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SPARSE SIGNAL RECONSTRUCTION IN COMPRESSIVE SENSING VIA DERIVATIVE-FREE ITERATIVE METHOD

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Abstract. Finding sparse solutions to under-determined or ill-conditioned equations is a problem that usually arise in compressive sensing. In this article, a derivative-free iterative method is presented for recovering sparse signal in compressive sensing by approximating the solution to a convex constrained nonlinear equation. The proposed method is derived from the modified Polak-Ribiere-Polyak conjugate gradient method for unconstrained optimization. The global convergence is established under mild assumptions. Preliminary numerical results in recovering sparse signal are given to show that the proposed method is efficient.

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1. INTRODUCTION

The task of signal recovery is to recover a high quality signal from its degraded measurement, which is known to be an ill-posed inverse problem. The under-determined problem needs to be limited by successful priors for ensuring appropriate solutions. The mathematical formulation of signal recovery can be generally modeled as:

$$b = Nv + k, \tag{1}$$

where $b \in \mathbb{R}^m$ is representing the observed data, $v \in \mathbb{R}^n$ is the unknown image, k is the noise and N is a linear mapping such that $N \in \mathbb{R}^{m\times n}(m < n)$. In order to address problem (1), one of the tools usually employed is the ℓ_1-*regularization*. The restoration is obtained by approximating the following unconstrained optimization

$$\min_v \frac{1}{2}\|Nv - b\|_2^2 + \mu\|v\|_1, \tag{2}$$

where μ is a positive regularization parameter and $\|\cdot\|_1$ is the ℓ_1-regularization term.

Several numerical methods have been developed in recent years for finding solution to model (2), among which iterative shrinkage thresholding (IST) [1], fast iterative shrinkage thresholding algorithm (FISTA) [2] are one of the most common ones widely known due to their simplicity and efficiency.

Gradient methods are also common methods to solve the model (2). For instance, Figueiredo [3] proposed a gradient based projection algorithm to solve (2). Motivated by the work in [3], Xiao and Xhu [4, 5] derived an approximate equivalence of (2) as a nonlinear monotone operator equation. Referring to [6], we briefly present a review on the reformulation procedure of (2) into a convex quadrtic program problem.

Let v be a vector in the Euclidean space $\mathbb{R}^n$. The vector v can be rewritten as

$$v = \alpha - \beta,\ \alpha \geq 0,\ \beta \geq 0,$$

where $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^n$ and $\alpha_i = (v_i)_+$, $\beta_i = (-v_i)_+$ for all $i \in n$ with $(\cdot)_+ = \max\{0, \cdot\}$. Subsequently, the ℓ_1-norm of a vector can be represented as $\|v\|_1 = e_n^T \alpha + e_n^T \beta$, where e_n is an n-dimensional vector with all elements one. Hence the ℓ_1-norm problem (2) was transformed into

$$\min_{\alpha,\beta} \frac{1}{2}\|b - N(\alpha - \beta)\|^2 + \mu e_n^T \alpha + \mu e_n^T \beta, \text{ such that } \alpha \geq 0,\ \beta \geq 0. \tag{3}$$

From [6], it also can be easily rewritten as the quadratic program problem with box constraints

$$\min \frac{1}{2} u^T J u + c^T u, \text{ such that } u \geq 0, \tag{4}$$

where $q = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, $y = N^T b$, $c = \mu e_{2n} + \begin{bmatrix} -y \\ y \end{bmatrix}$, *and* J= $\begin{bmatrix} N^T N & -N^T N \\ -N^T N & N^T N \end{bmatrix}$.

It can be observed that J is a semi-definite positive matrix. Hence, (4) is a convex quadratic program problem, and it is equivalent to the following nonlinear convex constrained equation

$$\chi(q) = \min\{q,\ Jq + c\} = 0, \text{ such that } q \in \mathscr{D}, \tag{5}$$

where $\mathscr{D} = \mathbb{R}_+^{2n}$ is a convex set.

