Available online at http://scik.org
J. Math. Comput. Sci. 2022, 12:51
https://doi.org/10.28919/jmcs/6547
ISSN: 1927-5307

THE FORCING STAR EDGE CHROMATIC NUMBER OF A GRAPH

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Abstract. Let S be a χ'_{st}-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique χ'_{st}-set containing T. The forcing star-edge chromatic number $f_{\chi'_{st}}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The *forcing star-edge chromatic number* $f_{\chi'_{st}}(G)$ of G is the smallest forcing number of all χ'_{st}*-sets* of G. Some general properties satisfied by this concept are studied. It is shown that for every pair a and b of integers with $0 \leq a < b$ and $b > a+2$ there exists a connected graph G such that $f_{\chi'_{st}}(G) = a$ and $\chi'_{st}(G) = b$, where $\chi'_{st}(G)$ is the star edge chromatic number of a graph.

Keywords: forcing star edge chromatic number; star edge chromatic number; edge chromatic number.

2010 AMS Subject Classification: 05C15.

1. INTRODUCTION

By a graph $G = (V,E)$, we mean a finite,undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. If $uv \in E(G)$, we say that u is a *neighbor* of v and denote by $N(v)$, the set of

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Received July 23, 2021

neighbors of v. The *degree* of a vertex $v \in V$ is $deg(v) = \mid N(v) \mid$. A vertex v is said to be a *universal vertex* if $deg(v) = n-1$.

A p-vertex coloring of is an assignment of p colors, $1,2,...,p$ to the vertices of G, the coloring is proper if no two distinct adjacent vertices have the same color. If $\chi(G) = p$, G is said to be p - *chromatic*, where $p \leq k$. A set $C \subseteq V(G)$ is called *chromatic set* if C contains all vertices of distinct colors in G. The Chromatic number of G is the minimum cardinality among all the chromatic sets of G. That is $\chi(G)$ = $min\{|C_i|/ \ C_i$ is a chromatic set of G $\}$. The concept of the chromatic number was studied in [1,2,4,5,8,9,13]. A *star colouring* of a graph G is proper colouring such that no path of length 4 is bicolourable. The minimum colours needed for a star coloring of G is called star chromatic number and is denoted by $\chi_s(G)$. Let G be a star colourable. A set $S \subseteq V(G)$ is called a *star chromatic set* if S contains all vertices of distinct colours in G. Any star chromatic set of order $\chi_s(G)$ is called a χ_s-set of G. The *edge-chromatic number* $\chi_e(G)$ of G is defined to be the least number of colours needed to colour the edges of G in such a way that no two adjacent edges have the same colour. The concept of edge chromatic number was studied in [1,14]. A *star edge colouring* of a graph G is a proper colouring without bichromatic 4-paths and 4-cycles and is denoted by $\chi'_{st}(G)$. Let G be a star edge colourable graph. A set $S \subseteq E(G)$ is called a star edge chromatic set if S contains all edges of distinct colours in G. Any star edge chromatic set of order $\chi'_{st}(G)$ is called a χ'_{st}-set of G. The concept of the star edge chromatic number was studied in [3,6,7,10]. The chromatic number has application in Time Table Scheduling, Map coloring, channel assignment problem in radio technology, town planning, GSM mobile phone networks etc [8,9]. The following theorem is used in the sequel.

Theorem 1.1. [**14**] For a complete graph $G = K_n$ $(n \geq 2)$, $\chi'_{st}(G) = n$.

2. The Forcing Star Edge Chromatic Number of a Graph

Definition 2.1. Let S be a χ'_{st}-set of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique χ'_{st}-set containing T. The *forcing star-edge chromatic number* $f_{\chi'_{st}}(S)$ of S in G is the minimum cardinality of a forcing subset for S. The *forcing star-edge chromatic number* $f_{\chi'_{st}}(G)$ of G is the smallest forcing number of all χ'_{st}*-sets* of G.

