Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4118-4128
https://doi.org/10.28919/jmcs/5865
ISSN: 1927-5307

APPLICATION OF VARIATIONAL ITERATION METHOD TO FIND THE SOLUTION OF COUPLED SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract: Present work is devoted to the application of variational iteration method to obtain the solutions of coupled system of first order differential equations. Numerical examples are taken to test the efficiency of this method. We have shown that the successive approximations of each problem are converging to their exact solution. Further we have shown graphically the third, fourth, fifth and sixth approximation values and exact values.

Keywords: variational iteration method; coupled system; successive approximation.

2010 AMS Subject Classification: 65F10.

1. INTRODUCTION

Many problems in various fields of sciences and engineering yield partial differential equations.

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Received April 15, 2021

For their physical interpretation we need their solutions. It may not possible to find the exact solution of certain partial differential equations. There are many methods to find the approximate solutions of such equations. For example, numerical methods, Adomain decomposition method, Homotopy analysis method etc. J.H.He[3-4] developed the variational iteration method to solve linear and non-linear ordinary and partial differential equations. Several researchers [1,2,5] are working on application of this method to find the solutions of linear and nonlinear partial differential equations. E.Rama, K.Somaiah and K.Sambaiah [6] have used this method for obtaining solution of various types of problems. In this paper an attempt is made to find the successive approximate solutions of system of first order coupled differential equations using Variational iteration method. Further it was shown that these solutions are converging to their exact solutions.

2. DESCRIPTION OF VARIATIONAL ITERATION METHOD

Let L be a linear operator, N be a non-linear operator and g(x) is known continuous function. Consider a differential equation of the type

$$L[y(t)]+N[y(t)]=g(t) \tag{2.1}$$

According to VIM the correction functional of (2.1) is given by

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda(s,t)\left[L[y_n(s)] + N[\tilde{y}_n(s)] - g(s)\right]ds \tag{2.2}$$

where $\lambda(s,t)$ is the Lagrange Multiplier. Let $\lambda^f(t,x)$ be the computed Lagrange multiplier. Let $y_n(t)$ be the nth approximate solution and $\tilde{y}_n(t)$ denotes restricted variation, that is $\tilde{\delta} y_n(t) = 0$. The successive approximations $y_{n+1}(t)$ will be computed by applying the obtained Lagrange multiplier $\lambda^f(t,x)$ and properly chosen initial approximation $y_0(x)$. The solution of (2.1) is obtained as y(t)= $\underset{n\to\infty}{Lt}\, y_n(t)$.

The method of finding the solution of (2.1) by the process stated above is known as Variational iteration method.

3. VIM FOR SOLVING A SYSTEM OF DIFFERENTIAL EQUATIONS

Let the system of differential equations be

$$L_1(y_1(t),y_2(t),\ldots\ldots\ldots,y_m(t))+N_1(y_1(t),y_2(t),\ldots\ldots\ldots,y_m(t))=g_1(y_1,y_2,\ldots y_m,t) \tag{3.1}$$

$$L_2(y_1(t),y_2(t),\ldots\ldots\ldots,y_m(t))+N_2(y_1(t),y_2(t),\ldots\ldots\ldots,y_m(t))=g_2(y_1,y_2,\ldots y_m,t) \tag{3.2}$$

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$$L_m(y_1(t),y_2(t),\ldots\ldots\ldots,y_m(t))+N_m(y_1(t),y_2(t),\ldots\ldots\ldots,y_m(t))=g_m(y_1,y_2,\ldots y_m,t) \tag{3.m}$$

where $L_1,L_2,\ldots,L_m$ are linear operators and $N_1,N_2,\ldots..,N_m$ are non-linear operators.

