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CAPUTO-HADAMARD APPROACH APPLICATIONS: SOLVABILITY FOR AN INTEGRO-DIFFERENTIAL PROBLEM OF LANE AND EMDEN TYPE

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Abstract. The present paper is dealing with a new direction in the Caputo-Hadamrd approach. It is concerned with the solvability of an integro differential problem of type Lane and Emden. The studied problem involves Caputo-Hadamard derivative with new different fractional orders. The main results of existence of solutions are based on the contraction principle of Banach, however, for the existence of solutions, the use of Scheafer fixed point theorem is applied to prove the result. Three examples are discussed at the end of this work.

Keywords: Caputo-Hadamard derivative; existence of solution; fixed point.

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1. Introduction

In the present paper, we are concentrating on the investigation of the existence and uniqueness of solutions for the following problem of nonlinear fractional differential equation of Lane-Emden singular type:

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$$(1)\qquad \begin{cases} {}_{H}^{C}D^{\beta}\left({}_{H}^{C}D^{\alpha}+\frac{A}{(\log t)^{\mu}}\right)x(t)+Bf\left(t,x(t),{}_{H}^{C}D^{\sigma}x(t)\right) \\ \\ +g\left(t,x(t),I^{\rho}x(t)\right)=h(t),t\in\left]1,e\right], \\ \\ \left({}_{H}^{C}D^{\alpha}+A\right)x(e)=0,x(1)=x(e)=\sum\limits_{i=1}^{n}\lambda_{i}I^{\delta_{i}}x(\eta_{i}), \\ \\ 0<\mu,\sigma<\alpha<1,1<\beta<2,1<\eta_{i}<e,\rho,\delta_{i},\lambda_{i}>0, \end{cases}$$

where ${}_{H}^{C}D^{\beta},{}_{H}^{C}D^{\alpha}$ and ${}_{H}^{C}D^{\sigma}$ are the derivatives in the sense of Caputo-Hadamard, I^{ρ} denotes the Hadamard integral of order ρ, with: $A,B>0,J=[1,e]$, the functions $f,g\in C\left(J\times\mathbb{R}^{2},\mathbb{R}\right)$ and h is defined over J. To the best of our knowledge, this is the first time where such problem is investigated.

The structure of our paper is as follows: In Section 2, we will recall some preliminary related to fractional calculus and Caputo-Hadamard derivatives. In Section 3, we apply the integral inequality theory combined with the fixed point theory for study the questions of existence and uniqueness of solutions for the considered problem. In Section 4, three illustrative examples are presented and discussed in details.

2. Preliminaries on Fractional Calculus

In this section, we recall the basic definitions, properties and lemmas involving Caputo-Hadamard derivatives, for more details, one can consult the references $[15,16,18]$. We begin this section by the following definition:

Definition 2.1: The Hadamard fractional integral of order $\alpha>0$, for a function $f\in L^{1}(J)$, is defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s},$$

where

$$\Gamma(\alpha):=\int_{0}^{+\infty}e^{-s}s^{\alpha-1}ds.$$

Let

$$\delta = t\frac{d}{dt}, \alpha > 0, n = [\alpha] + 1,$$

with $[\alpha]$ denotes the integer part of a real number α. Define the space

$$AC_{\delta}^{n}([a,b]) = \left\{ f : [a,b] \to \mathbb{R}, \delta^{n-1} f \in AC[a,b] \right\}.$$

Definition 2.2: The Caputo-Hadamard fractional derivative of order α for a function $f \in AC_{\delta}^{n}([a,b],\mathbb{R})$ is defined by:

$${}_{H}^{C}D_{a^{+}}^{\alpha} f(t) = \tfrac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{n-\alpha-1} \left(t\frac{dt}{t}\right)^{n} f(s) \frac{ds}{s},$$

provided that the right-hand side integral exists.

Now, we recall the following lemmas:

Lemma 2.1: Let $\alpha, \beta > 0$ and $f \in L^{1}([a,b],\mathbb{R})$. Then $I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t)$ and $D^{\alpha} I^{\alpha} f(t) = f(t)$.

Lemma 2.2: Let $\beta > \alpha > 0$ and $f \in L^{1}([a,b],\mathbb{R})$. Then $D^{\alpha} I^{\beta} f(t) = I^{\beta-\alpha} f(t)$.

Lemma 2.3: Let $x \in AC_{\delta}^{n}([a,b],\mathbb{R})$, $n-1 < \alpha < n$. Then the Caputo–Hadamard fractional differential equation

$${}_{H}^{C}D^{\alpha} x(t) = 0,$$

has a solution

$$x(t) = \sum_{k=0}^{n-1} c_{k} \left(\log \frac{t}{a}\right)^{k}, t > a > 0,$$

and the following formula holds

$$I^{\alpha}\, {}_{H}^{C}D^{\alpha} x(t) = x(t) + \sum_{k=0}^{n-1} c_{k} \left(\log \frac{t}{a}\right)^{k},$$

where $c_{k} \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n-1$.