Example 2.2. For the graph G of Figure 2.1, $S_1 = \{e_1,e_2,e_3,e_4,e_5,e_8\}$, $S_2 = \{e_1,e_2,e_3,e_5,e_7,e_8\}$, $S_3 = \{e_1,e_3,e_4,e_5,e_6,e_8\}$ and $S_4 = \{e_1,e_3,e_5,e_6,e_7,e_8\}$ are the χ'_{st}-sets of G such that $f_{\chi'_{st}}(S_i) = 2$, for i = 1 to 4 so that $\chi'_{st}(G) = 6$ and $f_{\chi'_{st}}(G) = 2$.

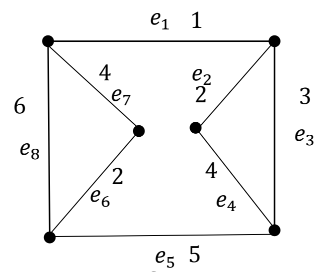

Figure 2.1

Observation 2.3. For every connected graph G, $0 \le f_{\chi'_{st}}(G) \le \chi'_{st}(G)$.

Remark 2.4. The bounds in Observation 2.3 are sharp. For the graph G given in Figure 2.2, $S = E(G)$ is the unique χ'_{st}-set of G such that $f_{\chi'_{st}}(G) = 0$. For the graph $G = C_6$ with edge set $E(G) = \{e_1,e_2,e_3,e_4,e_5,e_6\}$, $S_1 = \{e_1,e_2,e_3\}$, $S_2 = \{e_2,e_3,e_4\}$, $S_3 = \{e_1,e_2,e_6\}$, $S_4 = \{e_2,e_4,e_6\}$, $S_5 = \{e_1,e_3,e_5\}$, $S_6 = \{e_3,e_4,e_5\}$, $S_7 = \{e_1,e_5,e_6\}$ and $S_8 = \{e_4,e_5,e_6\}$ are the only eight χ'_{st}-sets of G such that $f_{\chi'_{st}}(S_i) = 3$ for $1 \le i \le 8$ so that $\chi'_{st}(G) = 3$ and $f_{\chi'_{st}}(G) = 3$. Also the bounds are strict. For the graph G given in Figure 2.1, $\chi'_{st}(G) = 6$, $f_{\chi'_{st}}(G) = 2$. Thus $0 < f_{\chi'_{st}}(G) < \chi'_{st}(G)$.

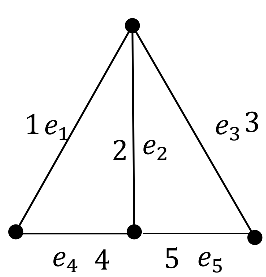

Figure 2.2

Definition 2.5. An edge e of a graph G is said to be a *star edge chromatic edge* of G if e belongs to every χ'_{st}-set of G.

Example 2.6. For the graph G of Figure 2.3, $S_1 = \{e_1, e_2, e_3, e_6\}$, $S_2 = \{e_1, e_3, e_4, e_6\}$, $S_3 = \{e_1, e_4, e_3, e_6\}$ are the χ'_{st}-sets of G such that e_1 and e_6 are a star edge chromatic edge of G.

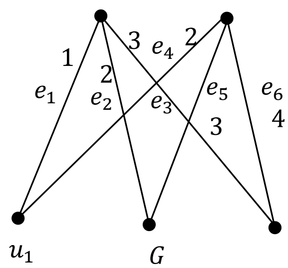

Figure 2.3

Theorem 2.7. Let G be a connected graph of order $n \geq 3$ with $\triangle(G) = n-1$. Let x be a universal vertex of G and e be an edge incident with x. Then e is a star edge chromatic edge of G.

Proof. On the contrary, suppose e is not a star edge chromatic edge of G. Then there exists a χ'_{st}-set S such that $e = xv$. Let $c(e) = c_1$. Since $e \notin S$, there exists $f = yz \in E(G)$ such that $c(f) = c_1$ and $y \neq x$, v and $z \neq x, v$. Since e and f are not adjacent, e and f are edges of a path P of length 4. Hence it follows that P is bi-colourable, which is a contradiction. $\square$

Observation 2.8. Let G be a connected graph. Then
(a) $f_{\chi'_{st}}(G) = 0$ if and only if G has a unique χ'_{st}-set.
(b) $f_{\chi'_{st}}(G) = 1$ if and only if G has at least χ'_{st}-set, containing one of its elements and
(c) $f_{\chi'_{st}}(G) = \chi'_{st}(G)$ if and only if χ'_{st}-set of G is the unique minimum χ'_{st}-set containing any of its proper subsets.

