,$ be λ_V, β_i-D-Lipschitz continuous, respectively. Let single-valued mapping $h : Y \to Y$ be r-strongly monotone and λ_h-Lipschitz continuous, and $k : Y \to Y$ be λ_k-Lipschitz continuous. Let mapping $H : Y \times Y \times Y \to Y$ be κ_1, κ_2 and κ_3-Lipschitz continuous in regards Q, R and S, respectively. Let φoG be α_i-Lipschitz continuous in regards i^{th} component, $i = 1,2,....$ Assume that the following condition is satisfy*

$$0 < (\kappa_1 + \kappa_2 + \kappa_3)\{\lambda_h + \lambda_k\lambda_V\} + \lambda\sum_{i=1}^{\infty}\alpha_i\beta_i + \lambda\lambda_k\lambda_V < \frac{(\ell - m\lambda)\left\{1 - \sqrt{1 - 2r + \lambda_h^2} - \lambda_k\lambda_V\right\}}{\tau};$$

Then there exist $q, v \in Y$, $a \in V(v)$ and $v_i \in W_i(v)$ that satisfy the resolvent equation problem (4.3). The iterative sequences $\{q_k\}, \{v_k\}, \{a_k\}$, and $\{w_i^k\}$, $i = 1,2,...$ and $k = 1,2,...,$ generated by Algorithm 4.3 converges strongly to the unique solution q, v, a, v_i, respectively.

Proof. Using Algorithms 4.3 and λ_V, β_i-D Lipschitz continuity of V, W_i, we have

$$\|a_k - a_{k-1}\| \leq D(V(v_k), V(v_{k-1})) + \varepsilon^k \|v_k - v_{k-1}\| \leq \{\lambda_V + \varepsilon^k\} \|v_k - v_{k-1}\| \tag{4.6}$$

$$\|v_i^k - v_i^{k-1}\| \leq D(W_1(v_k), W_i(v_{k-1})) + \varepsilon^k \|v_k - v_{k-1}\| \leq \{\beta_i + \varepsilon^k\} \|v_k - v_{k-1}\|, \tag{4.7}$$

where $k = 1,\ 2,\ .....$

Now, we compute

$$\begin{aligned}
\|q_{k+1} - q_k\| &= \|H(Q(h(v_k) - k(a_k)), R(Q(h(v_k) - k(a_k)), S(Q(h(v_k) - k(a_k)))\\
&\quad - H(Q(h(v_{k-1}) - k(a_{k-1})), R(h(v_{k-1}) - k(a_{k-1})), S(h(v_{k-1}) - k(a_{k-1})))\\
&\quad - \lambda\{\varphi o G(v_1^k, v_2^k, ...) + k(a_k) - \varphi o G(v_1^{k-1}, v_2^{k-1}, ...) - k(a_{k-1})\}\|\\
&\leq \|H(Q(h(v_k) - k(a_k)), R(Q(h(v_k) - k(a_k)), S(Q(h(v_k) - k(a_k)))\\
&\quad - H(Q(h(v_{k-1}) - k(a_{k-1})), R(h(v_{k-1}) - k(a_{k-1})), S(h(v_{k-1}) - k(a_{k-1})))\|\\
&\quad + \lambda\|\varphi o G(v_1^k, v_2^k, ...) - \varphi o G(v_1^{k-1}, v_2^{k-1}, ...)\| + \lambda\|k(a_k) - k(a_{k-1})\|. \qquad (4.8)
\end{aligned}$$

Now, we compute

$$\begin{aligned}
\|(h(v_k) - k(a_k)) - (h(v_{k-1}) - k(a_{k-1}))\| &\leq \|h(v_k) - h(v_{k-1})\| + \|k(a_k) - k(a_{k-1})\|\\
&\leq \lambda_h \|v_k - v_{k-1}\| + \lambda_k \|a_k - a_{k-1}\|\\
&\leq \lambda_h \|v_k - v_{k-1}\| + \lambda_k(\lambda_V + \varepsilon^k)\|v_k - v_{k-1}\|\\
&\leq \{\lambda_h + \lambda_k(\lambda_V + \varepsilon^k)\}\|v_k - v_{k-1}\| \qquad (4.9)
\end{aligned}$$

Since $H(Q,R,S)$ is $\kappa_1, \kappa_2, \kappa_3$-Lipschitz continuous in regards Q,R,S, respectively, We have

$$\begin{aligned}
&\|H(Q(h(v_k) - k(a_k)), R(Q(h(v_k) - k(a_k)), S(Q(h(v_k) - k(a_k)))\\
&\quad - H(Q(h(v_{k-1}) - k(a_{k-1})), R(h(v_{k-1}) - k(a_{k-1})), S(h(v_{k-1}) - k(a_{k-1})))\|\\
&\qquad \leq (\kappa_1 + \kappa_2 + \kappa_3)\|(h(v_k) - k(a_k)) - (h(v_{k-1}) - k(a_{k-1}))\|\\
&\qquad \leq (\kappa_1 + \kappa_2 + \kappa_3)\{\lambda_h + \lambda_k(\lambda_V + \varepsilon^k)\}\|v_k - v_{k-1}\|. \qquad (4.10)
\end{aligned}$$