Before proving our main results, we introduce the following important result. It deals with the integral representation of the above considered problem. We have:

Lemma 2.4: Let us take a function $G \in C(J,\mathbb{R})$. Therefore, the unique integral solution of the problem

$$
\begin{cases}
{}_{H}^{C}D^{\beta}\left({}_{H}^{C}D^{\alpha}+\frac{A}{(\log t)^{\mu}}\right)x(t)=G(t),t\in\left]1,e\right[,\\
\\
\left({}_{H}^{C}D^{\alpha}+A\right)x(e)=0,x(e)=x(1)=\sum_{i=1}^{n}\lambda_{i}I^{\delta_{i}}x(\eta_{i}),\\
\\
0<\mu<\alpha<1,1<\beta<2,1<\eta_{i}<e,\ \delta_{i},\lambda_{i}>0,
\end{cases} \tag{2}
$$

is given by

$$
\begin{aligned}
x(t) \ :\ &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\\
&-\frac{1}{\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\sum_{i=1}^{n}\lambda_i\frac{1}{\Gamma(\alpha+\delta_i)}\int_0^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\delta_i-1}\\
&\times\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\\
&+\left[\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\sum_{i=0}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right.\right.\\
&\left.\left.-\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\times\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\\
&\times\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\\
&+\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}-\frac{\left(1-\frac{1}{\alpha}\right)(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}+\left(1-\frac{1}{\alpha}\right)\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right.\\
&\left.\times\frac{1}{\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\right)\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau},
\end{aligned} \tag{3}
$$

such that $\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}\neq 1$.

Proof: By applying Lemma 2.3, we can reduce (2) to the following equivalent integral problem:

$$
\begin{aligned}
x(t) \quad : \quad &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau}\right. \\
&\left. - \frac{A}{(\log s)^{\mu}} x(s)\right) \frac{ds}{s} - \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} c_1 - \frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)} c_2 - c_3, \qquad (4)
\end{aligned}
$$

for some real constants c_1, c_2, c_3.

Since $\left({}_H^C D^{\alpha} + A\right) x(e) = 0$, we get

$$
c_1 + c_2 = \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau}, \qquad (5)
$$

now, by using the condition $x(e) = x(1)$ and (5), we obtain

$$
\begin{aligned}
c_1 \quad &= \quad \frac{\Gamma(\alpha+2)}{\alpha \Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau}\right. \qquad (6) \\
&\left. - \frac{A}{(\log s)^{\mu}} x(s)\right) \frac{ds}{s} - \frac{1}{\alpha \Gamma(\beta)} \int_1^e \left(\log \frac{e}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau},
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 \quad &= \quad \frac{1 - \frac{1}{\alpha}}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{\Gamma(\alpha+2)}{\alpha \Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \\
&\times \left(\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^{\mu}} x(s)\right) \frac{ds}{s}. \qquad (7)
\end{aligned}
$$

Now, the fact that $x(1) = \sum\limits_{i=0}^{n} \lambda_i I^{\delta_i} x(\eta_i)$ will allow us to get:

$$
\begin{aligned}
c_3 \quad &= \quad \frac{1}{\sum\limits_{i=0}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1} \left[\sum_{i=1}^{n} \lambda_i \frac{1}{\Gamma(\alpha+\delta_i)} \int_0^{\eta_i} \left(\log \frac{\eta_i}{s}\right)^{\alpha+\delta_i-1}\right. \\
&\times \left(\left(\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} G(\tau) \frac{d\tau}{\tau} - \frac{A}{(\log s)^{\mu}} x(s)\right) \frac{ds}{s}\right. \qquad (8) \\
&\left. - c_1 \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)} - c_2 \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right].
\end{aligned}
$$

Substituting (6) and (7) in (8), we get

$$
\begin{aligned}
c_3 &= \frac{1}{\sum\limits_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\left\{\sum_{i=1}^{n}\lambda_i\frac{1}{\Gamma(\alpha+\delta_i)}\int_0^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\delta_i-1}\right.\\
&\quad\times\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\\
&\quad-\left[\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}-\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right]\\
&\quad\times\left[\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}\right.\right.\\
&\quad\left.\left.-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\right]-\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\\
&\quad\left.\times\frac{1-\frac{1}{\alpha}}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}\right\}.
\end{aligned}
\tag{9}
$$

Then, replacing c_1, c_2, c_3 in (4), we obtain (3).

3. Main Results

For computational convenience, we need to introduce the following notions:

Let $E=\left\{x : x\in C(J,\mathbb{R}) \text{ and } {}_{H}^{C}D^{\sigma}x(t)\in C(J,\mathbb{R})\ \right\}$.

The space $(E,\|x\|_E)$ is a Banach space endowed with the norm

$\|x\|_E=\max\left\{\|x\|_\infty,\left\|{}_{H}^{C}D^{\sigma}x\right\|_\infty\right\}$, such that

$\|x\|_\infty=\sup\limits_{t\in J}|x(t)|$ and $\left\|{}_{H}^{C}D^{\sigma}x\right\|_\infty=\sup\limits_{t\in J}\left|{}_{H}^{C}D^{\sigma}x(t)\right|$.

Now, we define $S:E\to E$ as follows:

$$
\begin{aligned}
Sx(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}\right.\\
&\quad\left.-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}-\frac{1}{\sum\limits_{i=0}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\sum_{i=1}^{n}\lambda_i\frac{1}{\Gamma(\alpha+\delta_i)}\\
&\quad\times\int_0^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\delta_i-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}\right.
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&-\frac{A}{(\log s)^{\mu}}x(s)\Bigg)\frac{ds}{s}+\left[\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\right.\\
&\left.\times\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}-\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right)\Bigg]\\
&\times\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\Bigg(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}\\
&-\frac{A}{(\log s)^{\mu}}x(s)\Bigg)\frac{ds}{s}+\Bigg(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}-\frac{\left(1-\frac{1}{\alpha}\right)(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}\\
&+\left(1-\frac{1}{\alpha}\right)\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\frac{1}{\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1}\Bigg)\\
&\times\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\beta-1}G(\tau)\frac{d\tau}{\tau}.
\end{aligned}
$$

We begin by taking into account the following hypotheses:

(H_1) : The functions f, g are continuous over $J\times\mathbb{R}^2$ and h is continuous over J.

