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MINIMAL DECOMPOSITION THEOREMS AND MINIMAL EXTENSION PRINCIPLE FOR PICTURE FUZZY SETS

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Abstract: Picture fuzzy set theory was originally proposed as a mathematical tool to deal with uncertainty by taking yes, no, neutral memberships of an element of a universal set. It has been studied by a host of researchers theoretically and practically. But still now, the structural properties of picture fuzzy sets are not widely studied. In this article, we propose lower (α,γ,β)-cut and strong lower (α,γ,β)-cut of a picture fuzzy set and illustrate some of their properties. Three minimal decomposition theorems for picture fuzzy sets are introduced by lower (α,γ,β)-cut, strong lower (α,γ,β)-cut and level set of picture fuzzy sets with illustrations by a numerical example. Some properties of minimal extension principle are also described by using the lower (α,γ,β)-cut and the strong lower (α,γ,β)-cut of picture fuzzy sets. Finally, arithmetic operations for picture fuzzy sets are illustrated by using the minimal extension principle.

Keywords: picture fuzzy set; (α,γ,β)-cut; minimal decomposition theorems; minimal extension principle.

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1. INTRODUCTION

In many fields of science and engineering there are some data which are vague than exact. To give modeling these uncertainties a host of researchers have become involved recently. Fuzzy set theory [31] which was first concept to deal with uncertainty by allowing partial membership. Therefore, fuzzy set has been generalized by numerous researchers and many applications of fuzzy set theory have arisen over the years. One of generalization of fuzzy set is intuitionistic fuzzy set, proposed by Atanassov [1], is also capable of dealing with ambiguity by allowing membership degree and a non-membership degree, while a fuzzy set is characterized by only a membership degree. Later intuitionistic fuzzy set has been applied in diverse areas of science and engineering due to its more efficiency for dealing with ambiguity than the fuzzy set. But in the applications of intuitionistic fuzzy sets, there arise problems for the feature of neutrality of an element. To overcome these difficulties, Cuong and Kreinovich [3,4] initiated picture fuzzy set which is a direct extension of fuzzy set and intuitionistic fuzzy set by including the idea of positive, negative, and neutral membership degree of an element. Later a host of researchers studied its structural properties and applied them in many branches of science and engineering. Cuong and Kreinovich defined some basic operations of picture fuzzy sets such as union, intersection, complement, Cartesian product etc. and described some related properties of them. They also described picture fuzzy relation and Zadeh extension principle for picture fuzzy sets by applying Cartesian product. Dutta and Ganju [7] introduced (α,δ,β)-cut and strong (α,δ,β)-cut, level set for picture fuzzy sets and discussed some properties of them. They also described decomposition theorems by (α,δ,β)-cut and strong (α,δ,β)-cut and level set for picture fuzzy sets. Zadeh extension principle and arithmetic operations by using Zadeh extension principle are also illustrated by Dutta and Ganju. Besides these some other operations of picture fuzzy sets are also depicted by [see ([2], [5-6], [8-17], [19-30]).

In this work, we propose the lower (α,γ,β)-cut and the strong lower (α,γ,β)-cut of picture fuzzy sets and explore some properties of them. Minimal decomposition theorems for picture fuzzy set are introduced. Some properties of minimal extension principle for picture fuzzy sets are described by using the lower (α,γ,β)-cut and the strong lower (α,γ,β)-cut of a picture fuzzy set. Finally, some arithmetic operations for picture fuzzy sets are illustrated by using the minimal extension principle with numerical examples.

This article is organized as follows: in section 2, we give some definitions which are essential to rest of the paper. In section 3, the concept of the lower (α,γ,β)-cut and the strong lower (α,γ,β)-cut of a picture fuzzy set are proposed and described some properties. Here, we also introduce minimal decomposition theorems. In section 4, some properties of the minimal extension principle for picture fuzzy sets are depicted by using the lower (α,γ,β)-cut and strong lower (α,γ,β)-cut of picture fuzzy sets. Here, also the arithmetic operations for picture fuzzy sets by using the minimal extension principle are illustrated with numerical examples.

2. PRELIMINARIES

In this section, we recall some basic definitions for picture fuzzy sets which are used in later sections.

Definition 2.1: [31] Let X be non-empty set. A ***fuzzy set A*** in X is given by

$$A = \{(x, \mu_A(x)) : x \in X\},$$

where $\mu_A: X \to [0,1]$.