In recent years, numerous authors (see, e.g.,[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]) have proposed several algorithms based on the conjugate gradient methods to solve the convex constraint problem (5). Since solving (5) is equivalent to solving the ℓ_1-regularization problem (2), in this paper, we continue study on iterative algorithms based on conjugate gradient methods and extend our new method to solve the ℓ_1-regularization problem (2). In the next section, we will propose a derivative-free iterative method for solving the ℓ_1- regularization problem (2). The proposed method is inspired by the excellent numerical success of the NPRP method proposed in [31] for solving unconstrained optimization problem. Our proposed method does not require to compute gradient at each iteration, neither does it need to solve a linear equations using the Jacobian matrix or an approximation of it in per-iteration as in the case of the Newton method, Quasi-Newton and their variants (see, [32, 33, 34, 35]).

The outline of the paper is as follows. In the next Section, we recall the definition of the projection map and introduced our algorithm for solving the ℓ_1-regularization problem. In Section 3, the convergence analysis of the proposed method is analysed. Global convergence result for the proposed method is obtained. Finally, numerical experiments in recovering sparse signal in compressing sensing are illustrated.

2. THE METHOD

We begin this section with the definition of the projection map.

Definition 2.1. *Let $\mathscr{D} \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then for any $y \in \mathbb{R}^n$, its projection onto $\mathscr{D}$, denoted by $P_{\mathscr{D}}[y]$, is defined by*

$$P_{\mathscr{D}}[y] := \arg\min\{\|y-x\| \ : x \in \mathscr{D}\}.$$

The projection operator $P_{\mathscr{D}}$ has a well-known property, that is, for any $y, x \in \mathbb{R}^n$ the following nonexpansive property hold

$$\|P_{\mathscr{D}}[y] - P_{\mathscr{D}}[x]\| \leq \|y-x\|. \tag{6}$$

Inspired by the proposed conjugate gradient method in [31], we solve (5) by proposing a derivative-free method whose iterate takes the form

$$e_t = q_t + \varepsilon_t p_t \tag{7}$$

where ε_t is the steplength and p_t is the search direction computed by

$$p_t := \begin{cases} -\chi(q_t) & \text{if } t = 0, \\ -\chi(q_t) + \beta_t^{ENPRP} p_{t-1} & \text{if } t > 0, \end{cases} \tag{8}$$

where

$$\beta_t^{ENPRP} := \frac{\chi(q_t)^T y_{t-1}}{w_{t-1}} + \eta \frac{\|y_{t-1}\|^2}{w_{t-1}^2} \chi(q_t)^T p_{t-1}, \tag{9}$$

where

$$y_{t-1} := \chi(q_t) - \chi(q_{t-1}),\ w_{t-1} = \max\{r\|p_{t-1}\|, \|\chi(q_{t-1})\|\},\ \eta > \frac{1}{4}, r > 0. \tag{10}$$

The search direction defined by (8) satisfies the following Lemma.

Lemma 2.2. *Let p_t be the search direction generated by* (8)*, then p_t is a sufficient descent direction. That is for all $t \geq 0$,*

$$\chi(q_t)^T p_t \leq -\left(1 - \frac{1}{4\eta}\right) \|\chi(q_t)\|^2, \ \eta > \frac{1}{4}. \tag{11}$$

Proof. For $t = 0$, equation (11) obviously holds. For $t > 0$, we have

$$\begin{aligned}
\chi(q_t)^T p_t &= -\|\chi(q_t)\|^2 + \beta_k^{ENPRP} \chi(q_t)^T p_{t-1} \\
&\leq -\|\chi(q_t)\|^2 + \left(\frac{\chi(q_t)^T y_{t-1}}{w_{t-1}} + \eta \frac{\|y_{t-1}\|^2}{w_{t-1}^2} \chi(q_t)^T p_{t-1}\right) \chi(q_t)^T p_{t-1} \\
&\leq -\|\chi(q_t)\|^2 + \left(\frac{w_{t-1}\chi(q_t)^T p_{t-1}\chi(q_t)^T y_{t-1} + \eta \|y_{t-1}\|^2 (\chi(q_t)^T p_{t-1})^2}{w_{t-1}^2}\right)
\end{aligned}$$

