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SOME FUZZY SUBSETS VIA OPERATION

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Abstract. In this paper, we introduce and study some types of fuzzy subsets via operation on smooth fuzzy topological space in Sǒstak sense. Also, many interesting properties of this notions are presented.

Keywords: smooth fuzzy topology; fuzzy operation-open set.

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1. INTRODUCTION

The notion of fuzzy set was introduced by L.A. Zadeh in the year 1965. Since than it has been applied in almost all the branches of science and technology, where set theory and mathematical logic play an important role. This newly introduced concept opened lot of scope and directions for investigations in many ways in all the branches for research. Recently fuzzy set theory has been applied and fuzzy topological spaces have been studied by Kubiak [7] and Sǒstak [10] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang [1] and Goguen [6]). In this paper, we introduce and study some types of fuzzy subsets on smooth fuzzy topological space in Sǒstak sense. Also, many interesting properties of this notions are presented.

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2. Preliminaries

Definition 2.1. *A fuzzy point x_t in X is a fuzzy set taking value $t \in I_0$ at x and zero elsewhere, $x_t \in \lambda$ if and only if $t \leq \lambda(x)$. A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \bar{q} \mu$.*

Definition 2.2. [7, 10] *A function $\tau : I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions:*

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$;

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$ *for any* $\mu_1, \mu_2 \in I^X$.

(3) $\tau(\bigvee_{j \in \Gamma} \mu_j) \geq \bigvee_{j \in \Gamma} \tau(\mu_j)$ *for any* $\{\mu_j\}_{j \in \Gamma} \in I^X$.
The pair (X, τ) is called a smooth fuzzy topological space.

A fuzzy point in X with support $x \in X$ and the value $\alpha (0 < \alpha \leq 1)$ is denoted by x_α.

Definition 2.3. [9] *A fuzzy set $\lambda \in I^X$ is said to be q-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. It is known that $\lambda \leq \mu$ if and only if λ and $1 - \mu$ are not q-coincident, denoted by $\lambda \bar{q}(1 - \mu)$.*

Definition 2.4. [2] *A fuzzy set λ is said to be r-fuzzy Q-neighbourhood of x_p if $\tau(\lambda) \geq r$ such that $x_p q \lambda$. We will denote the set of all r-fuzzy open Q-neighbourhood of x_p by $\mathscr{Q}(x_p, r)$.*

Definition 2.5. [3] *Let (X, τ) ba a smooth fuzzy topological space. Then $\gamma : \tau \to \tau_r$ be an operation such that $\lambda^\gamma = \wedge \mu$, where $\lambda \leq \mu$ for $\tau(\mu) \geq r$, $\tau(\lambda) \geq r$ and $r \in I_0$.*

Definition 2.6. [3] *A smooth topological space (X, τ) with an operation γ is said to be a smooth fuzzy operation-topological space and is denoted by (X, τ, γ).*

Definition 2.7. [3] *Let (X, τ, γ) ba a smooth fuzzy operation-topological space. For $\delta \in I^X$ and $r \in I_0$, δ is called r-fuzzy γ-open if for each $\alpha \in I^X$ with $\alpha \leq \delta$ there exists an r-fuzzy open set $\lambda \in I^X$ such that $\alpha \leq \lambda$ and $\lambda^\gamma \leq \delta$. The complement of an r-fuzzy γ-open set is called an r-fuzzy γ-closed.*

Definition 2.8. [3] *Let (X, τ, γ) ba a smooth fuzzy operation-topological space. For any $\lambda \in I^X$, the r-fuzzy γ-interior of λ is defined by, γ-$I_\tau(\lambda, r) = \vee\{\mu : \mu \leq \lambda,\ \mu$ is r-fuzzy γ-open$\}$.*

Definition 2.9. [3] *Let (X, τ, γ) ba a smooth fuzzy operation-topological space. For any $\lambda \in I^X$, the r-fuzzy γ-closure of λ is defined by, $\gamma\text{-}C_\tau(\lambda, r) = \wedge\{\mu : \mu \geq \lambda,\ \mu$ is r-fuzzy γ-closed$\}$.*