(H_2) :There exist nonnegative constants M_i and $N_i, i=1,2$, such that for all $t\in J, x_i, y_i\in\mathbb{R}$:

$$
\begin{aligned}
|f(t,y_1,y_2)-f(t,x_1,x_2)| &\leq \sum_{i=1}^{2}M_i|y_i-x_i|,\\
|g(t,y_1,y_2)-g(t,x_1,x_2)| &\leq \sum_{i=1}^{2}N_i|y_i-x_i|,
\end{aligned}
$$

we put

$$\Delta_f=\max_{i=1,2}\{M_i\}\quad\text{and}\quad\Delta_g=\max_{i=1,2}\{N_i\}.$$

(H_3) : There exist nonnegative constants L_1, L_2, L_3, such that for all $t\in J, x_i\in\mathbb{R}, i=1,2$:

$$|f(t,x_1,x_2)|\leq L_1,\quad |g(t,x_1,x_2)|\leq L_2,\quad |h(t)|\leq L_3.$$

Setting the following quantities:

$$
\begin{aligned}
\Omega_1 \quad = \quad & \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|} \\
& + \left[\frac{\alpha+2}{\Gamma(\alpha+2)} + \frac{1}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|} \left(\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right.\right. \\
& \left.\left. + \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right] \frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)} + \left(\frac{\alpha+1+\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha+2)}\right. \\
& \left. + \left|1-\frac{1}{\alpha}\right| \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)} \frac{1}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|}\right) \\
& \times \frac{1}{\Gamma(\beta+1)},
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\Omega_2 = & \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)} + \frac{\sum_{i=1}^{n} \lambda_i \frac{\Gamma(1-\mu)(\log \eta_i)^{\alpha+\delta_i-\mu}}{\Gamma(\alpha+\delta_i-\mu+1)}}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|} + \left[\frac{\alpha+2}{\Gamma(\alpha+2)}\right. \\
& + \frac{1}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|} \left(\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right. \\
& \left.\left. + \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right] \frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\overline{\Omega}_1 \quad = \quad & \frac{1}{\Gamma(\alpha+\beta-\sigma+1)} + \left(\frac{1}{\Gamma(\alpha-\sigma+1)}\right. \\
& \left. + \frac{1}{\Gamma(\alpha-\sigma+2)}\right) \frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)} \\
& + \left(\frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha-\sigma+2)}\right) \frac{1}{\Gamma(\beta+1)},
\end{aligned} \tag{13}
$$

$$\overline{\Omega}_2 = \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu-\sigma+1)} + \left(\frac{1}{\Gamma(\alpha-\sigma+2)} + \frac{1}{\Gamma(\alpha-\sigma+1)}\right) \times \frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)}. \tag{14}$$

Now, we are in a good position to present our first uniqueness of solution for (1):

theorem 3.1: If (H_1) and (H_2) are satisfied and we have also:

$$\max\left\{\left(B\Delta_f+\Delta_g\right)\Omega_1 + A\Omega_2\,;\left(B\Delta_f+\Delta_g\right)\overline{\Omega}_1 + A\overline{\Omega}_2\right\} < 1. \tag{15}$$

Then, (1) has a unique solution on J.

Proof: We shall proceed to prove that S is contractive. For $x, y \in E$, we can write

$$\begin{aligned}
\|Sy-Sx\|_\infty \leq\ & \sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\right.\\
& \times\left[B\left|f\left(\tau,y(\tau),{}_H^C D^\sigma y(\tau)\right)-f\left(\tau,x(\tau),{}_H^C D^\sigma x(\tau)\right)\right|\right.\\
& \left.+\left|g\left(\tau,y(\tau),I^\rho y(\tau)\right)-g\left(\tau,x(\tau),I^\rho x(\tau)\right)\right|\right]\frac{d\tau}{\tau}\\
& \left.+\frac{A}{(\log s)^\mu}\left|y(s)-x(s)\right|\right)\frac{ds}{s}\\
& +\frac{1}{\left|\sum_{i=1}^n\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\sum_{i=1}^n\lambda_i\frac{1}{\Gamma(\alpha+\delta_i)}\\
& \times\int_1^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{\alpha+\delta_i-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
& \times\left[B\left|f\left(\tau,y(\tau),{}_H^C D^\sigma y(\tau)\right)-f\left(\tau,x(\tau),{}_H^C D^\sigma x(\tau)\right)\right|\right.\\
& \left.+\left|g\left(\tau,y(\tau),I^\rho y(\tau)\right)-g\left(\tau,x(\tau),I^\rho x(\tau)\right)\right|\right]\frac{d\tau}{\tau}\\
& \left.+\frac{A}{(\log s)^\mu}\left|y(s)-x(s)\right|\right)\frac{ds}{s}+\left[\left[\frac{(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}\right.\right.\\
& +\frac{1}{\left|\sum_{i=1}^n\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\left(\sum_{i=1}^n\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right.\\
& \left.\left.+\sum_{i=1}^n\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}
\end{aligned} \tag{16}$$