By defining $u_\alpha = \frac{1}{\sqrt{2\eta}} w_{t-1}\chi(q_t)$ and $u_\tau = \sqrt{2\eta}\chi(q_t)^T p_{t-1} y_{t-1}$, we get

$$\begin{aligned}
\chi(q_t)^T p_t &= -\|\chi(q_t)\|^2 + \left(\frac{u_\alpha^T u_\tau - (1/2)\|u_\tau\|^2}{\|w_{t-1}\|^2}\right), \\
&= -\left(1 - \frac{1}{4\eta}\right) \|\chi(q_t)\|^2 + \left(\frac{u_\alpha^T u_\tau - (1/2)\left(\|u_\alpha\|^2 + \|u_\tau\|^2\right)}{\|w_{t-1}\|^2}\right), \\
&\leq -\left(1 - \frac{1}{4\eta}\right) \|\chi(q_t)\|^2.
\end{aligned}$$

where $\eta > \frac{1}{4}$. Thus, (11) holds. □

In what follows, we state the iterative procedures/steps of our method.

Algorithm 1

Input. Set an initial point $q_0 \in \mathscr{D}$, the positive constants: $Tol > 0,\ r \in (0,1),\ m \in (0,2),\ a > 0, \rho > 0$. Set $t = 0$.

Step 0. Compute $\chi(q_t)$. If $\|\chi(q_t)\| \leq Tol$, stop. Otherwise, generate the search direction p_t using (8).

Step 1. Determine the step-size $\varepsilon_t = \max\{ar^i | i \geq 0\}$ such that

$$\chi(q_t + \varepsilon_t p_t)^T p_t \geq \rho \varepsilon_t \|p_t\|^2. \tag{12}$$

Step 2. Compute $e_t = q_t + \varepsilon_t p_t$, where e_t is a trial point.

Step 3. If $e_t \in \mathscr{D}$ and $\|\chi(e_t)\| = 0$, stop. Otherwise, compute the next iterate by

$$q_{t+1} = P_{\mathscr{D}}\left[q_t - m\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|^2}\chi(e_t)\right], \tag{13}$$

Step 4. Finally we set $t = t + 1$ and return to step 0.

3. Convergence Analysis

In this section, we obtain the global convergence property of Algorithm 1. We also make the following assumptions on the mapping χ.

Assumption 1.

(i) The solution set of the constrained nonlinear (5)*, denoted by $\mathscr{D}^*$, is nonempty.*

(ii) The mapping χ is Lipschitz continuous on $\mathbb{R}^n$. That is, there exists a constant $L > 0$ such that

$$\|\chi(x) - \chi(y)\| \leq L\|x - y\| \ \forall x, y \in \mathbb{R}^n \tag{14}$$

(iii) For any $y \in \mathscr{D}^*$ and $x \in \mathbb{R}^n$, it holds that

$$\chi(x)^T(x - y) \geq 0. \tag{15}$$

Lemma 3.1. *Let $\{p_t\}$ and $\{q_t\}$ be two sequences generated by Algorithm 1. Then, there exists a step size ε_t satisfying the line search* (12) *for all $t \geq 0$.*

Proof. For any $i \geq 0$, suppose (12) does not hold for the iterate t_0-th, then we have

$$-\chi(q_{t_0}+ar^i p_{t_0})^T p_{t_0} < \rho a r^i \|p_{t_0}\|^2.$$

Thus, by the continuity of χ and with $0 < r < 1$, it follows that by letting $i \to \infty$, we have

$$-\chi(q_{t_0})^T p_{t_0} \leq 0,$$

which contradicts (11). □

Lemma 3.2. *Let the sequences $\{q_t\}$ and $\{e_t\}$ be generated by the Algorithm 1 method under Assumption 1, then*

$$\varepsilon_t \geq \max\left\{a, \frac{rc\|\chi(q_t)\|^2}{(L+\rho)\|p_t\|^2}\right\}. \tag{16}$$