Theorem 2.10. *Let (X, τ, γ) ba a smooth fuzzy operation-topological space and $\lambda, \mu \in I^X$, then the following holds:*

(1) $\gamma\text{-}I_\tau(\lambda, r) < \lambda < \gamma\text{-}C_\tau(\lambda, r)$,

(2) $\lambda < \mu$ *implies* $\gamma\text{-}I_\tau(\lambda, r) < \gamma\text{-}C_\tau(\mu, r)$,

(3) $\gamma\text{-}I_\tau(\gamma\text{-}I_\tau(\lambda, r), r) = \gamma\text{-}I_\tau(\lambda, r)$,

(4) $\gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\lambda, r), r) = \gamma\text{-}C_\tau(\lambda, r)$,

(5) $\gamma\text{-}I_\tau(\overline{1} - \lambda, r) = \overline{1} - \gamma\text{-}C_\tau(\lambda, r)$,

(6) $\gamma\text{-}C_\tau(\overline{1} - \lambda, r) = \overline{1} - \gamma\text{-}I_\tau(\lambda, r)$.

3. SOME FUZZY SUBSETS VIA OPERATION

Throughout this section X be a nonempty set and $r \in (0, 1] = I_0$.

Proposition 3.1. *Let (X, τ, γ) be a smooth fuzzy operation-topological space. Then $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r), r) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$ and $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r), r), r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$.*

Proof. Let $\lambda \in I^X$. We have $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r) \leq \gamma\text{-}C_\tau(\lambda, r)$. Then $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r) \leq \gamma\text{-}C_\tau(\lambda, r)$ and hence $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r), r) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$. On the other hand $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r)$. Therefore, $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r), r)$. Hence $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r), r) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$. □

Definition 3.2. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\lambda \in I^X$. Then*

(1) $\lambda \in ro(\tau, r)$ *if* $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$ [4].

(2) $\lambda \in rc(\tau, r)$ *if* $\lambda = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$ [4].

(3) $\lambda \in \alpha(\tau, r)$ *if* $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r), r)$.

(4) $\lambda \in \sigma(\tau, r)$ *if* $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$.

(5) $\lambda \in \pi(\tau, r)$ *if* $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$.

(6) $\lambda \in \rho(\tau, r)$ *if* $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r) \vee \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$.

(7) $\lambda \in \beta(\tau,r)$ *if* $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$.

Remark 3.3. *Let* (X,τ,γ) *be a smooth fuzzy operation-topological space, we have* $ro(\tau,r) \leq \omega(r) \leq \alpha(\tau,r) \leq \sigma(\tau,r) \leq \rho(\tau,r) \leq \beta(\tau,r)$ *and* $\alpha(\tau,r) \leq \pi(\tau,r) \leq \rho(\tau,r)$.

Theorem 3.4. *Arbitrary union of any members of* $\pi(\tau,r)$ *(resp.* $\sigma(\tau,r)$, $\alpha(\tau,r)$, $\rho(\tau,r)$, $\beta(\tau,r)$*) is* $\pi(\tau,r)$ *(resp.* $\sigma(\tau,r)$, $\alpha(\tau,r)$, $\rho(\tau,r)$, $\beta(\tau,r)$*).*

Proof. We will give the proof for the case of $\pi(\tau,r)$, the other cases are similar. Let $\{\lambda_i : i \in \Delta\} \in \pi(\tau,r)$. Then for all $i \in \Delta$, $\lambda_i \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda_i,r),r)$. Then $\bigvee_{i\in\Delta} \lambda_i \leq \bigvee_{i\in\Delta} \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda_i,r),r) \leq \gamma\text{-}I_\tau(\bigvee_{i\in\Delta} \gamma\text{-}C_\tau(\lambda_i,r),r) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\bigvee_{i\in\Delta} \lambda_i,r),r)$. Hence $\bigvee_{i\in\Delta} \lambda_i \in \pi(\tau,r)$. The rest of the proof is similar. □