The correctional functional for the above system of differential equations are given by

$$y_{1,n+1} = y_{1,n} + \int_0^t \lambda_1(t,s)\left[L_1(y_{1,n}(s), y_{2.n}(s),\ldots..,y_{m,n}(s)) + N_1(\tilde{y}_{1,n}(s),\tilde{y}_{2,n}(s),\ldots.,\tilde{y}_{m,n}(s)) - g_1(s)\right]ds \tag{3.1.a}$$

$$y_{2,n+1} = y_{2,n} + \int_0^t \lambda_2(t,s)\left[L_2(y_{1,n}(s), y_{2.n}(s),\ldots..,y_{m,n}(s)) + N_2(\tilde{y}_{1,n}(s),\tilde{y}_{2,n}(s),\ldots.,\tilde{y}_{m,n}(s)) - g_2(s)\right]ds \tag{3.2.a}$$

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$$y_{m,n+1} = y_{m,n} + \int_0^t \lambda_m(t,s)\left[L_m(y_{1,n}(s), y_{2.n}(s),\ldots..,y_{m,n}(s)) + N_m(\tilde{y}_{1,n}(s),\tilde{y}_{2,n}(s),\ldots.,\tilde{y}_{m,n}(s)) - g_m(s)\right]ds \tag{3.m.a}$$

where $\lambda_i (i=1,2,\ldots.,m)$ are Lagrangian multipliers which are to be obtained using the variational theory. $y_{i,j}$ denotes jth approximation of dependent variable y_i and $\tilde{y}_n$ denotes restricted variation. i.e., $\tilde{\delta} y_n=0$.

4. APPLICATIONS

We consider few examples to know the efficiency and accuracy of the variational iteration method for obtaining the solution of coupled system of equations of first order differential equations.

4.1 Example: Consider the following coupled system of first order differential equations

$$\frac{dy_1}{dt} = y_1 + 3y_2 \tag{4.1}$$

$$\frac{dy_2}{dt} = 3y_1 + y_2 \tag{4.2}$$

with the initial conditions $y_1(0)=2$ and $y_2(0)=0$. Note that y_1 and y_2 are dependent variables and t is the independent variable. The exact solution of this system is $y_1(t)=e^{4t}+e^{-2t}$ and $y_2(t)= e^{4t} - e^{-2t}$. These $y_1(t)$ and $y_2(t)$ in terms of series are as follows.

$$y_1(t) = 2 + 2t + 10t^2 + \frac{28}{3}t^3 + \frac{34}{3}t^4 + ... \tag{4.3}$$

$$y_2(t) = 6t + 6t^2 + 12t^3 + 10t^4 + ... \tag{4.4}$$

To solve (4.1) and (4.2) by variational iteration method we consider the terms involving y_1 and y_2 as restricted variation except their derivative terms. Hence the correction functional of (4.1) and (4.2) are

$$y_{1,n+1} = y_{1,n} + \int_0^t \lambda_1(s,t)\left[\frac{dy_{1,n}(s)}{ds} - \tilde{y}_{1,n}(s) - 3\tilde{y}_{2,n}(s)\right]ds \tag{4.5}$$

$$y_{2,n+1} = y_{2,n} + \int_0^t \lambda_2(s,t)\left[\frac{dy_{2,n}(s)}{ds} - 3\tilde{y}_{1,n}(s) - \tilde{y}_{2,n}(s)\right]ds \tag{4.6}$$

From (4.5) and (4.6) we obtain the following stationery conditions.

$$1 + \lambda_1(s,t) = 0, \qquad [\frac{\partial}{\partial s}\lambda_1(s,t)]_{s=t} = 0$$

and

$$1+\lambda_2(s,t)=0, \qquad [\frac{\partial}{\partial s}\lambda_2(s,t)]_{s=t}=0$$

From the above equations the values of Lagrangian multipliers are

$$\lambda_1=-1 \tag{4.7}$$

$$\lambda_2=-1 \tag{4.8}$$

Hence, the variational iteration formulae of (4.5) and (4.6) become

$$y_{1,n+1}=y_{1,n}-\int_0^t\left[\frac{dy_{1,n}(s)}{ds}-y_{1,n}(s)-3y_{2,n}(s)\right]ds \tag{4.9}$$

$$y_{2,n+1}=y_{2,n}+\int_0^t\lambda_2(s,t)\left[\frac{dy_{2,n}(s)}{ds}-3y_{1,n}(s)-y_{2,n}(s)\right]ds \tag{4.10}$$