Proof. Let $\hat{\varepsilon}_t = \varepsilon_t r^{-1}$. Assume $\varepsilon_t \neq a$, from (12), $\hat{\varepsilon}_t$ does not satisfy (12). That is,

$$-\chi(q_t+\hat{\varepsilon}_t p_t)^T p_t < \rho\hat{\varepsilon}_t\|p_t\|^2.$$

From (14) and (11), it can be obviously seen that

$$\begin{aligned}
c\|\chi(q_t)\|^2 &\leq -\chi_t^T p_t \\
&= (\chi(q_t+\hat{\varepsilon}_t p_t)-\chi(q_t))^T p_t - \chi(q_t+\hat{\varepsilon}_t p_t)^T p_t \\
&\leq L\hat{\varepsilon}_t\|p_t\|^2+\rho\hat{\varepsilon}_t\|p_t\|^2 \\
&\leq \hat{\varepsilon}_t(L+\rho)\|p_t\|^2.
\end{aligned}$$

This gives the desired inequality (16). □

Lemma 3.3. *Suppose that Assumption* 1 *holds. Let $\{q_t\}$ and $\{e_t\}$ be sequences generated by the Algorithm 1, then for any solution q^* contained in the solution set $\mathscr{D}^*$ the inequality*

$$\|q_{t+1}-q^*\|^2 \leq \|q_t-q^*\|^2-\rho^2\|q_t-e_t\|^4. \tag{17}$$

holds. In addition, $\{q_t\}$ is bounded and

$$\sum_{t=0}^{\infty}\|q_t-e_t\|^4 < +\infty. \tag{18}$$

Proof. First, we begin by using the weakly monotonicity assumption (Assumption 1 (iii)) on the mapping χ. Thus, for any solution $q^* \in \mathscr{D}^*$,

$$\chi(e_t)^T(q_t - q^*) \geq \chi(e_t)^T(q_t - e_t).$$

The above inequality together with (12) gives

$$\chi(q_t + \varepsilon_t p_t)^T(q_t - e_t) \geq \rho\alpha_t^2\|p_t\|^2 \geq 0. \tag{19}$$

From (6) and (19), we have the following

$$\begin{aligned}
\|q_{t+1} - q^*\|^2 &= \left\| P_{\mathscr{D}}\left[q_t - m\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|^2}\chi(e_t)\right] - q^*\right\|^2 \\
&\leq \left\|\left[q_t - m\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|^2}\chi(e_t)\right] - q^*\right\|^2 \\
&= \|q_t - q^*\|^2 - 2m\left(\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|^2}\right)\chi(e_t)^T(q_t - q^*) + m^2\left(\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|}\right)^2 \\
&= \|q_t - q^*\|^2 - m\left(\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|^2}\right)\chi(e_t)^T(q_t - e_t) + m^2\left(\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|}\right)^2 \\
&= \|q_t - q^*\|^2 - m(2 - m)\left(\frac{\chi(e_t)^T(q_t - e_t)}{\|\chi(e_t)\|}\right)^2 \\
&\leq \|q_t - q^*\|^2.
\end{aligned}$$

Thus, the sequence $\{\|q_t - q^*\|\}$ has a nonincreasing and convergent property. Therefore, this makes $\{q_t\}$ to be bounded by a positive constant say k_b and therefore the following holds.

$$\rho^2\sum_{t=0}^{\infty}\|q_t - e_t\|^4 < \|q_0 - q^*\|^2 < +\infty.$$

□

Remark 3.4. *Taking into account of the definition of e_t and also by (18), it can be deduced that*

$$\lim_{t\to\infty}\varepsilon_t\|p_t\| = 0. \tag{20}$$

Theorem 3.5. *Suppose Assumption 1 holds. Let $\{q_t\}$ and $\{e_t\}$ be sequences generated by Algorithm 1, then*

$$\liminf_{t\to\infty}\|\chi(q_t)\| = 0. \tag{21}$$

Proof. Suppose (21) is not valid, that is, there exist a constant say $s > 0$ such that $s \leq \|\chi(q_t)\|$, $t \geq 0$. Then this along with (11) implies that

$$\|p_t\| \geq cs, \quad \forall t \geq 0. \tag{22}$$

From Lemma 3.3, having in view that the sequences $\{q_t\}$ is bounded by a positive constant say k_b. In addition with the continuity of χ, it further implies that $\{\|\chi(q_t)\|\}$ is bounded by a constant say u.