Theorem 2.9. Let G be a connected graph and W be the set of all star edge chromatic edges of G. Then $f_{\chi'_{st}}(G) \leq \chi'_{st}(G) - \mid W \mid$.

Proof. Let S be any χ'_{st}-set of G. Then $\chi'_{st}(G) = \mid S \mid$, $W \subseteq S$ and S is the unique χ'_{st}-set containing $S - W$. Thus $f_{\chi'_{st}}(G) \leq \mid S - W \mid = \mid S \mid - \mid W \mid = \chi'_{st}(G) - \mid W \mid$. $\square$

Observation 2.10. (a) For the complete graph $G = K_n$ $(n \geq 2)$, $f_{\chi'_{st}}(G) = 0$.
(b) For the star $G = K_{1,n-1}$ $(n \geq 3)$, $f_{\chi'_{st}}(G) = 0$.
(c) For the double star $G = K_{2,r,s}$, $f_{\chi'_{st}}(G) = 0$.

Theorem 2.11. For the complete bipartite graph $G = K_{r,s}$ $(1 \leq r \leq s)$,

$$f_{\chi'_{st}}(G) = \begin{cases} 0 & \text{if } r = 1,2,\ s \geq 2 \\ s-1 & \text{if } r = 2,\ s \geq 3 \\ s & \text{if } r = 3,\ s \geq 3 \\ s+r-3 & \text{if } r \geq 4, s \geq 4 \end{cases}$$

Proof. If $r = 1$ and $s \geq 2$, then the result follows from Observation 2.10(b). For $r = 2$ and $s = 2$, the result follows from Theorem 2.13. So, let $2 \leq r \leq s$. Let $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$ be the bipartite sets of G. Let $r = 2$ and $s \geq 3$. Let $e_{1j} = x_1y_j$ and $e_{2j} = x_2y_j$ $(1 \leq j \leq s)$, assign $c(e_{1j}) = c_j$ $(1 \leq j \leq s)$ and $c(e_{2j}) = c_{j+1}$ $(1 \leq j \leq s-1)$ and $c(e_{2s}) = s+1$ so that $\chi'_{st}(G) = s+1$. Since $\{e_{11}, e_{2s}\}$ is the set of all star edge chromatic edge of G, by Theorem 2.9, $f_{\chi'_{st}}(G) \leq s+1-2 = s-1$. Let S be a star edge chromatic edge set of G. We prove that $f_{\chi'_{st}}(S) = s-1$. On the contrary suppose that $f_{\chi'_{st}}(G) \leq s-2$. Then there exists a forcing subset T of S such that $\mid T \mid \leq s-2$. Let $e \in S$ such that $e \notin T$. Then e is not a star edge chromatic edge of G. Without loss of generality, let us assume that $c(e) = c_1$. Since $s \geq 3$, there exists $f \in E(G)$ such that $c(f) = c_1$. Let $S' = [S - \{e\}] \cup \{f\}$. Then S' is a χ'_{st}-set of G. Hence T is a proper subset of a χ'_{st}-set S' of G, which is a contradiction. Therefore $f_{\chi'_{st}}(G) = s-1$.