Remark 3.5. *Intersection of any two members of* $\pi(\tau,r)$ *(resp.* $\sigma(\tau,r)$, $\alpha(\tau,r)$, $\rho(\tau,r)$, $\beta(\tau,r)$*) is not in* $\pi(\tau,r)$ *(resp.* $\sigma(\tau,r)$, $\alpha(\tau,r)$, $\rho(\tau,r)$, $\beta(\tau,r)$*).*

Theorem 3.6. *Let* (X,τ,γ) *be a smooth fuzzy operation-topological space and* $\lambda \in I^X$, $\lambda \in ro(\tau,r)$ *if, and only if* $\lambda \in \alpha(\tau,r)$ *and* $\overline{1}-\lambda \in \beta(\tau,r)$.

Proof. Let $\lambda \in ro(\tau,r)$. We have $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r)$. By Theorem 2.10, $\gamma\text{-}I_\tau(\lambda,r) = \gamma\text{-}I_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r) = \lambda$. Then $\lambda = \gamma\text{-}I_\tau(\lambda,r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r)$. So $\lambda = \gamma\text{-}I_\tau(\lambda,r) = \gamma\text{-}I_\tau(\gamma\text{-}I_\tau(\lambda,r),r) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r),r)$. Thus, $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r),r)$ and hence $\lambda \in \alpha(\tau,r)$. On the other hand, since $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r)$, then $\overline{1}-\lambda = 1-\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r)$. By Theorem 2.10 (5) and (6), we have $\overline{1}-\lambda = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(1-\lambda,r),r)$ and $\gamma\text{-}C_\tau(\overline{1}-\lambda,r) - \gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\overline{1}-\lambda,r),r),r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\overline{1}-\lambda,r),r) = \overline{1}-\lambda$. Moreover, $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\overline{1}-\lambda,r),r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\overline{1}-\lambda,r),r),r)$. Then $\overline{1}-\lambda = \gamma\text{-}C_\tau(\overline{1}-\lambda,r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\overline{1}-\lambda,r),r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\overline{1}-\lambda,r),r),r)$. Thus, $\overline{1}-\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\overline{1}-\lambda,r),r),r)$ and hence $\overline{1}-\lambda \in \beta(\tau,r)$. Conversely, let $\lambda \in \alpha(\tau,r)$ and $\overline{1}-\lambda \in \beta(\tau,r)$. We have $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r),r)$ and $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r),r) \leq \lambda$. Thus, $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r),r)$ and hence, $\lambda \in ro(\tau,r)$ by Proposition 3.1. □

Theorem 3.7. *Let* (X,τ,γ) *be a smooth fuzzy operation-topological space and* $\lambda \in I^X$, $\lambda \in ro(\tau,r)$ *if, and only if* $\lambda \in \pi(\tau,r)$ *and* $\overline{1}-\lambda \in \sigma(\tau,r)$.

Proof. Let $\lambda \in \pi(\tau, r)$ and $\overline{1} - \lambda \in \sigma(\tau, r)$. Then $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$, $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r) \leq \lambda$. Thus $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$ and hence $\lambda \in ro(\tau, r)$. The converse follows from the fact that $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$. □

Theorem 3.8. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $A \in I^X$. If $\lambda \in \pi(\tau, r)$, then there exists a $\mu \in ro(\tau, r)$ such that $\lambda \leq \mu$ and $\gamma\text{-}C_\tau(\lambda, r) = \gamma\text{-}C_\tau(\mu, r)$.*

Proof. If $\lambda \in \pi(\tau, r)$, then $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$. If we take $\mu = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$, then $\mu \in ro(\tau, r)$ and also $\lambda \leq \mu$ and $\gamma\text{-}C_\tau(\lambda, r) = \gamma\text{-}C_\tau(\mu, r)$. □

Definition 3.9. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\lambda \in I^X$. Then λ is called r-fuzzy γ-dense if $\gamma\text{-}C_\tau(\lambda, r) = \overline{1}$.*