Definition 2.2: [1] Let X be non-empty set. An ***intuitionistic fuzzy set A*** in X is given by

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$, with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1; \forall x \in X$.

The values $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A.

For any intuitionistic fuzzy set A on the universal set X, let

$$\pi_A(x) = 1 - \left(\mu_A(x) + \nu_A(x)\right)$$

which is called the hesitancy degree (or intuitionistic fuzzy index) of an element x in A. It is the degree of indeterminacy membership of the element x whether belonging to A or not.

Obviously, $0 \le \pi_A(x) \le 1$ for any $x \in X$.

Definition 2.3: [3,4] A ***picture fuzzy set*** $\boldsymbol{A}$ on a universe of discourse X is of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of x in A, $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of x in A and $\nu_A(x) \in [0,1]$ is called the degree of negative membership of x in A, and where $\mu_A(x), \eta_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1; \forall x \in X.$$

Here $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)); \forall x \in X$ is called the degree of refusal membership of x in A.

The set of all picture fuzzy sets in X will be denoted by $PFS(X)$.

Definition 2.4: [3,4] Let $A, B \in PFS(X)$, then ***the subset, equality, union, intersection and complement*** are defined as follows:

1. $A \subseteq B$ iff $\forall x \in X, \mu_A(x) \leq \mu_B(x), \eta_A(x) \leq \eta_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;
2. $A = B$ iff $\forall x \in X, \mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$;
3. $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))) : x \in X\}$;
4. $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))) : x \in X\}$;
5. $A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) : x \in X\}$.

Definition 2.5: [7] Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\} \in PFS(X)$ and $\alpha, \gamma, \beta \in [0,1]$, $\alpha + \gamma + \beta \leq 1$. Then we call

$$A^{(\alpha,\gamma,\beta)} = \{x : \mu_A(x) \geq \alpha, \eta_A(x) \leq \gamma, \nu_A(x) \leq \beta\} \text{ and}$$
$$A^{(\alpha,\gamma,\beta)+} = \{x : \mu_A(x) > \alpha, \eta_A(x) < \gamma, \nu_A(x) < \beta\},$$

the $(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$***- cut set, the strong*** $(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$***- cut of*** $\boldsymbol{A}$ respectively.

3. LOWER $(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$-CUT AND MINIMAL DECOMPOSITION THEOREMS OF PICTURE FUZZY SETS

In this section, the concept of the lower (α, γ, β)-cut and strong lower (α, γ, β)-cut of picture fuzzy sets are introduced and some of their properties are described. Minimal decomposition theorems by these lower (α, γ, β)-cuts and level set are also established. Throughout this article, we denote $\wedge$ for minimum operator and $\vee$ for maximum operator.

Definition 3.1: Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\}$ be a picture fuzzy set on X and $\alpha, \gamma, \beta \in [0,1]$, $\alpha + \gamma + \beta \leq 1$, then the ***lower*** $(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$***-cut of*** $\boldsymbol{A}$ is given by

$A_{(\alpha,\gamma,\beta)} = \{x \in X\colon \mu_A(x) \leq \alpha, \eta_A(x) \leq \gamma, \nu_A(x) \geq \beta\}$.

That is, $\alpha_{\underline{\mu}_A} = \{x\colon \mu_A(x) \leq \alpha\}$, $\gamma_{\underline{\eta}_A} = \{x\colon \eta_A(x) \leq \gamma\}$ and $\beta_{\underline{\nu}_A} = \{x\colon \nu_A(x) \geq \beta\}$ are α, γ and β- lower cuts of positive membership, neutral membership and negative membership of a picture fuzzy set A respectively.

Example 3.1 (a): Let$X = \{1, 2, 3, 4, 5\}$ and

$A = \{(1, 0.3,0.4,0.2), (2, 0.2,0.6,0.1), (3, 0.4,0.1,0.5), (4, 0.4, 0.2, 0.2), (5, 0.2, 0.3, 0.4)\}$ be a picture fuzzy set in X.