Using the α_i-Lipschitz continuity of $\varphi o G_i$, $i = 1,2,..$, and β_i-D-Lipschitz continuity of W_i's, we have

$$\begin{aligned}
&\|\varphi oG(v_1^k, v_2^k, ...) - \varphi oG(v_1^{k-1}, v_2^{k-1}, ...)\| \\
&\quad = \|\varphi oG(v_1^k, v_2^k, ...) - \varphi oG(v_1^{k-1}, v_2^k, ...) + \varphi oG(v_1^{k-1}, v_2^k, ...) + ...\| \\
&\quad \leq \|\varphi oG(v_1^k, v_2^k, ...) - \varphi oG(v_1^{k-1}, v_2^k, ...)\| \\
&\qquad + \|\varphi oG(v_1^{k-1}, v_2^k, ...) - \varphi oG(v_1^{k-1}, v_2^{k-1}, ...)\| + ... \\
&\quad \leq \alpha_1 \|v_1^k - v_1^{k-1}\| + \alpha_2 \|v_2^k - v_2^{k-1}\| + ... \\
&\quad \leq \alpha_1(\beta_1 + \varepsilon^k)\|v_k - v_{k-1}\| + \alpha_2(\beta_2 + \varepsilon^k)\|v_k - v_{k-1}\| + ... \\
&\quad \leq \sum_{i=1}^{\infty} \alpha_i(\beta_i + \varepsilon^k)\|v_k - v_{k-1}\|. \qquad (4.11)
\end{aligned}$$

Using (4.6), (4.10) and (4.11) in (4.8), we have

$$\begin{aligned}
\|q_{k+1} - q_k\| &\leq (\kappa_1 + \kappa_2 + \kappa_3)\{\lambda_h + \lambda_k(\lambda_V + \varepsilon^k)\}\|v_k - v_{k-1}\| \\
&\quad + \lambda \sum_{i=1}^{\infty} \alpha_i(\beta_i + \varepsilon^k)\|v_k - v_{k-1}\| \\
&\quad + \{\lambda_h + \lambda \lambda_k(\lambda_V + \varepsilon^k)\}\|v_k - v_{k-1}\| \\
&\leq \Big\{(\kappa_1 + \kappa_2 + \kappa_3)\{\lambda_h + \lambda_k(\lambda_V + \varepsilon^k)\} + \lambda \sum_{i=1}^{\infty} \alpha_i(\beta_i + \varepsilon^k) + \lambda \lambda_k(\lambda_V + \varepsilon^k)\Big\} \\
&\qquad \times \Big\|v_k - v_{k-1}\Big\|. \qquad (4.12)
\end{aligned}$$

By Lipschitz continuity of resolvent operator and condition (4.6), we have

$$\begin{aligned}
\|v_k - v_{k-1}\| &= \Big\|\{v_k - v_{k-1} - (h(v_k) - h(v_{k-1}))\} + \{k(a_k) - k(a_{k-1})\} \\
&\quad + R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q_k) - R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q_{k-1})\Big\| \\
&\leq \Big\|v_k - v_{k-1} - (h(v_k) - h(v_{k-1}))\Big\| + \Big\|R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q_k) - R_{M,\lambda,\varphi}^{H(.,.,.)-\eta}(q_{k-1})\Big\| \\
&\quad + \Big\|k(a_k) - k(a_{k-1})\Big\| \\
&\leq \|v_k - v_{k-1} - (h(v_k) - h(v_{k-1}))\| + \frac{\tau}{\ell - m\lambda}\|q_k - q_{k-1}\| \\
&\quad + \lambda_k(\lambda_V + \varepsilon^k)\|v_k - v_{k-1}\| \qquad (4.13)
\end{aligned}$$

$$\begin{aligned}
&\Big\|v_k - v_{k-1} - (h(v_k) - h(v_{k-1}))\Big\|^2 \\
&\qquad = \Big\|v_k - v_{k-1}\Big\|^2 - 2[h(v_k) - h(v_{k-1}), v_k - v_{k-1}] + \Big\|h(v_k) - h(v_{k-1})\Big\|^2 \\
&\qquad \leq \Big\|v_k - v_{k-1}\Big\|^2 - 2r\Big\|v_k - v_{k-1}\Big\|^2 + \lambda_h^2\Big\|v_k - v_{k-1}\Big\|^2 \\
&\qquad \leq (1 - 2r + \lambda_h^2)\Big\|v_k - v_{k-1}\Big\|^2. \qquad (4.14)
\end{aligned}$$