$$
\begin{aligned}
&\times\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\left[B\left|f\left(\tau,y(\tau),{}_H^CD^{\sigma}y(\tau)\right)-f\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)\right|\right.\\
&\left.+\left|g\left(\tau,y(\tau),I^{\rho}y(\tau)\right)-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\right|\right]\frac{d\tau}{\tau}\\
&\left.+\frac{A}{(\log s)^{\mu}}\left|y(s)-x(s)\right|\right)\frac{ds}{s}+\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right.\\
&+\frac{\left|1-\frac{1}{\alpha}\right|(\log t)^{\alpha+1}}{\Gamma(\alpha+2)}+\left|1-\frac{1}{\alpha}\right|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\\
&\left.\times\frac{1}{\left|\sum\limits_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right)\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\beta-1}\\
&\times\left[B\left|f\left(\tau,y(\tau),{}_H^CD^{\sigma}y(\tau)\right)-f\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)\right|\right.\\
&\left.\left.+\left|g\left(\tau,y(\tau),I^{\rho}y(\tau)\right)-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\right|\right]\frac{d\tau}{\tau}\right\},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|Sy-Sx\|_{\infty} \leq{}& \left(B\Delta_f+\Delta_g\right)\left\{\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{\sum\limits_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}}{\left|\sum\limits_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right.\\
&+\left[\frac{\alpha+2}{\Gamma(\alpha+2)}+\frac{1}{\left|\sum\limits_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right.\right.\\
&\left.\left.+\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)}\\
&+\left(\frac{\alpha+1+\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha+2)}+\left|1-\frac{1}{\alpha}\right|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right.\\
&\left.\left.\times\frac{1}{\left|\sum\limits_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right)\frac{1}{\Gamma(\beta+1)}\right\}\|y-x\|_E
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&+A\left\{\frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}+\frac{\sum_{i=1}^{n}\lambda_i\frac{\Gamma(1-\mu)(\log\eta_i)^{\alpha+\delta_i-\mu}}{\Gamma(\alpha+\delta_i-\mu+1)}}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right.\\
&+\left[\frac{\alpha+2}{\Gamma(\alpha+2)}+\frac{1}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right.\\
&\left.\times\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}+\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\\
&\left.\times\frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)}\right\}\|y-x\|_E\\
\leq\;&\left[\left(B\Delta_f+\Delta_g\right)\Omega_1+A\Omega_2\right]\|y-x\|_E.
\end{aligned}
$$

Also, one can see that

$$
\begin{aligned}
D^{\sigma}Sx(t) \;=\;& \frac{1}{\Gamma(\alpha-\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\sigma-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\Big[h(\tau)-Bf\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)\\
&\left.-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\Big]\frac{d\tau}{\tau}-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\\
&+\left(\frac{(\log t)^{\alpha-\sigma+1}}{\Gamma(\alpha-\sigma+2)}-\frac{(\log t)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}\right)\\
&\times\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\Big[h(\tau)-Bf\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\Big]\frac{d\tau}{\tau}\\
&\left.-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}+\left(\frac{(\log t)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}-\frac{\left(1-\frac{1}{\alpha}\right)(\log t)^{\alpha-\sigma+1}}{\Gamma(\alpha-\sigma+2)}\right)\\
&\times\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\beta-1}\Big[h(\tau)-Bf\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)\\
&-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\Big]\frac{d\tau}{\tau}.
\end{aligned}
\tag{18}
$$

We have also

$$
\begin{aligned}
\|D^{\sigma}Sy-D^{\sigma}Sx\|_{\infty} \leq\ & \sup_{t\in J}\Bigg\{\frac{1}{\Gamma(\alpha-\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-\sigma-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\Big[B\left|f\left(\tau,y(\tau),{}_H^C D^{\sigma}y(\tau)\right)-f\left(\tau,x(\tau),{}_H^C D^{\sigma}x(\tau)\right)\right|\\
&+|g(\tau,y(\tau),I^{\rho}y(\tau))-g(\tau,x(\tau),I^{\rho}x(\tau))|\Big]\frac{d\tau}{\tau}\\
&\left.+\frac{A}{(\log s)^{\mu}}|y(s)-x(s)|\right)\frac{ds}{s}+\left(\frac{(\log t)^{\alpha-\sigma+1}}{\Gamma(\alpha-\sigma+2)}+\frac{(\log t)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}\right)\\
&\times\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\Big[B\left|f\left(\tau,y(\tau),{}_H^C D^{\sigma}y(\tau)\right)-f\left(\tau,x(\tau),{}_H^C D^{\sigma}x(\tau)\right)\right|\\
&+|g(\tau,y(\tau),I^{\rho}y(\tau))-g(\tau,x(\tau),I^{\rho}x(\tau))|\Big]\frac{d\tau}{\tau}\\
&\left.+\frac{A}{(\log s)^{\mu}}|y(s)-x(s)|\right)\frac{ds}{s}+\left(\frac{(\log t)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}\right.\\
&\left.+\frac{\left|1-\frac{1}{\alpha}\right|(\log t)^{\alpha-\sigma+1}}{\Gamma(\alpha-\sigma+2)}\right)\frac{1}{\Gamma(\beta)}\int_1^e\left(\log\frac{e}{\tau}\right)^{\beta-1}\\
&\times\Big[B\left|f\left(\tau,y(\tau),{}_H^C D^{\sigma}y(\tau)\right)-f\left(\tau,x(\tau),{}_H^C D^{\sigma}x(\tau)\right)\right|\\
&+|g(\tau,y(\tau),I^{\rho}y(\tau))-g(\tau,x(\tau),I^{\rho}x(\tau))|\Big]\frac{d\tau}{\tau}\Bigg\}\\
\leq\ & \left(B\Delta_f+\Delta_g\right)\left[\frac{1}{\Gamma(\alpha+\beta-\sigma+1)}+\left(\frac{1}{\Gamma(\alpha-\sigma+1)}\right.\right.\\
&\left.+\frac{1}{\Gamma(\alpha-\sigma+2)}\right)\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)}\\
&\left.+\left(\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha-\sigma+2)}\right)\frac{1}{\Gamma(\beta+1)}\right]\|y-x\|_E\\
&+A\left[\frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu-\sigma+1)}+\left(\frac{1}{\Gamma(\alpha-\sigma+2)}+\frac{1}{\Gamma(\alpha-\sigma+1)}\right)\right.\\
&\left.\times\frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)}\right]\|y-x\|_E\\
\leq\ & \left[\left(B\Delta_f+\Delta_g\right)\overline{\Omega}_1+A\overline{\Omega}_2\right]\|y-x\|_E .
\end{aligned}
\tag{19}
$$

Thus, we obtain the inequality:

$$\|Sy - Sx\|_{\infty} \leq \max\left\{\left(B\Delta_f + \Delta_g\right)\Omega_1 + A\Omega_2 \,;\; \left(B\Delta_f + \Delta_g\right)\overline{\Omega}_1 + A\overline{\Omega}_2\right\}\|y - x\|_E\,. \tag{20}$$

We conclude that S is contractive. As a consequence of Banach contraction principle, we deduce that S has a unique fixed point which is the exact solution of (1).