Let $\alpha = 0.6$, $\gamma = 0.2$, $\beta = 0.1$, then

$$A_{(\alpha,\gamma,\beta)} = \{x \in X\colon \mu_A(x) \leq 0.6, \eta_A(x) \leq 0.2, \nu_A(x) \geq 0.1\}$$

$$= \{3,4\}$$

Definition 3.2: Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\}$ be a picture fuzzy set on X and $\alpha, \gamma, \beta \in [0,1]$, $\alpha + \gamma + \beta \leq 1$, then the ***strong lower*** $(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$***-cut of*** $\boldsymbol{A}$ is given by

$$A_{(\alpha,\gamma,\beta)+} = \{x \in X\colon \mu_A(x) < \alpha, \eta_A(x) < \gamma, \nu_A(x) > \beta\}.$$

That is, $\alpha^+{}_{\underline{\mu}_A} = \{x\colon \mu_A(x) < \alpha\}$, $\gamma^+{}_{\underline{\eta}_A} = \{x\colon \eta_A(x) < \gamma\}$ and $\beta^+{}_{\underline{\nu}_A} = \{x\colon \nu_A(x) > \beta\}$ are α, γ and β- strong lower cuts of positive membership, neutral membership and negative membership of a picture fuzzy set A respectively.

Example 3.2 (a): Let $X = \{1, 2, 3, 4, 5\}$ and $A = \{(1, 0.3,0.4,0.2), (2, 0.2,0.6,0.1), (3, 0.4,0.1,0.5), (4, 0.4, 0.2, 0.2), (5, 0.2, 0.3, 0.4)\}$ be a picture fuzzy set in X.

Let $\alpha = 0.6$, $\gamma = 0.2$, $\beta = 0.1$, then

$$A_{(\alpha,\gamma,\beta)+} = \{x \in X\colon \mu_A(x) < 0.6 \,, \eta_A(x) < 0.2 \,, \nu_A(x) > 0.1\},$$

$$= \{3\}$$

It can be noted that, $A_{(\alpha,\gamma,\beta)+} \subseteq A_{(\alpha,\gamma,\beta)}$.

Theorem 3.3: Let $A \in PFS(X)$, then $A_{(\alpha,\gamma,\beta)+} \subseteq A_{(\alpha,\gamma,\beta)}$.

Proof: Let $x \in A_{(\alpha,\gamma,\beta)+}$, then

$$\mu_A(x) < \alpha, \eta_A(x) < \gamma \,, \nu_A(x) > \beta$$

$$\Rightarrow \mu_A(x) \le \alpha \,, \eta_A(x) \le \gamma \,, \nu_A(x) \ge \beta$$

$$\Rightarrow x \in A_{(\alpha,\gamma,\beta)}.$$

$\therefore\ A_{(\alpha,\gamma,\beta)+} \subseteq A_{(\alpha,\gamma,\beta)}$.

Theorem 3.4: Let $A, B \in PFS(X)$, then $A \subseteq B$ implies

1. $B_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)}$
2. $B_{(\alpha,\gamma,\beta)+} \subseteq A_{(\alpha,\gamma,\beta)+}$

Proof: 1. Let $x \in B_{(\alpha,\gamma,\beta)}$, then

$$\mu_B(x) \le \alpha \,, \eta_B(x) \le \gamma \,, \nu_B(x) \ge \beta.$$

As $A \subseteq B$, we have

$$\mu_A(x) \le \mu_B(x) \le \alpha \,, \eta_A(x) \le \eta_B(x) \le \gamma \,, \nu_A(x) \ge \nu_B(x) \ge \beta$$

$$\Rightarrow \mu_A(x) \le \alpha \,, \eta_A(x) \le \gamma \,, \nu_A(x) \ge \beta$$

$$\Rightarrow x \in A_{(\alpha,\gamma,\beta)}.$$

$\therefore\ B_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)}$.

2. Proof is similar to 1.

Theorem 3.5: Let $A, B \in PFS(X)$, then

1. $(A \cup B)_{(\alpha,\gamma,\beta)} = A_{(\alpha,\gamma,\beta)} \cap B_{(\alpha,\gamma,\beta)}$
2. $(A \cup B)_{(\alpha,\gamma,\beta)+} = A_{(\alpha,\gamma,\beta)} \cap B_{(\alpha,\gamma,\beta)+}$
3. $(A \cup B)_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)} \cup B_{(\alpha,\gamma,\beta)}$
4. $(A \cup B)_{(\alpha,\gamma,\beta)+} \subseteq A_{(\alpha,\gamma,\beta)} \cup B_{(\alpha,\gamma,\beta)+}$

Proof: 1.Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so from the theorem 3.4 we have,

$(A \cup B)_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)}$ and $(A \cup B)_{(\alpha,\gamma,\beta)} \subseteq B_{(\alpha,\gamma,\beta)}$

$\Rightarrow (A \cup B)_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)} \cap B_{(\alpha,\gamma,\beta)}$.