Using (4.14) in (4.13), we have

$$\Big\|v_k - v_{k-1}\Big\| \leq \sqrt{1 - 2r + \lambda_h^2}\Big\|v_k - v_{k-1}\Big\| + \frac{\tau}{(\ell - m\lambda)}\Big\|q_k - q_{k-1}\Big\| + \lambda_k(\lambda_V + \varepsilon^k)\Big\|v_k - v_{k-1}\Big\|.$$

$$\|v_k - v_{k-1}\| \leq \frac{\tau}{\Big\{1 - \Big\{\sqrt{1 - 2r + \lambda_h^2} + \lambda_k(\lambda_V + \varepsilon^k)\Big\}\Big\}(\ell - m\lambda)}\|q_k - q_{k-1}\|. \qquad (4.15)$$

Using (4.13) in (4.12), we have

$$\|q_{k+1} - q_k\| \leq \Theta(\varepsilon^k)\|q_k - q_{k-1}\|, \text{ where} \qquad (4.16)$$

$$\Theta(\varepsilon^k) = \frac{\tau\Big\{(\kappa_1 + \kappa_2 + \kappa_3)\{\lambda_h + \lambda_k(\lambda_V + \varepsilon^k)\} + \lambda\sum_{i=1}^{\infty}\alpha_i(\beta_i + \varepsilon^k) + \lambda\lambda_k(\lambda_V + \varepsilon^k)\Big\}}{\Big\{1 - \Big\{\sqrt{1 - 2r + \lambda_h^2} + \lambda_k(\lambda_V + \varepsilon^k)\Big\}\Big\}(\ell - m\lambda)}.$$

Since $0 < \varepsilon < 1$, this implies that $\Theta(\varepsilon^k) \to \Theta$ as $k \to \infty$, where

$$\Theta = \frac{\tau\Big\{(\kappa_1 + \kappa_2 + \kappa_3)\{\lambda_h + \lambda_k\lambda_V\} + \lambda\sum_{i=1}^{\infty}\alpha_i\beta_i + \lambda\lambda_k\lambda_V\Big\}}{\Big\{1 - \Big\{\sqrt{1 - 2r + \lambda_h^2} + \lambda_k\lambda_V\Big\}\Big\}(\ell - m\lambda)}.$$

It is given that $\Theta < 1$, then $\{q_k\}$ is a Cauchy sequence in Banach space Y, then $q_k \to q$ as $k \to \infty$. From (4.15), $\{v_k\}$ is also Cauchy sequence in Banach space Y, then there exist v such that $v_k \to v$.

From equation (4.5)-(4.6) and Algorithm 4.3, the sequences $\{v_i^k\}$ and $\{a_k\}$ are also Cauchy sequences in Y. Thus, there exist v_i and a such that $v_i^k \to v_i$ and $a_k \to a$ as $k \to \infty$. Next we will prove that $v_i \in W_i(v)$. Since $v_i^k \in W_i(v)$, then

$$\begin{aligned}
d(v_i,\, W_i(v)) &\leq \|v_i - v_i^k\| + d(v_i^k,\, W_i(v)) \\
&\leq \|v_i - v_i^k\| + D(W_i(v_k),\, W_i(v)) \\
&\leq \|v_i - v_i^k\| + \beta_i\,\|v_k - v\| \to 0, \text{ as } k \to \infty,
\end{aligned}$$

which gives $d(v_i,\ W_i(v)) = 0$. Due to $W_i(v) \in CB(Y)$, we have $v_i \in W_i(v)$, $i = 1,2,....$ In the same manner, we easily show that $a \in V(v)$.

By the continuity of $R^{H(.,..,.)-\eta}_{M,\lambda,\varphi}$, Q, R, S, V, W_i, φoG, k, h, η and M and Algorithms 4.3, we know that $(q,v,a,(v_1,v_2,...))$ satisfy

$$q_{k+1} = [H(Q(h(v_k)-k(a_k)),R(h(v_k)-k(a_k),S(h(v_k)-k(a_k)))-\lambda\{\varphi oG(v_1^k,v_2^k,...)+k(a_k)\}],$$

$$\to q = [H(Q(h(v)-k(a)),R(h(v)-k(a),S(h(v)-k(a)))-\lambda\{\varphi oG(v_1,v_2,...)+k(a)\}] \text{ as } k\to\infty$$

$$R^{H(.,..,.)-\eta}_{M,\lambda,\varphi}(q_k) = h(v_k)-k(a_k) \to h(v)-k(a) = R^{H(.,..,.)-\eta}_{M,\lambda,\varphi}(q) \text{ as } k\to\infty.$$

By using Lemma 4.2, we have

$$\varphi oG(v_1,v_2,...)+\lambda^{-1}(q-H(Q(R^{H(.,..,.)-\eta}_{M,\lambda,\varphi}(q)),R(R^{H(.,..,.)-\eta}_{M,\lambda,\varphi}(q)),S(R^{H(.,..,.)-\eta}_{M,\lambda,\varphi}(q))) = 0,$$

Thus we have

$$\varphi oG(v_1,v_2,...)+\lambda^{-1}J^{H(.,..,.)-\eta}_{M,\lambda,\varphi}(q) = 0. \tag{4.17}$$

Hence $(q,v,a,(v_1,v_2,....))$ is a solution of the problem (4.3).

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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