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SOME PROPERTIES OF THE LOC-TAI EXTENSION OF THE GAMMA FUNCTION

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Abstract. In this paper, we derive digamma and polygamma functions for the Loc – Tai extension of the Gamma function. Some inequalities associated with the derived functions are also proved.

Keywords: Loc-Tai gamma function; digamma and polygamma functions; inequality.

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1. Introduction

The Gamma function $\Gamma(s)$ is defined for $s \in \mathbb{R}^+$ as (see [1, 2])

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{1}$$

It was introduced by Leonhard Euler, a renown mathematician. His aim was to generalize the factorial function $f(n) = n!$, $n = 1, 2, 3, ...$ to non integer values.

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The digamma function $\psi(s)$ and the polygamma function $\psi^m(s)$ are defined respectively for $s>0$ and $m\in\mathbb{N}$ as (see [3, 4])

$$\psi(s)=-\gamma-\frac{1}{s}+\sum_{n=1}^{\infty}\frac{s}{n(n+s)}=-\gamma+\int_0^{\infty}\frac{e^{-t}-e^{-st}}{1-e^{-t}}dt. \tag{2}$$

$$\psi^m(s)=(-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+s)^{m+1}}=(-1)^{m+1}\int_0^{\infty}\frac{t^me^{-st}}{1-e^{-t}}dt, \tag{3}$$

where

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)$$

is the Euler-Mascheroni constant.

The Loc - Tai extension of the gamma function (see [5]) is defined for $s>0$ and $k\in\mathbb{N}$ as

$$\Gamma_{t^k}(s)=\int_0^{\infty}e^{-t}t^{k(s-1)}dt. \tag{4}$$

It also satisfies the following basic properties:

1. $\Gamma_{t^k}(1)=1$
2. $\Gamma_t(s)=\Gamma(s)$
3. $\Gamma_{t^k}(s)=\Gamma\left(k(s-1)+1\right)$.

2. Preliminaries

In [5], the authors established the following results:

Lemma 2.1. *Let $s>0$ and $k\in\mathbb{N}$. Then*

$$\frac{1}{\Gamma_{t^k}(s)}=(k(s-1)+1)\,e^{\gamma(k(s-1)+1)}\prod_{i=1}^{\infty}\left(1+\frac{(k(s-1)+1)}{i}\right)e^{-\frac{(k(s-1)+1)}{i}}, \tag{5}$$

where γ is the Euler-Mascheroni constant.

Lemma 2.2. Let $s>0$ and $k\in\mathbb{N}$. Then

$$\Gamma_{t^k}(s)=\lim_{n\to\infty}\frac{n!n^{k(s-1)+1}}{\prod_{i=0}^{n}(k(s-1)+i+1)}. \tag{6}$$

The following results are also well known in the literature (see [7, 8]).

Lemma 2.3 (Hölder's inequality). Let $p > 1$ and $q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let f and g be continuous functions on $[a,b]$. Then

$$\int_0^\infty f(t)g(t)dt \leq \left(\int_0^\infty f(t)^p dt\right)^{\frac{1}{p}} \left(\int_0^\infty g(t)^q dt\right)^{\frac{1}{q}}. \tag{7}$$

Lemma 2.4 (Minkowski's inequality). Let $p > 1$ and f and g be continuous functions on $[a,b]$. Then

$$\left(\int_0^\infty |f(t)+g(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_0^\infty |g(t)|^p dt\right)^{\frac{1}{p}}. \tag{8}$$

Lemma 2.5. Let $p,q \geq 0$ and $k \geq 1$. Then the inequality

$$p^k+q^k \leq (p+q)^k \tag{9}$$

holds.

Definition 2.6. A function $f : I \to (0,\infty)$ is said to be log convex if $\ln f$ is convex on I. That is

$$\ln f(\alpha x+\beta y) \leq \alpha \ln f(x)+\beta \ln f(y), \tag{10}$$

or equivalently,

$$f(\alpha x+\beta y) \leq (f(x))^\alpha (f(y))^\beta, \tag{11}$$

for each $x,y \in I$ and $\alpha,\beta \in (0,1)$ such that $\alpha+\beta=1$.

The main aim of this paper is to derive digamma and polygamma functions for the Loc – Tai extension of the Gamma function, and prove some associated inequalities.

3. Main Results

Proposition 3.1. Let $t,s > 0$ and $k \in \mathbb{N}$. Then the digamma function $\psi_{t^k}(s)$ has the following series representation:

$$\psi_{t^k}(s) = -k\left(\frac{1}{k(s-1)+1}+\gamma\right) - k\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{k(s-1)+i+1}\right). \tag{12}$$

Proof. By relation (5), we have

$$\frac{1}{\Gamma_{t^k}(s)} = (k(s-1)+1)\, e^{\gamma(k(s-1)+1)} \prod_{i=1}^{\infty} \left(1+\frac{k(s-1)+1}{i}\right) e^{-\frac{k(s-1)+1}{i}}$$

where γ is the Euler-Mascheroni constant.