Let $r = 3$, $s \geq 3$. Let $e_{ij} = x_1y_j$, $e_{2j} = x_2y_j$, $e_{3j} = x_3y_j$ $(1 \leq j \leq s)$. Assign $c(e_{1j}) = c_j$ $(1 \leq j \leq s)$,c(e$_{2j}$) $= c_{j+1}$ $(1 \leq j \leq s-1)$, $c(e_{3j}) = c_{j+2}$ $(1 \leq j \leq s-2)$ and $c(e_{3s}) = s+2$ so that $\chi'_{st}(G) = s+2$. Since $\{e_{11}, e_{3s}\}$ are the star edge chromatic edges of G, by Theorem 2.9, $f_{\chi'_{st}}(G) \leq s+2-2 = s$. Let S be a star edge chromatic edge set of G. We prove that $f_{\chi'_{st}}(G) = s$. On the contrary, suppose that $f_{\chi'_{st}}(G) \leq s-1$. Then there exists a forcing subset T of S such that $\mid T \mid \leq s-1$. Let $e \in S$ such that $e \notin T$. Then e is not a star edge chromatic edge of G. Without loss of generality, let us assume that $c(e) = c_1$. Since $s \geq 3$, there exists $f \in E(G)$ such that $c(f) = c_1$. Let $S' = [S - \{e\}] \cup \{f\}$. Then S' is a χ'_{st}-set of G, which is a contradiction.

Therefore Hence T is a proper subset of a χ'_{st}-set S' of G, which is a contradiction. Therefore $f_{\chi'_{st}}(S) = s$. Since this is true for all χ'_{st}-set S of G, $f_{\chi'_{st}}(G) = s$.

Let $r \geq 4$, $s \geq 4$. Let $e_{ij} = x_1y_j$, $e_{2j} = x_2y_j$,...,$e_i = x_iy_j$ $(1 \leq i \leq r)$, $(1 \leq j \leq s)$. Assign $c(e_{1j}) = c_j$ $(1 \leq j \leq s)$, $c(e_{2j}) = c_{j+1}$ $(1 \leq j \leq s-1)$,...,$c(e_{ij}) = c_{j+i-1}$ $(1 \leq i \leq r)$ $(1 \leq j \leq s-i+1)$ and $c(e_{is}) = s+r-1$ so that $\chi'_{st}(G) = s+r-1$. Since $\{e_{11}, e_{rs}\}$ is the set of all star edge chromatic edges of G, by Theorem 2.9, $f_{\chi'_{st}}(G) \leq s+r-3$. Let S be a star edge chromatic edge set of G. We prove that $f_{\chi'_{st}}(S) = s+r-3$. On the contrary, suppose that $f_{\chi'_{st}}(S) \leq s+r-4$. Then there exists a forcing subset T of S such that $\mid T \mid \leq s+r-4$. Let $e \in S$ such that $e \notin T$. Then e is not a star edge chromatic edge of G. Without loss of generality, let us assume that $c(e) = c_1$. Since $s \geq 3$, there exists $f \in E(G)$ such that $c(f) = c_1$. Let $S' = [S - \{e\}] \cup \{f\}$. Then S' is a χ'_{st}-set of G. which is a contradiction. Therefore $f_{\chi'_{st}}(S) = s+r-3$. Since this is true for all χ'_{st}-set S of G, $f_{\chi'_{st}}(G) = s+r-3$. □

Theorem 2.12. For the path $G = P_n$ $(n \geq 3)$, $f_{\chi'_{st}}(G) = \begin{cases} 0 & \text{if } n = 3,4 \\ 1 & \text{if } n = 5 \\ 2 & \text{if } n = 6 \\ 3 & \textit{otherwise} \end{cases}$

Proof. Let P_n be $v_1, v_2, ..., v_n$ and $e_i = v_{i-1}v_i$ $(2 \leq i \leq n)$. For $n = 3$ and $n = 4$, $S = E(G)$ is the unique χ'_{st}-set then the result follows from Observation 2.8 (a). For $n = 5$, $S_1 = \{e_1, e_2, e_3\}$ and $S_2 = \{e_2, e_3, e_4\}$ are the only χ'_{st}-sets of G such that $f_{\chi'_{st}}(G) = 1$. For $n = 6$, $S_1 = \{e_1, e_2, e_3\}$, $S_2 = \{e_2, e_3, e_4\}$, $S_3 = \{e_2, e_3, e_4\}$, $S_4 = \{e_3, e_4, e_5\}$ are the only χ'_{st}-sets of G such that $f_{\chi'_{st}}(G) = 2$. For $n \geq 7$, we consider the following cases.