Again, let $x \in A_{(\alpha,\gamma,\beta)} \cap B_{(\alpha,\gamma,\beta)}$

$\Rightarrow x \in A_{(\alpha,\gamma,\beta)}$ and $x \in B_{(\alpha,\gamma,\beta)}$

$\Rightarrow \mu_A(x) \leq \alpha$, $\mu_B(x) \leq \alpha \Rightarrow max\{\mu_A(x), \mu_B(x)\} \leq \alpha$

$\eta_A(x) \leq \gamma$, $\eta_B(x) \leq \gamma \Rightarrow min\{\eta_A(x), \eta_B(x)\} \leq \gamma$

$\nu_A(x) \geq \beta$, $\nu_B(x) \geq \beta \Rightarrow min\{\nu_A(x), \nu_B(x)\} \geq \beta$

$\Rightarrow x \in (A \cup B)_{(\alpha,\gamma,\beta)}$.

Therefore, $A_{(\alpha,\gamma,\beta)} \cap B_{(\alpha,\gamma,\beta)} \subseteq (A \cup B)_{(\alpha,\gamma,\beta)}$.

Hence, $(A \cup B)_{(\alpha,\gamma,\beta)} = A_{(\alpha,\gamma,\beta)} \cap B_{(\alpha,\gamma,\beta)}$.

2. Proof is similar to 1.
3. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so from the theorem 3.4 we have,

$(A \cup B)_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)}$ and $(A \cup B)_{(\alpha,\gamma,\beta)} \subseteq B_{(\alpha,\gamma,\beta)}$

$\Rightarrow (A \cup B)_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)} \cup B_{(\alpha,\gamma,\beta)}$.

4. Proof is similar to 3.

Theorem 3.6: Let $A, B \in PFS(X)$, then $A = B$ implies

1. $A_{(\alpha,\gamma,\beta)} = B_{(\alpha,\gamma,\beta)}$

2. $A_{(\alpha,\gamma,\beta)+} = B_{(\alpha,\gamma,\beta)+}$

Proof: 1. Let $x \in A_{(\alpha,\gamma,\beta)}$, then

$$\mu_A(x) \leq \alpha \,, \eta_A(x) \leq \gamma \,, \nu_A(x) \geq \beta.$$

As $A = B$, we have

$$\mu_B(x) = \mu_A(x) \leq \alpha \,, \eta_B(x) = \eta_A(x) \leq \gamma \,, \nu_B(x) = \nu_A(x) \geq \beta$$

$$\Rightarrow \mu_B(x) \leq \alpha \,, \eta_B(x) \leq \gamma \,, \nu_B(x) \geq \beta$$

$$\Rightarrow x \in B_{(\alpha,\gamma,\beta)}.$$

$\therefore A_{(\alpha,\gamma,\beta)} \subseteq B_{(\alpha,\gamma,\beta)}$.

Again,

Let $x \in B_{(\alpha,\gamma,\beta)}$, then

$$\mu_B(x) \leq \alpha \,, \eta_B(x) \leq \gamma \,, \nu_B(x) \geq \beta.$$

As $A = B$, we have

$$\mu_A(x) = \mu_B(x) \leq \alpha \,, \eta_A(x) = \eta_B(x) \leq \gamma \,, \nu_A(x) = \nu_B(x) \geq \beta$$

$$\Rightarrow \mu_A(x) \leq \alpha \,, \eta_A(x) \leq \gamma \,, \nu_A(x) \geq \beta$$

$$\Rightarrow x \in A_{(\alpha,\gamma,\beta)}.$$

$\therefore B_{(\alpha,\gamma,\beta)} \subseteq A_{(\alpha,\gamma,\beta)}$.

Thus, $A_{(\alpha,\gamma,\beta)} = B_{(\alpha,\gamma,\beta)}$.