By taking log on both sides, we have

$$\ln \Gamma_{t^k}(s) = -\ln(k(s-1)+1) - \gamma k(s-1)+1 - \sum_{i=1}^{\infty} \ln\left(1+\frac{k(s-1)+1}{i}\right) + \sum_{i=1}^{\infty} \frac{k(s-1)+1}{i}.$$

Now

$$\begin{aligned}
\psi_{t^k}(s) &= \frac{d}{ds}\left(\ln \Gamma_{t^k}(s)\right) \\
&= -\frac{k}{k(s-1)+1} - \gamma k - \sum_{i=1}^{\infty}\left(\frac{k}{k(s-1)+i+1}\right) + \sum_{i=1}^{\infty}\frac{k}{i} \\
&= -\frac{k}{k(s-1)+1} - \gamma k - k\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{k(s-1)+i+1}\right) \\
&= -k\left(\frac{1+\gamma}{k(s-1)+1}\right) - k\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{k(s-1)+i+1}\right).
\end{aligned}$$

□

Alternatively, we have

Proof. By relation (6), we have

$$\Gamma_{t^k}(s) = \lim_{n\to\infty} \frac{n!\, n^{k(s-1)+1}}{\prod_{i=0}^{n}(k(s-1)+i+1)}.$$

By taking log on both sides, we have

$$\ln \Gamma_{t^k}(s) = \lim_{n\to\infty}\left(\ln n! + (k(s-1)+1)\ln n - \sum_{i=0}^{n} \ln(k(s-1)+i+1)\right).$$

Differentiating with respect to s yields

$$\begin{aligned}\psi_{t^k}(s) &= \lim_{n\to\infty}\left(k\ln n-\sum_{i=0}^{n}\frac{k}{(k(s-1)+i+1}\right)\\ &= k\lim_{n\to\infty}\ln n-\sum_{i=0}^{\infty}\frac{k}{k(s-1)+i+1}.\\ &= k\lim_{n\to\infty}\ln n-\frac{k}{k(s-1)+1}-\sum_{i=1}^{\infty}\frac{k}{k(s-1)+i+1}.\end{aligned}$$

Replacing $\lim_{n\to\infty}\ln n$ with $\sum_{i=1}^{\infty}(\frac{1}{i}-\gamma)$ in the above expression gives

$$\psi_{t^k}(s) = -k\left(\frac{1}{k(s-1)+1}+\gamma\right)-k\left(\sum_{i=1}^{\infty}\frac{1}{i}-\frac{1}{k(s-1)+i+1}\right).$$

□

Proposition 3.2. Let $t,s>0$ and $k\in\mathbb{N}$. Then the digamma function $\psi_{t^k}(s)$ has the integral representation

$$\psi_{t^k}(s) = -k\left(\frac{1}{k(s-1)+1}+\gamma\right)-k\int_0^{\infty}\left(\frac{1-e^{-(k(s-1)+1)x}}{1-e^{-ix}}\right)dx \tag{13}$$

Proof. From (12), we have

$$\psi_{t^k}(s) = -k\left(\frac{1}{k(s-1)+1}+\gamma\right)-k\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{k(s-1)+i+1}\right)$$

$$= -\frac{k}{k(s-1)+1}-\gamma k-k\sum_{i=1}^{\infty}\int_0^{\infty}\left(e^{-ix}-e^{-(k(s-1)+i+1)x}\right)dx$$

$$= -\frac{k}{k(s-1)+1}-\gamma k-k\int_0^{\infty}\sum_{i=1}^{\infty}\left(e^{-ix}-e^{-(k(s-1)+i+1)x}\right)dx$$

$$= -\frac{k}{k(s-1)+1}-\gamma k-k\int_0^{\infty}\sum_{i=1}^{\infty}e^{-ix}\left(1-e^{-(k(s-1)+1)x}\right)dx$$

$$= -\frac{k}{k(s-1)+1} - \gamma k - k\int_0^\infty \left(1 - e^{-(k(s-1)+1)x}\right)\sum_{i=1}^{\infty} e^{-ix}dx$$

$$= -k\left(\frac{1}{k(s-1)+1} + \gamma\right) - k\int_0^\infty \left(\frac{1 - e^{-(k(s-1)+1)x}}{1 - e^{-ix}}\right)dx.$$

□

Proposition 3.3. Let $s > 0$ and $k, m \in \mathbb{N}$. Then the polygamma function $\psi_{t^k}^{m}(s)$ has the series representation

$$\psi_{t^k}^{m}(s) = (-1)^{m+1} m! \sum_{i=0}^{\infty} \frac{k^{m+1}}{(k(s-1)+i+1)^{m+1}}. \tag{14}$$