Theorem 3.10. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\lambda, \mu \in I^X$. Then we have the following*

(1) $\gamma\text{-}I_\tau(\lambda \wedge \mu, r) \leq \gamma\text{-}I_\tau(\lambda, r) \wedge \gamma\text{-}I_\tau(\mu, r)$.
(2) $\gamma\text{-}I_\tau(\lambda \vee \mu, r) \geq \gamma\text{-}I_\tau(\lambda, r) \wedge \gamma\text{-}I_\tau(\mu, r)$.
(3) $\gamma\text{-}C_\tau(\lambda, r) \vee \gamma\text{-}C_\tau(\mu, r) \leq \gamma\text{-}C_\tau(\lambda \vee \mu, r)$.
(4) $\gamma\text{-}C_\tau(\lambda, r) \wedge \gamma\text{-}C_\tau(\mu, r) \geq \gamma\text{-}C_\tau(\lambda \vee \mu, r)$.

Remark 3.11. *Let (X, τ, γ) be a smooth fuzzy operation-topological space. For any $\lambda, \mu \in I^X$, $\gamma\text{-}I_\tau(\lambda \wedge \mu, r) = \gamma\text{-}I_\tau(\lambda, r) \wedge \gamma\text{-}I_\tau(\mu, r)$ and $\gamma\text{-}C_\tau(\lambda, r) \vee \gamma\text{-}C_\tau(\mu, r) = \gamma\text{-}C_\tau(\lambda \vee \mu, r)$ are not true in general.*

Theorem 3.12. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\lambda, \mu, \delta \in I^X$. If $\lambda \in \pi(\tau, r)$, then $\lambda = \mu \wedge \delta$, where $\mu \in ro(\tau, r)$ and δ is an r-fuzzy γ-dense set.*

Proof. Let $\lambda \in \pi(\tau, r)$. By Theorem 3.8, there exists a $\mu \in ro(\tau, r)$ such that $\lambda \leq \mu$ and $\gamma\text{-}C_\tau(\lambda, r) = \gamma\text{-}C_\tau(\mu, r)$. If we take $\delta = \lambda \vee (\overline{1} - \mu)$, then by Theorem 3.10, we have $\overline{1} \leq \gamma\text{-}C_\tau(\mu, r) \vee \gamma\text{-}C_\tau(\overline{1} - \mu, r) = \gamma\text{-}C_\tau(\lambda, r) \vee \gamma\text{-}C_\tau(\overline{1} - \lambda, r) \leq \gamma\text{-}C_\tau(\lambda \vee (\overline{1} - \mu), r) = \gamma\text{-}C_\tau(\delta, r)$. Thus, δ is an r-fuzzy γ-dense set and hence $\lambda = \mu \wedge \delta$. □

Theorem 3.13. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\lambda \in I^X$. If λ is r-fuzzy γ-open and r-fuzzy γ-closed, then $\lambda \in \alpha(\tau, r)$ and $\overline{1} - \lambda \in \pi(\tau, r)$.*

Proof. Let λ be an r-fuzzy γ-open and r-fuzzy γ-closed set. By Proposition 3.1, $\lambda = \gamma\text{-}I_\tau(\lambda, r)$ and $\lambda = \gamma\text{-}C_\tau(\lambda, r)$. We have $\lambda = \gamma\text{-}I_\tau(\lambda, r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$. Then $\lambda = \gamma\text{-}I_\tau(\lambda, r) = \gamma\text{-}I_\tau(\gamma\text{-}I_\tau(\lambda, r), r) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r), r)$. It follows that $\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r), r)$ and hence, $\lambda \in \alpha(\tau, r)$. On the other hand, since $\lambda = \gamma\text{-}I_\tau(\lambda, r)$ and $\lambda = \gamma\text{-}C_\tau(\lambda, r)$, then $\overline{1} - \lambda = 1 - \gamma\text{-}I_\tau(\lambda, r) = \gamma\text{-}C_\tau(\overline{1} - \lambda, r)$ and $\overline{1} - \lambda = 1 - \gamma\text{-}C_\tau(\lambda, r) = \gamma\text{-}I_\tau(\overline{1} - \lambda, r)$. This implies $\overline{1} - \lambda = \gamma\text{-}I_\tau(\overline{1} - \lambda) = \gamma\text{-}I_\tau(\gamma\text{-}I_\tau(\overline{1} - \lambda, r), r) \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\overline{1} - \lambda, r), r)$. Thus, $\overline{1} - \lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\overline{1} - \lambda, r), r)$ and hence, $\overline{1} - \lambda \in \pi(\tau, r)$. □