2. Let $x \in A_{(\alpha,\gamma,\beta)+}$, then

$$\mu_A(x) < \alpha \,, \eta_A(x) < \gamma \,, \nu_A(x) > \beta.$$

As $A = B$, we have

$$\mu_B(x) = \mu_A(x) < \alpha \,, \eta_B(x) = \eta_A(x) < \gamma \,, \nu_B(x) = \nu_A(x) > \beta$$

$$\Rightarrow \mu_B(x) < \alpha \,, \eta_B(x) < \gamma \,, \nu_B(x) > \beta$$

$$\Rightarrow x \in B_{(\alpha,\gamma,\beta)+}.$$

$\therefore A_{(\alpha,\gamma,\beta)+} \subseteq B_{(\alpha,\gamma,\beta)+}$.

Again,

$x \in B_{(\alpha,\gamma,\beta)+}$, then

$$\mu_B(x) < \alpha\ ,\eta_B(x) < \gamma\ ,\nu_B(x) > \beta.$$

As $A = B$, we have

$$\mu_A(x) = \mu_B(x) < \alpha\ ,\eta_A(x) = \eta_B(x) < \gamma\ ,\nu_A(x) = \nu_B(x) > \beta$$

$$\Rightarrow \mu_A(x) < \alpha\ ,\eta_A(x) < \gamma\ ,\nu_A(x) > \beta$$

$$\Rightarrow x \in A_{(\alpha,\gamma,\beta)+}.$$

$\therefore B_{(\alpha,\gamma,\beta)+} \subseteq A_{(\alpha,\gamma,\beta)+}$.

Thus, $A_{(\alpha,\gamma,\beta)+} = B_{(\alpha,\gamma,\beta)+}$.

Theorem 3.7: Let $A \in PFS(X)$.

1. If $\alpha_1 \leq \alpha_2\ ,\gamma_1 \leq \gamma_2\ ,\beta_1 \geq \beta_2$, then $A_{(\alpha_1,\gamma_1,\beta_1)} \subseteq A_{(\alpha_2,\gamma_2,\beta_2)}$
2. If $\alpha_1 \leq \alpha_2\ ,\gamma_1 \leq \gamma_2\ ,\beta_1 \geq \beta_2$, then $A_{(\alpha_1,\gamma_1,\beta_1)+} \subseteq A_{(\alpha_2,\gamma_2,\beta_2)+}$

Proof: 1. Let $x \in A_{(\alpha_1,\gamma_1,\beta_1)}$, then

$$\mu_A(x) \leq \alpha_1\ ,\eta_A(x) \leq \gamma_1\ ,\nu_A(x) \geq \beta_1$$

$$\Rightarrow \mu_A(x) \leq \alpha_1 \leq \alpha_2\ ,\eta_A(x) \leq \gamma_1 \leq \gamma_2\ ,\nu_A(x) \geq \beta_1 \geq \beta_2$$

$$\Rightarrow \mu_A(x) \leq \alpha_2\ ,\eta_A(x) \leq \gamma_2\ ,\nu_A(x) \geq \beta_2$$

$$\Rightarrow x \in A_{(\alpha_2,\gamma_2,\beta_2)}.$$

$\therefore A_{(\alpha_1,\gamma_1,\beta_1)} \subseteq A_{(\alpha_2,\gamma_2,\beta_2)}$.

2. Let $x \in A_{(\alpha_1,\gamma_1,\beta_1)+}$, then

$$\mu_A(x) < \alpha_1\ ,\eta_A(x) < \gamma_1\ ,\nu_A(x) > \beta_1$$

$$\Rightarrow \mu_A(x) < \alpha_1 \leq \alpha_2\ ,\eta_A(x) < \gamma_1 \leq \gamma_2\ ,\nu_A(x) > \beta_1 \geq \beta_2$$

$$\Rightarrow \mu_A(x) < \alpha_2\ ,\eta_A(x) < \gamma_2\ ,\nu_A(x) > \beta_2$$

$$\Rightarrow x \in A_{(\alpha_2,\gamma_2,\beta_2)+}.$$

$\therefore A_{(\alpha_1,\gamma_1,\beta_1)+} \subseteq A_{(\alpha_2,\gamma_2,\beta_2)+}$.