Moreover,from the proposed conjugate gradient parameter (9), by using Lipschitz continuity and the triangular inequality, we can deduce that

$$\begin{aligned}
\|\beta_t^{ENPRP}\| &= \left\| \frac{\chi(q_t)^T y_{t-1}}{w_{t-1}} + \eta \frac{\|y_{t-1}\|^2}{w_{t-1}^2} \chi(q_t)^T p_{t-1} \right\| \\
&\leq \frac{\|\chi(q_t)\| \|y_{t-1}\|}{r\|p_{t-1}\|} + \eta \frac{\|y_{t-1}\|^2}{r^2\|p_{t-1}\|^2} \|\chi(q_t)\| \|p_{t-1}\| \\
&\leq \frac{2LR\|\chi(q_t)\|}{r\|p_{t-1}\|} + \eta \frac{(2LR)^2}{r^2\|p_{t-1}\|^2} \|\chi(q_t)\| \|p_{t-1}\| \\
&= \left(\frac{2LR}{r} + \frac{4\eta L^2R^2}{r^2} \right) \frac{\|\chi(q_t)\|}{\|p_{t-1}\|} \qquad (23)
\end{aligned}$$

Let $Q = \left(\frac{2LR}{r} + \frac{4\eta L^2R^2}{r^2} \right)$, thus, from (8), (10) and (23), it follows that for all $t \geq 1$,

$$\|p_t\| = \|\chi(q_t)\| + \|\beta_t^{ENPRP}\| \|p_{t-1}\| \leq Q\|\chi(q_t)\| \leq Qu \triangleq \gamma,$$

From (16), we have

$$\begin{aligned}
\varepsilon_t \|p_t\| &\geq \max\left\{ a, \frac{rc\|\chi(q_t)\|^2}{(L+\rho)\|p_t\|^2} \right\} \|p_t\| \\
&\geq \max\left\{ acs, \frac{rcs^2}{(L+\rho)\gamma} \right\} > 0,
\end{aligned}$$

which contradicts (20). Hence (21) is valid. □

4. Numerical Result

The numerical section of this article focuses on evaluating the performance efficiency of the proposed method in recovering sparse signal in compressive sensing. In what follows, the proposed method is reffered to as ENPRP. All numerical results are obtained by implementing the methods in Matlab R2020b on a HP laptop with 8GB RAM and 2.40 GHz processor. ENPRP method is compared with some other efficient methods such as the conjugate gradient method for solving convex constrained monotone equations with applications in compressive sensing by Xiaoh et al. [5] and the projection method for convex constrained monotone nonlinear equations with applications by Liu et al. [36].

The experiment considered a typical compressive sensing scenario where the main aim is to reconstruct a length-n sparse signal from m observations. We used the mean squared error (MSE) to evaluate the quality of the signal restoration. Mathematically, the MSE is computed using the formula

$$MSE := \frac{1}{n}\|\bar{q}-q\|, \tag{24}$$

where $\bar{q}$ denotes the original signal and q denotes the restored signal. The experiment is implemented with chosen signal size of $n = 2^{12}$ and $m = 2^{10}$ where the original signal contains 2^6 randomly non-zeros elements. We note that, the matrix N in (1) is generated in MATLAB via the command $randn(m,n)$ and the noise k (Gaussian noise) is distributed as 10^{-3}

In this paper, the parameters for implementation of our method are specified by $\eta = 0.5,\ r = 0.5,\ m = 1,\ \rho = 0.8,\ a = 1$ and the following merit function

$$f(q) = \mu\|q\|_1 + \frac{1}{2}\|Nq-b\|_2^2 \tag{25}$$

is employed. The methods are implemented using the same initial point and the regularization parameter μ is selected based on the approach in [36], that is,

$$\mu = 0.005\|N^T b\|_\infty.$$

A measurement image $q_0 = N^T b$ is used in starting the experiment and the stopping criterion

$$\frac{\|f_t - f_{t-1}\|}{\|f_{t-1}\|} < 10^{-5}$$

where f_t is the the function value at q_t is employed.