Now, we shall use the following lemma to prove the second main result.

Lemma 3.1: Let E be a Banach space and $S : E \to E$ be a completely continuous operator. If one has bounded the set

$$F = \{x \in E : x = \mu Sx,\ 0 < \mu < 1\},$$

then S has a fixed point in E.

Theorem 3.2: Assume that the hypotheses (H_1) and (H_3) are satisfied. Then, (1) has at least one solution defined over J.

Proof: We need the bounded ball: $B_r = \{x \in E : \|x\|_E \leq r\}$ and we proceed as follows:

Step 1: The application S is continuous on E.

The proof is trivial and then it can be omitted.

Step 2: Uniform boundedness: For all $x \in C_r$ and by (H_3) we have

$$\begin{aligned}
\|Sx\|_{\infty} \leq\ & [BL_1 + L_2 + L_3]\left\{\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|}\right. \\
& + \left[\frac{\alpha+2}{\Gamma(\alpha+2)} + \frac{1}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|}\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right.\right. \\
& \left.\left. + \sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)} \\
& + \left(\frac{\alpha+1+\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha+2)} + \left|1 - \frac{1}{\alpha}\right|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right.
\end{aligned}$$

$$
\begin{aligned}
& \times \frac{1}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|}\Bigg) \frac{1}{\Gamma(\beta+1)} \Bigg\} \\
& + Ar \left\{ \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)} + \frac{\sum_{i=1}^{n} \lambda_i \frac{\Gamma(1-\mu)(\log \eta_i)^{\alpha+\delta_i-\mu}}{\Gamma(\alpha+\delta_i-\mu+1)}}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|} \right. \\
& + \left[\frac{\alpha+2}{\Gamma(\alpha+2)} + \frac{1}{\left|\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\delta_i}}{\Gamma(\delta_i+1)} - 1\right|} \right. \\
& \left. \times \left(\sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)} + \sum_{i=1}^{n} \lambda_i \frac{(\log \eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)} \right) \right] \\
& \left. \times \frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)} \right\} \\
\leq \quad & (BL_1 + L_2 + L_3)\,\Omega_1 + Ar\Omega_2 < +\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\|D^{\sigma} Sx\|_{\infty} \leq \quad & (BL_1 + L_2 + L_3) \left[\frac{1}{\Gamma(\alpha+\beta-\sigma+1)} + \left(\frac{1}{\Gamma(\alpha-\sigma+1)} \right. \right. \\
& \left. + \frac{1}{\Gamma(\alpha-\sigma+2)} \right) \frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)} \\
& \left. + \left(\frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha-\sigma+2)} \right) \frac{1}{\Gamma(\beta+1)} \right] \\
& + Ar \frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu-\sigma+1)} + \left(\frac{1}{\Gamma(\alpha-\sigma+2)} \right. \\
& \left. + \frac{1}{\Gamma(\alpha-\sigma+1)} \right) \frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)} \Bigg] \\
\leq \quad & (BL_1 + L_2 + L_3)\,\overline{\Omega}_1 + Ar\overline{\Omega}_2 < +\infty.
\end{aligned} \tag{21}
$$

Hence, for any $x \in C_r$, we obtain $\|Sx\|_E < +\infty$, which means in particular that the operator S is uniformly bounded on C_r.

Step 3: The application S maps bounded sets into equicontinuous sets of E:

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and let C_r be the above bounded set of E, for all $x \in C_r$, we have