Case (i) $n = 3r+1$, $r \geq 2$. Assign $c(e_i) = 1$, $i = 1, 4, ..., 3r-2$, $c(e_j) = 2$, $j = 2, 5, ..., 3r-1$, $c(e_k) = 3$, $k = 3, 6, ..., 3r$. Then $S_{ijk} = \{e_i, e_j, e_k\}$ and $S_{ik} = \{e_i, e_{3r-2}, e_k\}$ are the χ'_{st}-sets of G such that $\chi'_{st}(S_{ijk}) = \chi'_{st}(S_{ik}) = 3$ for i, j, k $(i = 1, 4, ..., 3r-2, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r)$ so that $\chi'_{st}(G) = 3$. By Observation 2.3, $0 \leq f_{\chi'_{st}}(G) \leq 3$. Since χ'_{st}-set of G is not unique $f_{\chi'_{st}}(G) \geq 1$. It is easily observed that no singleton subsets or two elements subsets of S_{ijk} for all i, j, k $(i = 1, 4, ..., 3r-2, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r)$ is a forcing subset of S_{ijk} so that $f_{\chi'_{st}}(S_{ijk}) = 3$. Similarly no singleton or two element subsets of S_{jk} is a forcing subset of S_{ik}

so that $f_{\chi'_{st}}(S_{jk}) = 3$. Since this is true for all χ'_{st}-set S_{ijk} for all i, j, k ($i = 1, 4, 3r-2, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r$) so that $f_{\chi'_{st}}(G) = 3$.

Case (ii) $n = 3r+2$, $r \geq 2$. Assign $c(e_i) = 1$, $i = 1, 4, ..., 3r+1$, $c(e_j) = 2$, $j = 2, 5, ..., 3r-1$, $c(e_k) = 3$, $k = 3, 6, ..., 3r$. Then $S_{ijk} = \{e_i, e_j, e_k\}$, $S_{ij} = \{e_i, e_j, e_{3r-2}\}$, $S_i = \{e_i, e_{3r+1}, e_{3r-1}\}$ are the χ'_{st}-sets of G such that $\chi'_{st}(S_{ijk}) = \chi'_{st}(S_{ik}) = \chi'_{st}(S_{ij}) = \chi'_{st}(S_i) = 3$ for i, j, k ($i = 1, 4, ..., 3r+1, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r$) so that $\chi'_{st}(G) = 3$. By Observation 2.3, $0 \leq f_{\chi'_{st}}(G) \leq 3$. Since χ'_{st}-set of G is not unique $f_{\chi'_{st}}(G) \geq 1$. It is easily observed that no singleton subsets or two elements subsets of S_{ijk} for all i, j, k ($i = 1, 4, ..., 3r+1, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r$) is a forcing subset of S_{ijk} so that $f_{\chi'_{st}}(S_{ijk}) = 3$. Similarly no singleton or two element subsets of S_{jk} for all i, k ($i = 1, 4, ..., 3r+1, k = 3, 6, ..., 3r$) is a forcing subset of S_{ik} so that $f_{\chi'_{st}}(S_{ik}) = 3$. Similarly no singleton subsets or two elements subsets of S_{ij} for all i, j, k ($i = 1, 4, ..., 3r+1, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r$) is a forcing subset of S_{ijk} so that $f_{\chi'_{st}}(S_{ijk)} = 3$. Similarly no singleton or two element subsets of S_{ij} for all i, j ($i = 1, 4, ..., 3r+1, j = 2, 5, ..., 3r-1$) is a forcing subset of S_{ij} so that $f_{\chi'_{st}}(S_{ij}) = 3$. Similarly no singleton subsets or two elements subsets of S_i for all i ($i = 1, 4, ..., 3r+1$) is a forcing subset of S_i so that $f_{\chi'_{st}}(S_{i)=3}$. Since this is true for all χ'_{st}-sets S_{ijk}, S_{ik}, S_{ij} and S_i for all i, j, k ($i = 1, 4, ..., 3r+1, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r$) so that $f_{\chi'_{st}}(G) = 3$.