Proof. From (12), we have

$$\psi_{t^k}(s) = -\frac{k}{k(s-1)+1} - \gamma k - k\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{k(s-1)+i+1}\right). \tag{15}$$

By differentiating (15) termwisely with respect to s, we have

$$\psi_{t^k}^{(1)}(s) = \frac{d}{ds}\left(\psi_{t^k}(s)\right) = \frac{k^2}{(k(s-1)+1)^2} + \sum_{i=1}^{\infty}\frac{k^2}{(k(s-1)+i+1)^2}$$

$$\psi_{t^k}^{(2)}(s) = \frac{d^2}{ds^2}\left(\psi_{t^k}(s)\right) = -\frac{2k^3}{(k(s-1)+1)^3} - \sum_{i=1}^{\infty}\frac{2k^3}{(k(s-1)+i+1)^3}$$

$$\psi_{t^k}^{(3)}(s) = \frac{d^3}{ds^3}\left(\psi_{t^k}(s)\right) = \frac{6k^4}{(k(s-1)+1)^4} + \sum_{i=1}^{\infty}\frac{6k^4}{(k(s-1)+i+1)^4}$$

$$\psi_{t^k}^{(4)}(s) = \frac{d^4}{ds^4}\left(\psi_{t^k}(s)\right) = -\frac{24k^5}{(k(s-1)+1)^5} - \sum_{i=1}^{\infty}\frac{24k^5}{(k(s-1)+i+1)^5}$$

$$\vdots$$

$$\psi_{t^k}^m(s) = (-1)^{m+1} m! k^{m+1} \frac{1}{(k(s-1)+1)^{m+1}} + \sum_{i=1}^{\infty} \frac{1}{(k(s-1)+i+1)^{m+1}}$$

Thus

$$\psi_{t^k}^m(s) = (-1)^{m+1} m! \sum_{i=0}^{\infty} \frac{k^{m+1}}{(k(s-1)+i+1)^{m+1}}.$$

□

Proposition 3.4. Let $s > 0$ and $k, m \in \mathbb{N}$. Then the polygamma function $\psi_{t^k}^m(s)$ has the integral representation

$$\psi_{t^k}^m(s) = (-1)^{m+1} (k)^{m+1} \int_0^{\infty} \frac{t^m e^{-(k(s-1)+1)t}}{1-e^{-t}} dt. \tag{16}$$

Proof. From (14), we have

$$\psi_{t^k}^m(s) = (-1)^{m+1} m! \sum_{i=0}^{\infty} \frac{k^{m+1}}{(k(s-1)+i+1)^{m+1}}$$

By applying the inverse Laplace transform on the summand, we have

$$\psi_{t^k}^m(s) = (-1)^{m+1} k^{m+1} \sum_{i=0}^{\infty} \int_0^{\infty} t^m e^{-(k(s-1)+i+1)t} dt$$

$$= (-1)^{m+1} k^{m+1} \int_0^{\infty} \sum_{i=0}^{\infty} t^m e^{-(k(s-1)+i+1)t} dt$$

$$= (-1)^{m+1} k^{m+1} \int_0^{\infty} t^m e^{-k(s-1)t} \sum_{i=0}^{\infty} e^{-(i+1)t} dt$$

$$= (-1)^{m+1} (k)^{m+1} \int_0^{\infty} \frac{t^m e^{-(k(s-1)+1)t}}{1-e^{-t}} dt.$$

□

Theorem 3.5. Let $r > 0$, $s > 0$ and $\alpha, \beta \in (0,1)$ such that $\alpha + \beta = 1$ and $\alpha m + \beta n \in \mathbb{N}$. Then the inequality

$$\mid \psi_{t^k}^{(\alpha m+\beta n)}(\alpha r + \beta s) \mid \leq \mid \psi_{t^k}^{(m)}(r) \mid^{\alpha} \mid \psi_{t^k}^{(n)}(s \mid^{\beta} \tag{17}$$

holds.