Theorem 3.14. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\lambda \in I^X$. If there exists an r-fuzzy γ-open set μ such that $\mu \leq \lambda \leq \gamma\text{-}C_\tau(\mu)$, then $\lambda \in \sigma(\tau, r)$.*

Proof. Let $\mu \leq \lambda \leq \gamma\text{-}C_\tau(\mu, r)$ for an r-fuzzy γ-open set μ. Since $\mu \leq \lambda$, then $\mu \leq \gamma\text{-}I_\tau(\lambda, r)$. This implies $\gamma\text{-}C_\tau(\mu, r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$ and $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda, r), r)$. Thus, $\lambda \in \sigma(\tau, r)$. □

Theorem 3.15. *Let (X, τ, γ) be a smooth fuzzy operation-topological space and $\delta \in I^X$. If $\delta \in \beta(\tau, r)$, then $\delta = \lambda \wedge \mu$, where $\lambda \in \sigma(\tau, r)$ and μ is r-fuzzy γ-dense.*

Proof. Let $\delta \in \beta(\tau, r)$. Then $\delta \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\delta, r), r), r)$. Then we have $\gamma\text{-}C_\tau(\delta, r) \leq \gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\delta, r), r), r), r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\delta, r), r), r)$. Moreover, $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\delta, r), r) \leq \gamma\text{-}C_\tau(\delta, r)$ and then $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\delta, r), r), r) \leq \gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\delta, r), r) = \gamma\text{-}C_\tau(\delta, r)$. We have $\gamma\text{-}C_\tau(\delta, r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\delta, r), r), r)$. This implies that $\lambda = \gamma\text{-}C_\tau(\delta, r) \in \sigma(\tau, r)$. If we take $\mu = \delta \vee (1 - \gamma\text{-}C_\tau(\delta, r))$, then μ is r-fuzzy γ-dense and $\delta = \lambda \wedge \mu$. □

Theorem 3.16. *For a smooth fuzzy operation-topological space (X, τ, γ), the following properties are equivalent:*

(1) $\lambda \in \beta(\tau, r)$,
(2) *there exists* $\mu \in \pi(\tau, r)$ *such that* $\mu \leq \gamma\text{-}C_\tau(\lambda, r) \leq \gamma\text{-}C_\tau(\mu, r)$,
(3) $\gamma\text{-}C_\tau(\lambda, r) \in rc(\tau, r)$.

Proof. (1)⇒(2): Let $\lambda \in \beta(\tau, r)$. We have $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r)$. Put $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r)$. Then $\lambda \in \pi(\tau, r)$. Since $\lambda = \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r) \leq \gamma\text{-}C_\tau(\lambda, r)$ and put $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda, r), r), r)$, then $\mu \leq \gamma\text{-}C_\tau(\lambda, r)$ and $\lambda \leq \gamma\text{-}C_\tau(\mu, r)$. By Theorem 2.10 (4), $\gamma\text{-}C_\tau(\lambda, r) \leq \gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\mu, r), r) = \gamma\text{-}C_\tau(\mu, r)$. Hence $\mu \leq \gamma\text{-}C_\tau(\lambda, r) \leq \gamma\text{-}C_\tau(\mu, r)$.