We choose

$$y_{1,0}(t)=y_1(0)=2 \tag{4.11}$$

and

$$y_{2,0}(t)=y_2(0)=0 \tag{4.12}$$

Substituting n=0 in equation (4.9) we obtain

$$y_{1,1}=y_{1,0}-\int_0^t\left[\frac{dy_{1,0}(s)}{ds}-y_{1,0}(s)-3y_{2,0}(s)\right]ds$$

Using (4.11) and (4.12) we get

$$y_{1,1}=2+2t$$

Similarly from the equations (4.10),(4.11) and (4.12) we obtain

$$y_{2,1}=6t$$

Continuing this process for n=1,2,3 we have the following approximations of y_1 and y_2.

$$y_{1,2}=2+2t+10t^2 \quad , \qquad y_{2,2}=6t+6t^2$$

$$y_{1,3}=2+2t+10t^2+\frac{28}{3}t^3 \quad , \qquad y_{2,3}=6t+6t^2+12t^3$$

$$y_{1,4} = 2 + 2t + 10t^2 + \frac{28}{3}t^3 + \frac{34}{3}t^4 \;, \qquad y_{2,4} = 6t + 6t^2 + 12t^3 + 10t^4$$

$$y_{1,5} = 2 + 2t + 10t^2 + \frac{28}{3}t^3 + \frac{34}{3}t^4 + \frac{124}{15}t^5 \;, \qquad y_{2,5} = 6t + 6t^2 + 12t^3 + 10t^4 + \frac{44}{5}t^5$$

$$y_{1,6} = 2 + 2t + 10t^2 + \frac{28}{3}t^3 + \frac{34}{3}t^4 + \frac{124}{15}t^5 + \frac{52}{9}t^6 \;, \qquad y_{2,6} = 6t + 6t^2 + 12t^3 + 10t^4 + \frac{44}{5}t^5 + \frac{28}{5}t^6$$

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In view of (4.3) and (4.4) we have