Case (iii) $n = 3r$, $r \geq 3$. Assign $c(e_i) = 1$, $i = 1, 4, ..., 3r-2$, $c(e_j) = 2$, $j = 2, 5, ..., 3r-1$, $c(e_k) = 3$, $k = 3, 6, ..., 3r-3$. Then $S_{ijk} = \{e_i, e_j, e_k\}$ is a χ'_{st}-set of G such that $\chi'_{st}(S_{ijk}) = 3$ for i, j, k ($i = 1, 4, ..., 3r-2, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r-3$) so that $\chi'_{st}(G) = 3$. By Observation 2.3, $0 \leq f_{\chi'_{st}}(G) \leq 3$. Since χ'_{st}-set of G is not unique $f_{\chi'_{st}}(G) \geq 1$. It is easily observed that no singleton subsets or two elements subsets of S_{ijk} for all i, j, k ($i = 1, 4, ..., 3r-2, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r-3$) is a forcing subset of S_{ijk} so that $f_{\chi'_{st}}(S_{ijk}) = 3$. Since this is true for all χ'_{st}-set S_{ijk} for all i, j, k ($i = 1, 4, ..., 3r-2, j = 2, 5, ..., 3r-1, k = 3, 6, ..., 3r-3$) so that $f_{\chi'_{st}}(G) = 3$. $\square$

Theorem 2.13. For the cycle $G = C_n$ ($n \geq 4$), $f_{\chi'_{st}}(G) = \begin{cases} 0 & \text{if } n = 4,5 \\ 3 & otherwise \end{cases}$

Proof. Let C_n be $v_1, v_2, ..., v_n, v_1$ and $e_i = v_{i-1}v_i$ $(2 \leq i \leq n)$, $e_n = v_nv_1$. For $n = 4$ and 5, $S = E(G)$ is the unique $\chi_{st}^{'}$-set so that $f_{\chi_{st}'}(G) = 0$. For $n \geq 6$, we consider the following cases.

Case (i) $n = 3r,\ r \geq 2$. Assign $c(e_i) = 1,\ i = 1,4,...,3r-2,\ c(e_j) = 2,\ j = 2,5,...,3r-1,\ c(e_k) = 3,\ k = 3,6,...,3r$. Then $S_{ijk} = \{e_i, e_j, e_k\}$ is a $\chi_{st}^{'}$-set of G such that $\chi_{st}^{'}(S_{ijk}) = 3$ for all i, j, k $(i = 1,4,...,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r)$ so that $\chi_{st}^{'}(S_{ijk}) = 3$. By Observation 2.3, $0 \leq f_{\chi_{st}'}(G) \leq 3$. Since $\chi_{st}^{'}$-set of G is not unique $f_{\chi_{st}'}(G) \geq 1$. It is easily observed that no singleton subsets or two elements subsets of S_{ijk} for all i, j, k $(i = 1,4,...,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r)$ is a forcing subset of S_{ijk} so that $f_{\chi'_{st}}(S_{ijk}) = 3$. Since this is true for all $\chi_{st}^{'}$-set S_{ijk} for all i, j, k $(i = 1,4,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r)$ so that $f_{\chi_{st}'}(G) = 3$.

Case (ii) $n = 3r+1,\ r \geq 2$. Assign $c(e_i) = 1,\ i = 1,4,...,3r-2,\ c(e_j) = 2,\ j = 2,5,...,3r-1,\ c(e_k) = 3,\ k = 3,6,...,3r,\ c(e_n) = 4,\ n = 3r+1$. Then $S_{ijkn} = \{e_i, e_j, e_k, e_n\}$ and $S_{ikn} = \{e_i, e_{3r-2}, e_k, e_n\}$ are the $\chi_{st}^{'}$-sets of G such that $\chi_{st}^{'}(S_{ijkn}) = \chi_{st}^{'}(S_{ikn}) = 4$ for all i, j, k, n $(i = 1,4,....,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r, n = 3r+1)$ so that $\chi_{st}^{'}(G) = 4$ and $f_{\chi_{st}'}(S_{ijkn}) = f_{\chi_{st}'}(S_{ikn}) = 3$. Since this is true for all i, j, k, n $(i = 1,4,....,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r, n = 3r+1)$, $f_{\chi_{st}'}(G) = 3$.