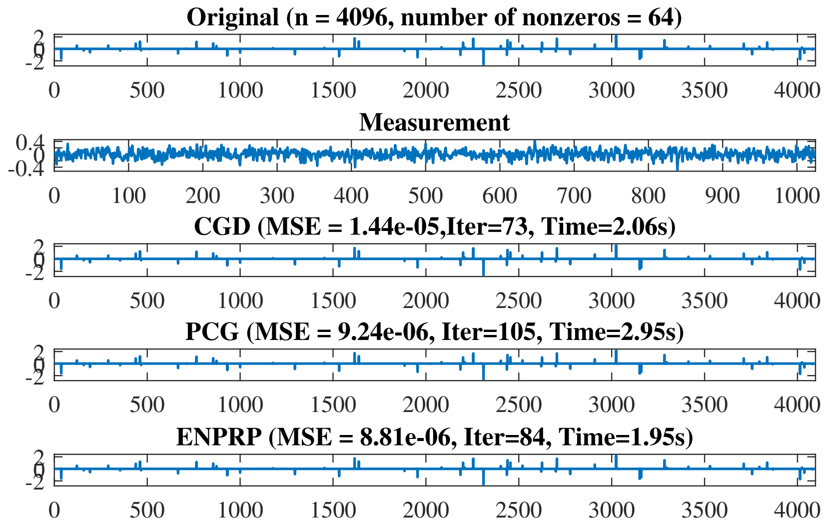

FIGURE 1. Reconstruction of sparse signal. From the top to the bottom is the original signal (First plot), the measurement (Second plot), and the reconstructed signals by CGD (Third plot), PCG (Fourth plot) and ENPRP (Fifth plot).

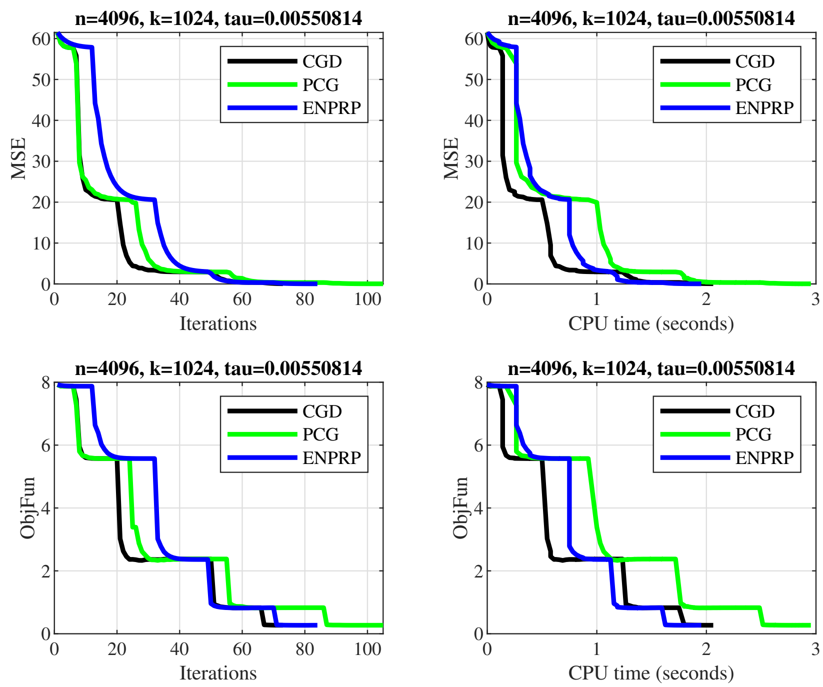

FIGURE 2. Comparison results of ENPRP, CGD and PCG algorithm. From left to right: the changed trend of MSE goes along with the number of iterations or CPU time in seconds, and the changed trend of the objective function values accompany the number of iterations or CPU time in seconds.

When comparing the efficiency of the methods, different noise samples were selected with the experiment repeated at least 20 times. We refer readers to Table 1. The figure 1 shows the distorted signal reconstructed by the various methods. Furthermore, the performance comparison of ENPRP method verses CGD and PCG in terms of their convergence behaviour from the trend of MSE and objective function values along with the number of iterations and CPU time increasing is illustrated on Figure 2.