$$y_1(t) = \underset{n\to\infty}{Lt}\, y_{1,n}(t) = e^{4t} + e^{-2t}$$

and

$$y_2(t) = \underset{n\to\infty}{Lt}\, y_{2,n}(t) = e^{4t} - e^{-2t}$$

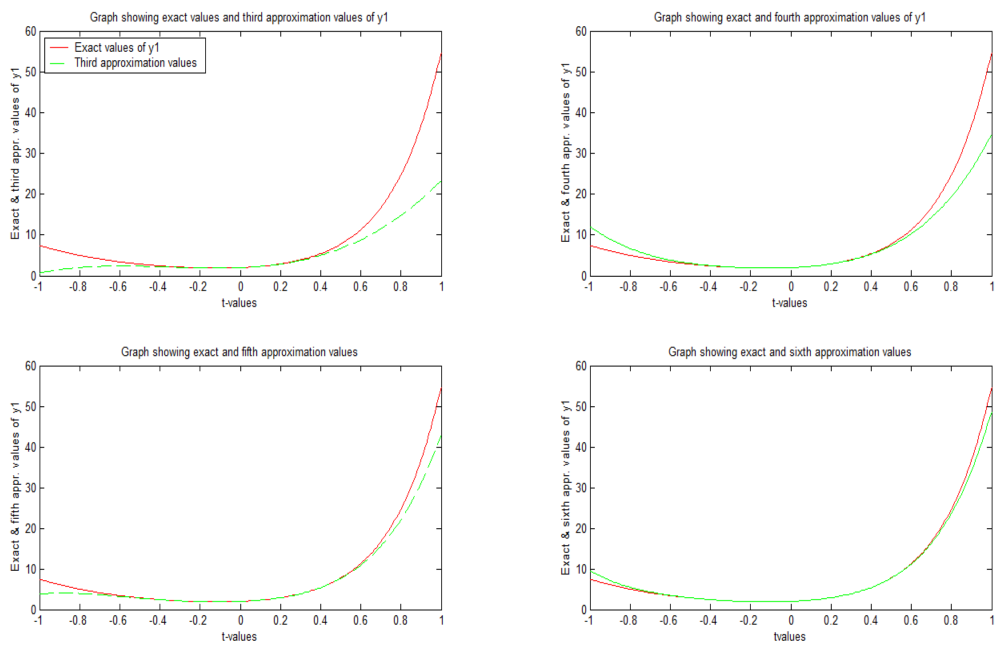

Fig.1

The exact values and the 3rd,4th,5th and 6th approximations of y_1 are shown in the Fig.1. and that of y_2 are shown in the Fig.2.

4.2 Example: Consider another coupled system of first order differential equations

$$\frac{dy_1}{dt} = 3y_1 + 2y_2 \tag{4.13}$$

$$\frac{dy_2}{dt} = -5y_1 - 3y_2 \tag{4.14}$$

with the initial conditions $y_1(0)=1$ and $y_2(0)= -1$. The exact solution this system is $y_1(t)=\cos(t)+\sin(t)$ and $y_2(t)= -2\sin(t)-\cos(t)$. These $y_1(t)$ and $y_2(t)$ in terms of series are as follows.

$$y_1(t) = 1 + t - \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} - \frac{t^7}{7!} + \dots\dots\dots\dots \tag{4.15}$$

$$y_2(t) = -1 - 2t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4!} - 2\frac{t^5}{5!} + \frac{t^6}{6!} + 2\frac{t^7}{7!} + \dots\dots\dots \tag{4.16}$$