Case (iii) $n = 3r+2,\ r \geq 2$. Assign $c(e_i) = 1,\ i = 1,4,...,3r-2,\ c(e_j) = 2,\ j = 2,5,...,3r-1,\ c(e_k) = 3,\ k = 3,6,...,3r,\ c(e_{n-1}) = 4,\ n = 3r+2,\ c(e_n) = 5,\ n = 3r+2$. Then $S_{ijkn} = \{e_i, e_j, e_k, e_{n-1}, e_n\}$, $S_{ikn} = \{e_i, e_{3r-2}, e_k, e_{n-1}, e_n\}$, $S_{ijn} = \{e_i, e_j, e_{3r-1}, e_{n-1}, e_n\}$, $S_{in} = \{e_i, e_{3r-2},$
$e_{3r-1}, e_{n-1}, e_n\}$ are the only $\chi_{st}^{'}$-sets of G such that $\chi_{st}^{'}(S_{ijkn}) = \chi_{st}^{'}(S_{ikn}) = \chi_{st}^{'}(S_{ijn}) = \chi_{st}^{'}(S_{in}) = 5$ for all i, j, k, n $(i = 1,4,....,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r, n = 3r+2)$ so that $\chi_{st}^{'}(G) = 5$ and $f_{\chi_{st}'}(S_{ijkn}) = f_{\chi_{st}'}(S_{ikn}) = f_{\chi_{st}'}(S_{ijn}) = f_{\chi_{st}'}(S_{in}) = 3$. Since this is true for all i, j, k, n $(i = 1,4,....,3r-2, j = 2,5,...,3r-1, k = 3,6,...,3r, n = 3r+2)$, $f_{\chi_{st}'}(G) = 3$. □

Theorem 2.14. For every pair a and b of integers with $0 \leq a < b$ and $b > a+2$ there exists a connected graph G such that $f_{\chi_{st}'}(G) = a$ and $\chi_{st}^{'}(G) = b$.

Proof. For $a = 0$ and $b \geq 2$, let $G = K_b$. Then by Observation 2.9(a) and Theorem 1.1, $f_{\chi_{st}'}(G) = 0$ and $\chi_{st}^{'}(G) = b$. For $a = 1$, $b = 3$, let $G = P_5$. Then by Theorem 2.12, $f_{\chi_{st}'}(G) = 1$ and $\chi_{st}^{'}(G) = 3$. Let $P_5 : v_1, v_2, v_3, v_4, v_5$. Let G be the graph obtained from P_5 by adding new vertices $z_1, z_2, ..., z_{b-3}$ and introducing edge v_1z_i $(1 \leq i \leq b-3)$. The graph G is shown in Figure 2.4.

Let $c(v_1z_1)=1$, $c(v_1z_2)=2$, $c(v_1z_{b-2})=b-3$, $c(v_1v_2)=b-2$, $c(v_2v_3)=b-1$, $c(v_3v_4)=b$, $c(v_4v_5)=b-2$. Then $Z=\{v_1z_1,v_2z_2,...,v_1z_{b-3},v_2v_1,v_3v_4\}$ is the set of all star edge chromatic edge of G. Then $S_1=Z\cup\{v_1v_2\}$ and $S_2=Z\cup\{v_4v_5\}$ are the only two $\chi_{st}^{'}$-sets of G such that $f_{\chi_{st}'}(S_1)$ =f$_{\chi_{st}'}(S_2)=1$ so that $f_{\chi_{st}'}(G)=1$ and $\chi_{st}^{'}(G)=b$. So, let $a\geq 2$ and $b\geq 4$. Let $H=K_3,a$ be a complete bipartite graph with bipartite sets $X_1=\{x_1,x_2,x_3\}$ and $Y_1=\{y_1,y_2,...,y_a\}$. Let G be the graph obtained from H by adding new vertices $z_1,z_2,...,z_{b-a-2}$ and introducing edges x_1z_i $(1\leq i\leq a-2)$. The graph G is shown in Figure 2.5.