From Table 1 and the provided plots, ENPRP method is the top performer as it has successfully reconstructed the sparse signal in most cases. In general, ENPRP method reported least mean squared error and required lesser number of iteration and CPU time in recovering the sparse signal. On the overall, in this experiment, based on the performance result obtained from the experiment, we can clearly see that ENPRP method best performed in recovering the sparse signal compared to CGD and PCG.

TABLE 1. Result of the Numerical Experiments

	CGD			PCG			ENPRP		
SN	CPU(s)	#ITN	MSE	CPU(s)	#ITN	MSE	CPU(s)	#ITN	MSE
1	3.45	130	7.20E-06	2.67	102	1.91E-05	2.23	91	6.27E-06
2	3.13	103	5.51E-06	3.25	107	4.19E-06	2.73	85	4.45E-06
3	3.20	115	9.26E-06	3.75	118	7.88E-06	2.63	89	7.50E-06
4	2.20	88	8.69E-06	3.55	106	6.41E-06	1.91	81	6.68E-06
5	2.84	115	8.19E-06	2.58	101	2.20E-05	2.44	91	8.29E-06
6	2.72	102	1.26E-05	3.38	108	1.19E-05	2.48	90	1.16E-05
7	2.92	96	1.14E-05	3.61	99	1.08E-05	2.78	77	1.14E-05
8	2.78	89	3.09E-05	2.86	99	9.42E-06	2.81	93	9.84E-06
9	2.63	102	7.88E-06	3.05	108	6.91E-06	2.02	81	7.32E-06
10	3.84	120	1.01E-05	3.03	104	9.57E-06	2.27	78	9.63E-06
11	2.38	94	7.15E-06	2.48	104	7.01E-06	2.09	87	7.55E-06
12	2.09	83	9.48E-06	3.08	116	8.81E-06	2.30	86	8.32E-06
13	3.09	113	1.32E-05	3.09	105	8.14E-06	2.06	86	8.21E-06
14	2.86	114	1.01E-05	3.09	113	8.87E-06	2.30	86	8.49E-06
15	3.14	100	1.06E-05	2.78	86	1.06E-05	2.63	89	1.11E-05
16	2.28	78	8.65E-06	2.59	100	5.71E-06	2.11	78	5.46E-06
17	2.95	98	2.01E-05	3.03	97	2.25E-05	2.94	95	1.20E-05
18	3.61	113	1.21E-05	3.50	101	1.11E-05	2.83	89	1.09E-05
19	2.50	94	1.00E-05	2.84	105	9.79E-06	2.00	73	1.00E-05
20	2.05	77	9.14E-06	2.98	108	8.34E-06	2.13	87	7.92E-06
21	2.41	95	7.02E-06	2.50	106	6.68E-06	1.89	80	6.09E-06
22	2.39	84	1.46E-05	2.47	97	1.69E-05	2.11	79	9.82E-06
23	2.31	76	6.64E-06	2.69	97	4.98E-06	2.02	80	5.19E-06
24	2.78	90	9.84E-06	3.17	106	9.01E-06	2.42	78	9.57E-06
25	3.11	99	9.81E-06	3.28	107	9.56E-06	2.67	90	9.90E-06
26	3.38	124	1.04E-05	2.95	110	1.02E-05	2.34	93	9.76E-06
27	2.20	89	8.69E-06	2.66	110	4.84E-06	2.08	81	5.09E-06
28	2.86	95	5.19E-06	3.13	98	8.02E-06	2.64	88	4.45E-06
29	2.59	88	1.33E-05	3.06	102	1.21E-05	2.31	83	1.24E-05
30	2.77	100	7.00E-06	2.59	102	2.08E-05	2.17	85	6.84E-06
31	2.41	90	9.29E-06	2.59	94	1.65E-05	2.59	87	9.07E-06
Average	2.77	99	1.05E-05	2.98	104	1.06E-05	2.35	85	8.42E-06

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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