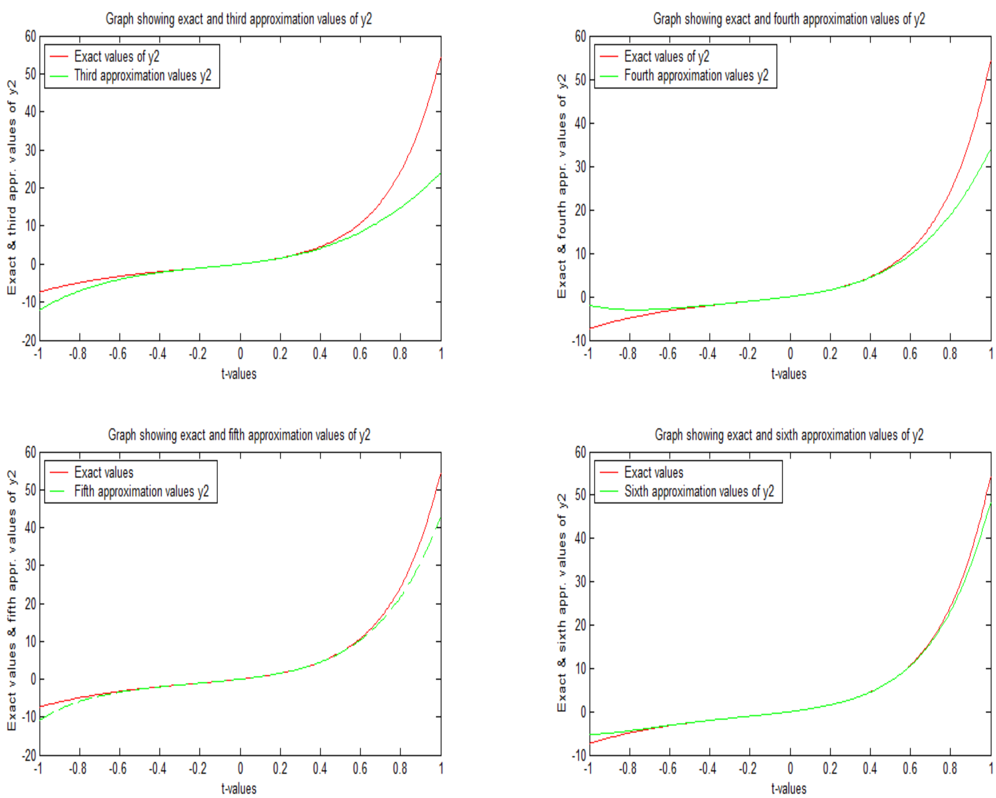

Fig.2

Proceeding as in the above example it can be shown that the Lagrangian multipliers are -1 and -1. Therefore the variational iteration formula for (4.13) and (4.14) become

$$y_{1,n+1} = y_{1,n} - \int_0^t \left[\frac{dy_{1,n}(s)}{ds} - 3y_{1,n}(s) - 2y_{2,n}(s)\right]ds \tag{4.17}$$

$$y_{2,n+1} = y_{2,n} - \int_0^t \left[\frac{dy_{2,n}(s)}{ds} + 5y_{1,n}(s) + 3y_{2,n}(s)\right]ds \tag{4.18}$$

We choose the initial approximations

$$y_{1,0}(t) = y_1(0) = 1 \tag{4.19}$$

and

$$y_{2,0}(t) = y_2(0) = -1 \tag{4.20}$$

Substituting n=0 in (4.17) we have

$$y_{1,,1} = y_{1,0} - \int_0^t \left[\frac{dy_{1,0}(s)}{ds} - 3y_{1,0}(s) - 2y_{2,0}(s)\right]ds$$

Using (4.19) and (4.20) we have

$$y_{1,1}(t) = 1 + t$$

Similarly it can be shown that

$$y_{2,1}(t) = -1 - 2t$$

The successive approximations of $y_1(t)$and $y_2(t)$are

$$y_{1,2} = 1 + t - \frac{t^2}{2} \quad , \qquad y_{2,2} = -1 - 2t + \frac{t^2}{2}$$

$$y_{1,3} = 1 + t - \frac{t^2}{2} - \frac{t^3}{3!} \quad , \qquad y_{2,3} = -1 - 2t + \frac{t^2}{2} + \frac{t^3}{3}$$

$$y_{1,4} = 1 + t - \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} \quad , \qquad y_{2,4} = -1 - 2t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4!}$$

$$y_{1,5} = 1 + t - \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \quad , \qquad y_{2,5} = -1 - 2t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4!} - 2\frac{t^5}{5!}$$

$$y_{1,6} = 1 + t - \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} \quad , \qquad y_{2,6} = -1 - 2t + \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4!} - 2\frac{t^5}{5!} + \frac{t^6}{6!}$$

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In view of (4.15) and (4.16) we have