$$
\begin{aligned}
|Sx(t_2) - Sx(t_1)| \leq\ & \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_1^{s} \left(\log \frac{s}{\tau}\right)^{\beta-1} \right.\right. \\
& \times \left[h(\tau) - Bf\left(\tau, x(\tau), {}_H^C D^{\sigma} x(\tau)\right) - g\left(\tau, x(\tau), I^{\rho} x(\tau)\right)\right] \frac{d\tau}{\tau} \\
& \left. - \frac{A}{(\log s)^{\mu}} x(s) \right) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} \\
& \times \left(\frac{1}{\Gamma(\beta)} \int_1^{s} \left(\log \frac{s}{\tau}\right)^{\beta-1} \left[h(\tau) - Bf\left(\tau, x(\tau), {}_H^C D^{\sigma} x(\tau)\right) \right.\right. \\
& \left.\left. - g\left(\tau, x(\tau), I^{\rho} x(\tau)\right)\right] \frac{d\tau}{\tau} - \frac{A}{(\log s)^{\mu}} x(s) \right) \frac{ds}{s} \\
& + \left[\frac{(\log t_2)^{\alpha+1} - (\log t_1)^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{(\log t_1)^{\alpha} - (\log t_2)^{\alpha}}{\Gamma(\alpha+1)} \right] \\
& \times \frac{\Gamma(\alpha+2)}{\alpha \Gamma(\alpha)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_1^{s} \left(\log \frac{s}{\tau}\right)^{\beta-1} \right. \\
& \times \left[h(\tau) - Bf\left(\tau, x(\tau), {}_H^C D^{\sigma} x(\tau)\right) - g\left(\tau, x(\tau), I^{\rho} x(\tau)\right)\right] \frac{d\tau}{\tau} \\
& \left. - \frac{A}{(\log s)^{\mu}} x(s) \right) \frac{ds}{s} + \left(\frac{(\log t_2)^{\alpha} - (\log t_1)^{\alpha}}{\Gamma(\alpha+1)} \right. \\
& \left. + \frac{\left(1-\frac{1}{\alpha}\right)\left[(\log t_1)^{\alpha+1} - (\log t_2)^{\alpha+1}\right]}{\Gamma(\alpha+2)} \right) \frac{1}{\Gamma(\beta)} \int_1^{e} \left(\log \frac{e}{\tau}\right)^{\beta-1} \qquad (22) \\
& \left. \times \left[h(\tau) - Bf\left(\tau, x(\tau), {}_H^C D^{\sigma} x(\tau)\right) - g\left(\tau, x(\tau), I^{\rho} x(\tau)\right)\right] \frac{d\tau}{\tau} \right| \\
\leq\ & [BL_1 + L_2 + L_3] \left\{ \left[\left| (\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right| \right.\right. \\
& + \left(\frac{\left| (\log t_2)^{\alpha+1} - (\log t_1)^{\alpha+1} \right|}{\Gamma(\alpha+2)} \right. \\
& \left.\left. + \frac{\left| (\log t_1)^{\alpha} - (\log t_2)^{\alpha} \right|}{\Gamma(\alpha+1)} \right) \frac{\Gamma(\alpha+2)}{\alpha} \right] \frac{1}{\Gamma(\alpha+\beta+1)} \\
& + \left(\frac{\left| (\log t_2)^{\alpha} - (\log t_1)^{\alpha} \right|}{\Gamma(\alpha+1)} \right. \\
& \left.\left. + \frac{\left|1-\frac{1}{\alpha}\right| \left| (\log t_1)^{\alpha+1} - (\log t_2)^{\alpha+1} \right|}{\Gamma(\alpha+2)} \right) \frac{1}{\Gamma(\beta+1)} \right\} \\
& + Ar \left[\left| (\log t_2)^{\alpha-\mu} - (\log t_1)^{\alpha-\mu} \right| \right. \\
& + \left(\frac{\left| (\log t_2)^{\alpha+1} - (\log t_1)^{\alpha+1} \right|}{\Gamma(\alpha+2)} \right.
\end{aligned}
$$

$$+\frac{\left|(\log t_1)^{\alpha}-(\log t_2)^{\alpha}\right|}{\Gamma(\alpha+1)}\Bigg)\Bigg]\frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}.$$

Similarly as before, we have

$$\begin{aligned}
|D^{\sigma}Sx(t_2)-D^{\sigma}Sx(t_1)| \leq\ & \Bigg|\frac{1}{\Gamma(\alpha-\sigma)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-\sigma-1}\left(\frac{1}{\Gamma(\beta)}\int_1^{s}\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\left[h(\tau)-Bf\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\right]\frac{d\tau}{\tau}\\
&\left.-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha_i-\sigma_i)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha_i-\sigma_i-1}\left(\frac{1}{\Gamma(\beta_i)}\int_1^{s}\left(\log\frac{s}{\tau}\right)^{\beta_i-1}\right.\\
&\times\left[h(\tau)-Bf\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\right]\frac{d\tau}{\tau}\\
&\left.-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s} \qquad (23)\\
&+\left[\frac{(\log t_2)^{\alpha-\sigma+1}-(\log t_1)^{\alpha-\sigma+1}}{\Gamma(\alpha-\sigma+2)}+\frac{(\log t_1)^{\alpha-\sigma}-(\log t_2)^{\alpha-\sigma}}{\Gamma(\alpha-\sigma+1)}\right]\\
&\times\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\left(\frac{1}{\Gamma(\beta)}\int_1^{s}\left(\log\frac{s}{\tau}\right)^{\beta-1}\right.\\
&\times\left[h(\tau)-Bf\left(\tau,x(\tau),{}_H^CD^{\sigma}x(\tau)\right)-g\left(\tau,x(\tau),I^{\rho}x(\tau)\right)\right]\frac{d\tau}{\tau}\\
&\left.-\frac{A}{(\log s)^{\mu}}x(s)\right)\frac{ds}{s}\Bigg|\\
\leq\ & [BL_1+L_2+L_3]\Bigg[\frac{\left|(\log t_2)^{\alpha+\beta-\sigma}-(\log t_1)^{\alpha+\beta-\sigma}\right|}{\Gamma(\alpha+\beta-\sigma+1)}\\
&+\left(\frac{\left|(\log t_2)^{\alpha-\sigma+1}-(\log t_1)^{\alpha-\sigma+1}\right|}{\Gamma(\alpha-\sigma+2)}+\frac{\left|(\log t_1)^{\alpha-\sigma}-(\log t_2)^{\alpha-\sigma}\right|}{\Gamma(\alpha-\sigma+1)}\right)\\
&\times\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)}\Bigg]\\
&+Ar\Bigg[\frac{\Gamma(1-\mu)\left|(\log t_2)^{\alpha-\sigma-\mu}-(\log t_1)^{\alpha-\sigma-\mu}\right|}{\Gamma(\alpha-\sigma-\mu+1)}\\
&+\left(\frac{\left|(\log t_2)^{\alpha-\sigma+1}-(\log t_1)^{\alpha-\sigma+1}\right|}{\Gamma(\alpha-\sigma+2)}+\frac{\left|(\log t_1)^{\alpha-\sigma}-(\log t_2)^{\alpha-\sigma}\right|}{\Gamma(\alpha-\sigma+1)}\right)\\
&\times\frac{\Gamma(\alpha+2)}{\alpha}\frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}\Bigg].
\end{aligned}$$

The right hand sides of (22) and (23) tend to zero independently of (u_1,u_2) as $t_1 \to t_2$.