(2)⇒(3): Let $\mu \in \pi(\tau,r)$ such that $\mu \leq \gamma\text{-}C_\tau(\lambda,r) \leq \gamma\text{-}C_\tau(\mu,r)$. Then $\mu \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r)$ and $\gamma\text{-}C_\tau(\mu,r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r),r)$. It follows that $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r),r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$. Since $\gamma\text{-}C_\tau(\lambda,r) \leq \gamma\text{-}C_\tau(\mu,r)$, $\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r) \leq \gamma\text{-}C_\tau(\mu,r)$ and then $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) \leq \gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\mu,r),r)$. Since $\mu \leq \gamma\text{-}C_\tau(\lambda,r)$, then $\gamma\text{-}C_\tau(\mu,r) \leq \gamma\text{-}C_\tau(\gamma\text{-}C_\tau(\lambda,r),r)$. This implies that $\gamma\text{-}C_\tau(\mu,r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r),r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) \leq \gamma\text{-}C_\tau(\mu,r) \leq \gamma\text{-}C_\tau(\lambda,r) \leq \gamma\text{-}C_\tau(\mu,r)$, $\gamma\text{-}C_\tau(\lambda,r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r),r)$. Hence $\gamma\text{-}C_\tau(\lambda,r) \in rc(\tau,r)$.

(3)⇒(1). Let $\gamma\text{-}C_\tau(\lambda,r) \in rc(\tau,r)$. We have $\gamma\text{-}C_\tau(\lambda,r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$. Since $\lambda \leq \gamma\text{-}C_\tau(\lambda,r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$, then $\lambda \in \beta(\tau,r)$. □

Theorem 3.17. *Let (X,τ,γ) be a smooth fuzzy operation-topological space. If $\lambda \leq \mu \leq \gamma\text{-}C_\tau(\lambda,r)$ and $\lambda \in \beta(\tau,r)$, then $\mu \in \beta(\tau,r)$.*

Proof. Let $\lambda \leq \mu \leq \gamma\text{-}C_\tau(\lambda,r)$ and $\lambda \in \beta(\tau,r)$. Then we have $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$. Since $\mu \leq \gamma\text{-}C_\tau(\lambda,r)$, we have $\mu \leq \gamma\text{-}C_\tau(\lambda,r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r),r)$. Thus, $\mu \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\mu,r),r),r)$ and hence $\mu \in \beta(\tau,r)$. □

Theorem 3.18. *For a smooth fuzzy operation-topological space (X,τ,γ), the following properties are equivalent:*

(1) $\lambda \in rc(\tau,r)$,
(2) $\overline{1}-\lambda \in \pi(\tau,r)$ *and* $\lambda \in \sigma(\tau,r)$,
(3) $\overline{1}-\lambda \in \alpha(\tau,r)$ *and* $\lambda \in \beta(\tau,r)$.

Proof. (1)⇒ (2): Let $\lambda \in rc(\tau,r)$. Then $\lambda = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r)$ and $\overline{1}-\lambda \in \pi(\tau,r)$ and $\lambda \in \sigma(\tau,r)$.

(2)⇒ (3): $\overline{1}-\lambda \in \pi(\tau,r)$ and $\lambda \in \sigma(\tau,r)$. Then $\overline{1}-\lambda \leq \gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\overline{1}-\lambda,r)$ and $\lambda \leq \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r)$. Then $\gamma\text{-}C_\tau(\lambda,r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r)$ and hence $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r),r),r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r) \leq \lambda$. This shows that $\overline{1}-\lambda \in \alpha(\tau,r)$. Since $\sigma(\tau,r) \leq \beta(\tau,r)$, it is obvious that $\lambda \in \beta(\tau,r)$.

(3)$\Rightarrow$ (1): $\overline{1}-\lambda \in \alpha(\tau,r)$ and $\lambda \in \beta(\tau,r)$. Then we have $\lambda = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r)$ and $\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\lambda,r),r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r),r),r) = \gamma\text{-}C_\tau(\gamma\text{-}I_\tau(\gamma\text{-}C_\tau(\lambda,r),r),r) = \lambda$. Therefore, $\lambda \in rc(\tau,r)$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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