$$y_1(t) = \underset{n\to\infty}{Lt}\; y_{1,n}(t) = \cos t + \sin t$$

and

$$y_2(t) = \underset{n\to\infty}{Lt}\; y_{2,n}(t) = -\cos t - 2\sin t$$

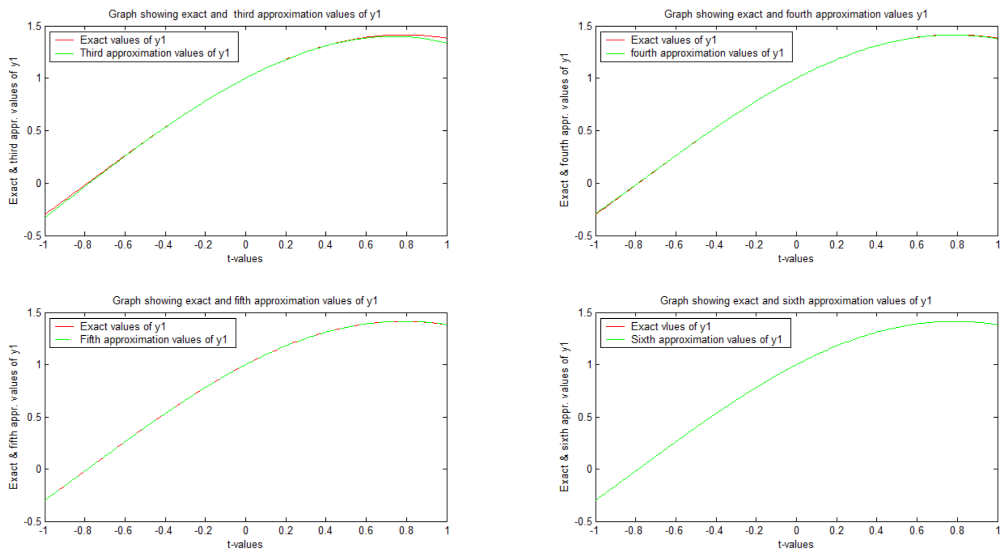
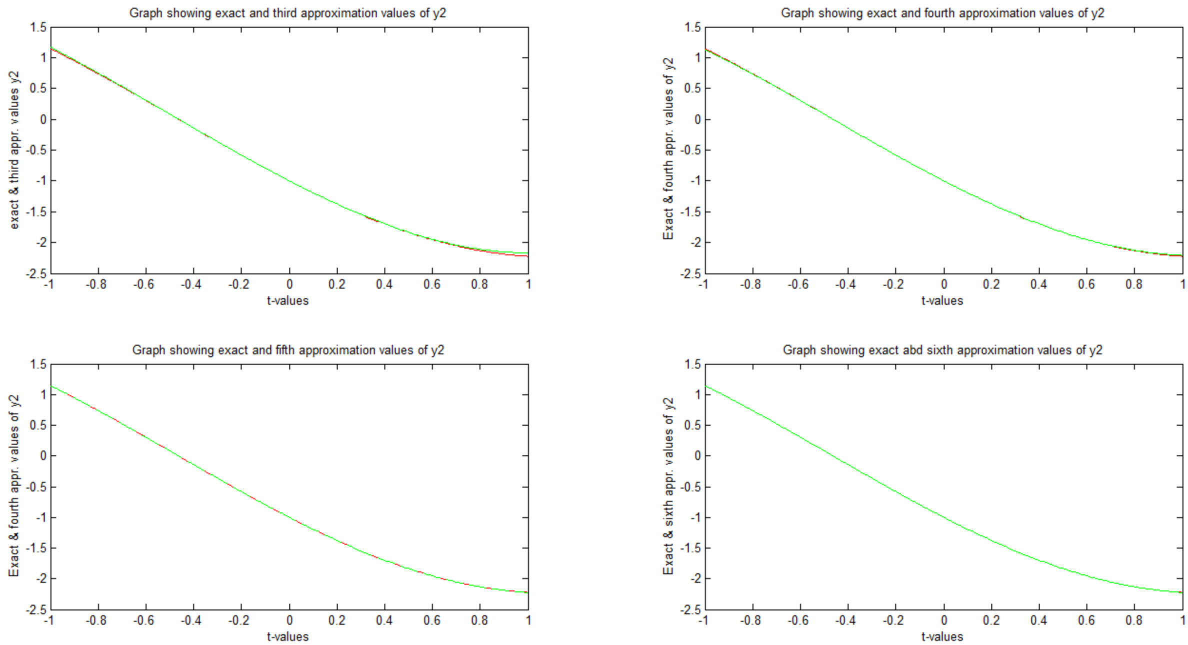


Fig.3

The exact values of y_1 and third, fourth, fifth, sixth approximations values are shown graphically in Fig.3. Further the exact values of y_2 and third, fourth, fifth, sixth approximations values are shown graphically in Fig.4.



F.g.4

CONCLUSION

Thus from the examples considered above it is understood that VIM can be easily applicable to obtain the solutions of coupled system of first order differential equations. It can be observed that the approximations are approaching the exact values in fewer iterations. The same can be observed from the graphs. This shows the efficacy of this method.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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