As a consequence of Steps $1,2$ and 3, thanks to Ascoli-Arzela theorem, we conclude that S is completely continuous.

Step 4: The set

$$F=\{x\in X : x=\omega Sx,\ 0<\omega<1\}$$

is bounded.

Let $x\in F$, then, we have $x=\omega Sx$ for some $0<\omega<1$. Hence, we can write

$$
\begin{aligned}
\|\omega Sx\|_\infty \leq\ & \omega\left[BL_1+L_2+L_3\right]\left\{\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\beta+\delta_i}}{\Gamma(\alpha+\beta+\delta_i+1)}}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right.\\
&+\left[\frac{\alpha+2}{\Gamma(\alpha+2)}+\frac{1}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}\right.\right.\\
&\left.\left.+\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)}\\
&+\left(\frac{\alpha+1+\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha+2)}+\left|1-\frac{1}{\alpha}\right|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right.\\
&\left.\left.\times\frac{1}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right)\frac{1}{\Gamma(\beta+1)}\right\}\\
&+\omega Ar\left\{\frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu+1)}+\frac{\sum_{i=1}^{n}\lambda_i\frac{\Gamma(1-\mu)(\log\eta_i)^{\alpha+\delta_i-\mu}}{\Gamma(\alpha+\delta_i-\mu+1)}}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right.\\
&+\left[\frac{\alpha+2}{\Gamma(\alpha+2)}+\frac{1}{\left|\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\delta_i}}{\Gamma(\delta_i+1)}-1\right|}\right.\\
&\left.\times\left(\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i}}{\Gamma(\alpha+\delta_i+1)}+\sum_{i=1}^{n}\lambda_i\frac{(\log\eta_i)^{\alpha+\delta_i+1}}{\Gamma(\alpha+\delta_i+2)}\right)\right]\\
&\left.\times\frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)}\right\}
\end{aligned}
\tag{24}
$$

$$\leq \quad \omega\left[\left(BL_1+L_2+L_3\right)\Omega_1+Ar\Omega_2\right]<+\infty,$$

and

$$\begin{aligned}
\|D^{\sigma}\omega Sx\|_{\infty} \quad \leq \quad & \omega\left(BL_1+L_2+L_3\right)\left[\frac{1}{\Gamma(\alpha+\beta-\sigma+1)}+\left(\frac{1}{\Gamma(\alpha-\sigma+1)}\right.\right. \\
& \left.+\frac{1}{\Gamma(\alpha-\sigma+2)}\right)\frac{\Gamma(\alpha+2)}{\alpha\Gamma(\alpha+\beta+1)}+\left(\frac{1}{\Gamma(\alpha-\sigma+1)}\right. \\
& \left.\left.+\frac{\left|1-\frac{1}{\alpha}\right|}{\Gamma(\alpha-\sigma+2)}\right)\frac{1}{\Gamma(\beta+1)}\right]+\omega Ar\left[\frac{\Gamma(1-\mu)}{\Gamma(\alpha-\mu-\sigma+1)}\right. \\
& \left.+\left(\frac{1}{\Gamma(\alpha-\sigma+2)}+\frac{1}{\Gamma(\alpha-\sigma+1)}\right)\frac{\Gamma(\alpha+2)\Gamma(1-\mu)}{\alpha\Gamma(\alpha-\mu+1)}\right] \\
\leq \quad & \omega\left[\left(BL_1+L_2+L_3\right)\overline{\Omega}_1+Ar\overline{\Omega}_2\right]<+\infty.
\end{aligned} \tag{25}$$

From (24) and (25), we see that $\|x\|_E<+\infty$. Consequently, F is bounded.

As a consequence of the above Schaefer theorem, we conclude that S has a fixed point which is a solution of (1).

4. Examples

We discuss thee examples. We begin by considering the problem:

Example 4.1:

$$\begin{cases}
{}_H^CD^{1.7}\left({}_H^CD^{0.54}+\frac{10^{-2}}{(\log t)^{0.5}}\right)x(t)+\frac{\left|t-{}_H^CD^{0.4}x(t)\right|}{30\pi^4e^{2t+1}\left(1+\left|t-{}_H^CD^{0.4}x(t)\right|\right)} \\
+\frac{\sin\left(t+x(t)+I^{0.3}x(t)\right)}{39e^t}=e^{\frac{1}{2}t+1}, t\in\,]1,e[, \\
\left({}_H^CD^{0.54}+3\right)x(e)=0, x(1)=x(e)=\sum_{i=1}^{2}\lambda_iI^{\delta_i}x(\eta_i),
\end{cases} \tag{26}$$

where $\beta=1.7, \alpha=0.54, \mu=0.5, \sigma=0.4, \rho=0.3, \delta_1=0.2, \delta_2=0.25, A=10^{-2}, B=\frac{1}{30\pi^4}, \lambda_1=1.5, \lambda_2=2, \eta_1=2, \eta_2=2.33,$

and $\left(\frac{1}{30\pi^4}\times\frac{1}{e^3}+\frac{1}{99e}\right)3.7966+10^{-2}\times10.461 : 0.11878$

$$\begin{aligned}
f\left(t,x(t),{}_H^CD^{0.5}x(t)\right) &= \frac{\left|t-{}_H^CD^{0.4}x(t)\right|}{e^{2t+1}\left(1+\left|t-{}_H^CD^{0.4}x(t)\right|\right)}, \\
g\left(t,x(t),I^{0.4}x(t)\right) &= \frac{\sin\left(t+x(t)+I^{0.3}x(t)\right)}{39e^t},
\end{aligned}$$