Assign $c(x_1y_i)=c_i$ $(1\leq i\leq a)$, $c(x_2y_i)=c_{i+1}$ $(1\leq i\leq a)$, $c(x_3y_i)=c_{i+2}$ $(1\leq i\leq a)$, $c(x_iz_i)=c_{a+2+i}$ $(1\leq i\leq b-a-2)$. Then C is a proper star edge colouring of G such that $\chi_{st}^{'}(G)=a+2+b-a-2=b$.

We prove that $f_{\chi_{st}'}(G)=a$. Let $Z=\{x_1z_1,x_1z_2,...,x_1z_{b-a-2},x_3y_a\}$ be the set of all star edge chromatic edge of G. By Theorem 2.9, $f_{\chi_{st}'}(G)\leq b-(b-a-2+2)=a$. Suppose that $f_{\chi_{st}'}(G)<a$. Then there exists a forcing subset T of S such that $\mid T\mid\leq a-1$. Let $e\in Z$ such that $e\notin T$. Then e is not a star edge chromatic edge of G. Without loss of generality, let us assume $c(e)=c_2$. Since $a\geq 2$, there exists $f\in E(G)$ such that $c(f)=c_2$. Let $Z^{'}=[Z-\{e\}]\cup\{f\}$. Then Z' is a $\chi_{st}^{'}$-set of G. Hence T is a proper subset of $\chi_{st}^{'}$-set of Z' of G, which is a contradiction. Therefore $f_{\chi_{st}'}(G)=a$. $\square$

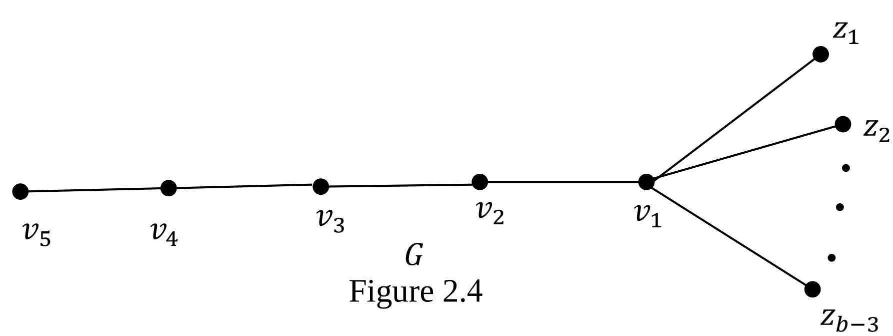

Figure 2.4

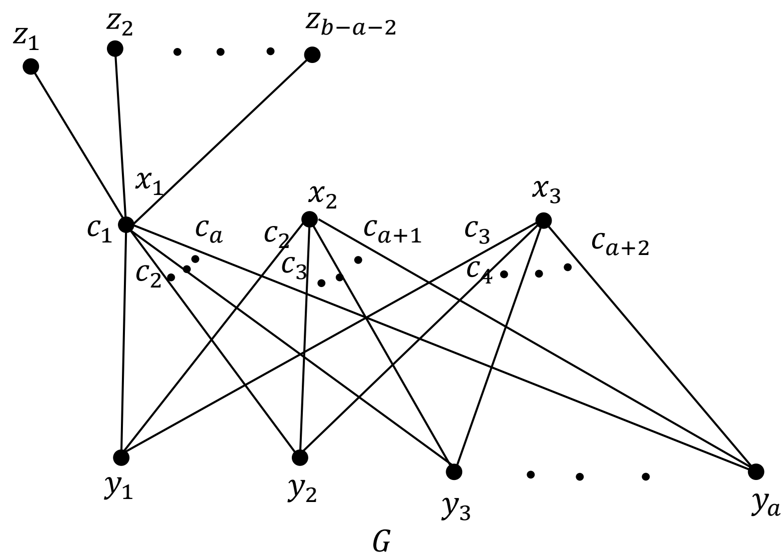

Figure 2.5

3. Conclusion

In this paper, we studied the concept of forcing star edge chromatic number of a graph. We extend this concept to graph products in future work.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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