Proof. From relation (16), we have

$$
\begin{aligned}
\mid \psi_{t^k}^{(\alpha m+\beta n)}(\alpha r+\beta s) \mid &= \int_0^\infty \frac{t^{(\alpha m+\beta n)} k^{(\alpha m+\beta n+1)} e^{-(k\alpha r+\beta s-1)+1)t}}{(1-e^{-t})} dt \\
&= \int_0^\infty \frac{t^{(\alpha m+\beta n)} k^{(\alpha m+\beta n+\alpha+\beta)} e^{-(k(\alpha r+\beta s-\alpha+\beta)+\alpha+\beta)t}}{(1-e^{-t})^{\alpha+\beta}} dt \\
&= \int_0^\infty \frac{e^{-(\alpha k(r-1)+1)t-(\beta k(s1)+1)]t} t^{\alpha m} k^{\alpha(m+1)} t^{\beta n} k^{\beta(n+1)}}{(1-e^{-t})^{\alpha+\beta}} dt \\
&= \int_0^\infty \frac{e^{-\alpha(k(r-1)+1)t} t^{\alpha m} k^{\alpha(m+1)} e^{-\beta(k(s-1)+1)t} t^{\beta n} k^{\beta(n+1)}}{(1-e^{-t})^{\alpha} (1-e^{-t})^{\beta}} dt \\
&= \int_0^\infty \left(\frac{e^{-(k(r-1)+1)t} t^m k^{(m+1)}}{(1-e^{-t})} \right)^{\alpha} \left(\frac{e^{-(k(s-1)+1)t} t^n k^{(n+1)}}{(1-e^{-t})} \right)^{\beta} dt
\end{aligned}
$$

By applying Hölder's inequality, we have

$$
\begin{aligned}
\mid \psi_{t^k}^{(\alpha m+\beta n)}(\alpha r+\beta s) \mid &\le \left[\int_0^\infty \frac{e^{-(k(r-1)+1)t} t^m k^{(m+1)}}{(1-e^{-t})} \right]^{\alpha} \left[\int_0^\infty \frac{e^{-(k(s-1)+1)t} t^n k^{(n+1)}}{(1-e^{-t})} \right]^{\beta} \\
&= \mid \psi_{t^k}^{(m)}(r) \mid^{\alpha} \mid \psi_{t^k}^{(n)}(s) \mid^{\beta} .
\end{aligned}
$$

□

Remark 3.6. A similar results was proved for the (p,k)-gamma function, see Theorem 2.5 of [6].

Theorem 3.7. Let $r > 0$, $s > 0$ and $b \ge 1$. Then the inequality

$$
\left(\mid \psi_{t^k}^{(m)}(r) \mid + \mid \psi_{t^k}^{(n)}(s) \mid \right)^{\frac{1}{b}} \le \mid \psi_{t^k}^{(m)}(r) \mid^{\frac{1}{b}} + \mid \psi_{t^k}^{(n)}(s) \mid^{\frac{1}{b}} \tag{18}
$$

holds.

Proof. From (16), we obtain

$$
\left[\mid \psi_{t^k}^{(m)}(r) \mid + \mid \psi_{t^k}^{(n)}(s) \mid \right]^{\frac{1}{b}} = \left[\int_0^\infty \frac{e^{-(k(r-1)+1)t} t^m k^{(m+1)}}{(1-e^{-t})} dt + \int_0^\infty \frac{e^{-(k(s-1)+1)t} t^n k^{(n+1)}}{(1-e^{-t})} dt \right]^{\frac{1}{b}} .
$$

By applying Lemma 2.5 followed by Minkowski's inequality, we have

$$
\begin{aligned}
\left[\mid \psi_{t^k}^{(m)}(r)\mid + \mid \psi_{t^k}^{(n)}(s)\mid\right]^{\frac{1}{b}} &= \left[\int_0^\infty \left(\frac{e^{-\frac{1}{b}(k(r-1)+1)t} t^{\frac{m}{b}} k^{\frac{m+1}{b}}}{(1-e^{-t})^{\frac{1}{b}}}\right)^b + \int_0^\infty \left(\frac{e^{-\frac{1}{b}(k(s-1)+1)t} t^{\frac{n}{b}} k^{\frac{(n+1)}{b}}}{(1-e^{-t})^{\frac{1}{b}}}\right)^b dt\right]^{\frac{1}{b}} \\
&\leq \left[\left(\int_0^\infty \frac{e^{-\frac{1}{b}(k(r-1)+1)t} t^{\frac{m}{b}} k^{\frac{m+1}{b}}}{(1-e^{-t})^{\frac{1}{b}}} + \int_0^\infty \frac{e^{-\frac{1}{b}(k(s)+1)t} t^{\frac{n}{b}} k^{\frac{(n+1)}{b}}}{(1-e^{-t})^{\frac{1}{b}}}\right)^b dt\right]^{\frac{1}{b}} \\
&\leq \left(\int_0^\infty \frac{e^{-(k(r-1)+1)t} t^m k^{(m+1)}}{(1-e^{-t})} dt\right)^{\frac{1}{b}} + \left(\int_0^\infty \frac{e^{-(k(s-1)+1)t} t^n k^{(n+1)}}{(1-e^{-t})} dt\right)^{\frac{1}{b}} \\
&= \mid \psi_{t^k}^{(m)}(r)\mid^{\frac{1}{b}} + \mid \psi_{t^k}^{(n)}(s)\mid^{\frac{1}{b}}.
\end{aligned}
$$

□

CONFLICT OF INTEREST

The author declares that there is no conflict of interests.

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