we have: $\Delta_f = \frac{1}{e^3}, \Delta_g = \frac{1}{99e}$. Since, it is found that $\Omega_1 = 5.6332, \Omega_2 = 20.653, \overline{\Omega}_1 = 3.7966, \overline{\Omega}_2 = 10.461$ and $\left(B\Delta_f + \Delta_g\right)\Omega_1 + A\Omega_2 = 0.22756$, $\left(B\Delta_f + \Delta_g\right)\overline{\Omega}_1 + A\overline{\Omega}_2 = 0.11878$. Note that

$$\max\left\{\left(B\Delta_f + \Delta_g\right)\Omega_1 + A\Omega_2\ ;\ \left(B\Delta_f + \Delta_g\right)\overline{\Omega}_1 + A\overline{\Omega}_2\right\} = 0.22756 < 1.$$

Thus, Theorem 3.1 implies that the problem (26) has a unique solution on $[1, e]$.

Next example illustrates Theorem 3.2

Example 4.2: Consider the Caputo-Hadamard fractional problem:

$$\begin{cases} {}_H^CD^{1.8}\left({}_H^CD^{0.7} + \frac{7^{-2}}{(\log t)^{0.6}}\right)x(t) + \frac{1}{4e^3}\frac{\sin\left({}_H^CD^{0.5}x(t)+2t+1\right)}{3t^3+1} \\ +\frac{\cos\left(x(t)-I^{0.4}x(t)+t\right)}{5\log(3t+1)} = \frac{1}{t(t+2)^4}, t \in]1, e[, \\ \left({}_H^CD^{0.7} + 7^{-2}\right)x(e) = 0, x(1) = x(e) = \sum\limits_{i=1}^{2}\lambda_i I^{\delta_i}x(\eta_i), \end{cases} \tag{27}$$

where $\beta = 1.8, \alpha = 0.7, \mu = 0.6, \sigma = 0.5, \rho = 0.4, \delta_1 = 0.2, \delta_2 = 0.3, A = 7^{-2}, B = \frac{1}{4e^3}, \lambda_1 = 1, \lambda_2 = 2, \eta_1 = 2.1, \eta_2 = 2.4, r = 0.99$ and

$$\begin{aligned} f\left(t, x(t), {}_H^CD^{0.5}x(t)\right) &= \frac{\sin\left({}_H^CD^{0.5}x(t) + 2t + 1\right)}{3t^3 + 1}, \\ g\left(t, x(t), I^{0.4}x(t)\right) &= \frac{\cos\left(x(t) - I^{0.4}x(t) + t\right)}{5\log(3t+1)}, h(t) = \frac{1}{t(t+2)^4}, \end{aligned}$$

We have: $L_1 = \frac{1}{4}, L_2 = \frac{1}{10\log 2}, L_3 = \frac{1}{81}$. So, we obtain: $\Omega_1 = 3.7053, \Omega_2 = 23.155, \overline{\Omega}_1 = 2.7075, \overline{\Omega}_2 = 11.763$ and $\|Sx\|_\infty \le 1.0597, \left\|D^{0.5}Sx\right\|_\infty \le 0.67012, \|\omega Sx\|_\infty \le 1.0597\omega, \|D^\sigma \omega Sx\|_\infty \le 0.67012\omega : 0 < \omega < 1$. Hence, all conditions of theorem 3.2 are holds true, witch implies that the problem (27) has at least one solution on $[1, e]$.

Example 4.3: Consider the following third problem

$$\begin{cases} {}_H^CD^{\frac{3}{2}}\left({}_H^CD^{\frac{1}{2}} + \frac{1}{(\log t)^{\frac{1}{4}}}\right)x(t) + \frac{1}{4+e}\left(\frac{{}_H^CD^{\frac{1}{4}}}{1 + {}_H^CD^{\frac{1}{4}}}\right) \\ +\frac{1}{\sqrt{25+t^2}}\left(\sin x(t) + \frac{|x|}{1+|x|} + I^{\frac{3}{4}}x(t)\right) = \frac{e^{-t}}{3\sqrt{t^2+9}}, \quad t \in]1, e[, \\ \left({}_H^CD^{\frac{1}{2}} + 1\right)x(e) = 0, x(1) = x(e) = \sum\limits_{i=1}^{2}\lambda_i I^{\delta_i}x(\eta_i), \end{cases} \tag{28}$$

For this third example, we note that $\beta = \dfrac{3}{2}, \quad \alpha = \dfrac{1}{2}, \quad \mu = \dfrac{1}{4}, \quad \rho = \dfrac{3}{4}, \quad \delta_1 = \delta_2 = 0,1,$
$A = 1, \quad B = \dfrac{1}{4+e},$

and

$$
\begin{aligned}
f\left(t,x(t),{}_{H}^{C}D^{\frac{1}{2}}x(t)\right) &= \frac{{}_{H}^{C}D^{\frac{1}{4}}}{1+{}_{H}^{C}D^{\frac{1}{4}}}, \\
g\left(t,x(t),I^{0.4}x(t)\right) &= \frac{1}{\sqrt{25+t^{2}}}\left(sinx(t)+\frac{|x|}{1+|x|}+I^{\frac{3}{4}}x(t)\right), \\
h(t) = \frac{e^{-t}}{3\sqrt{t^{2}+9}}.
\end{aligned}
\tag{29}
$$

It's clear f,g and h are continuous and bounded functions. Thus the conditions of Theorem 3.2 are valid, then the above example 4.3 has at least on solution on $[1,e]$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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