Definition 3.8: Let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\}$ be a picture fuzzy set on X. We now define a ***special picture fuzzy set*** denoted by $(\alpha, \gamma, \beta)A$, is defined by

$$(\alpha, \gamma, \beta)A\ (x) = (\alpha, \gamma, \beta)A_{(\alpha,\gamma,\beta)}(x) = (\alpha^{\underline{\mu_A}}, \gamma^{\underline{\eta_A}}, \beta^{\underline{\nu_A}}),$$

where the positive membership $\alpha^{\underline{\mu_A}}$, neutral membership $\gamma^{\underline{\eta_A}}$ and negative membership $\beta^{\underline{\nu_A}}$ are as follows:

$$\alpha^{\underline{\mu_A}}(x) = \begin{cases} \alpha & ;\ x \in \alpha_{\underline{\mu_A}} \\ 0 & ;\text{Otherwise} \end{cases}$$

$$\gamma^{\underline{\eta_A}}(x) = \begin{cases} \gamma & ;\ x \in \gamma_{\underline{\eta_A}} \\ 0 & ;\text{Otherwise} \end{cases}$$

$$\beta^{\underline{\nu_A}}(x) = \begin{cases} \beta & ;\ x \in \beta_{\underline{\nu_A}} \\ 0 & ;\text{Otherwise} \end{cases}$$

Definition 3.9: Let A be a picture fuzzy set, then the ***level set for positive membership*** is defined as

$$\curlywedge (A_+) = \{\alpha :\ \mu_A(x) = \alpha\ , \alpha \in [0,1]\}$$

level set for neutral membership is defined as

$$\curlywedge (A_\pm) = \{\gamma :\ \eta_A(x) = \gamma\ , \gamma \in [0,1]\}$$

and ***level set for negative membership*** is defined as

$$\curlywedge (A_-) = \{\beta :\ \nu_A(x) = \beta\ , \beta \in [0,1]\}.$$

Theorem3.10: (First minimal decomposition theorem):

Let X be a non-empty set. For a picture fuzzy set $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\}$ in X,

$$A = \left(\bigcap_{\alpha\in[0\,,1]} \alpha^{\underline{\mu_A}}\,, \bigcap_{\gamma\in[0\,,1]} \gamma^{\underline{\eta_A}}\,, \bigcup_{\beta\in[0\,,1]} \beta^{\underline{\nu_A}}\right).$$

Proof: Let x be an arbitrary element in X and let $\mu_A(x) = a, \eta_A(x) = c, \nu_A(x) = b$.

Then

$$\left(\left(\bigcap_{\alpha\in[0\,,1]} \alpha^{\underline{\mu_A}}\right), \left(\bigcap_{\gamma\in[0\,,1]} \gamma^{\underline{\eta_A}}\right), \left(\bigcup_{\beta\in[0\,,1]} \beta^{\underline{\nu_A}}\right)\right)$$

$$= \left(\bigwedge_{\alpha\in[0\,,1]} \alpha^{\underline{\mu_A}}\,, \bigwedge_{\gamma\in[0\,,1]} \gamma^{\underline{\eta_A}}\,, \bigvee_{\beta\in[0\,,1]} \beta^{\underline{\nu_A}}\right)$$

$$= \begin{pmatrix} max\left[\bigwedge_{\alpha\in[0\,,a)} \alpha^{\underline{\mu_A}}\,, \bigwedge_{\alpha\in[a,1]} \alpha^{\underline{\mu_A}}\right], max\left[\bigwedge_{\gamma\in[0\,,c)} \gamma^{\underline{\eta_A}}\,, \bigwedge_{\gamma\in[c\,,1]} \gamma^{\underline{\eta_A}}\right], \\ max\left[\bigvee_{\beta\in[0\,,b]} \beta^{\underline{\nu_A}}\,, \bigvee_{\beta\in(b\,,1]} \beta^{\underline{\nu_A}}\right] \end{pmatrix}.$$

For each $\alpha \in [0, a)$, we have $\mu_A(x) = a > \alpha$.

Therefore, $\alpha^{\underline{\mu_A}}(x) = 0$. On the other hand, for each $\alpha \in [a, 1]$, we have $\mu_A(x) = a \leq \alpha$ and $\alpha^{\underline{\mu_A}} = \alpha$.

Similarly, for each $\gamma \in [0, c)$, we have $\eta_A(x) = c > \gamma$.

Therefore, $\gamma^{\underline{\eta_A}}(x) = 0$. On the other hand, for each $\gamma \in [c, 1]$, we have $\eta_A(x) = c \leq \gamma$ and $\gamma^{\underline{\eta_A}} = \gamma$.

Again, for each $\beta \in [0, b]$, we have $\nu_A(x) = b \geq \beta$

Therefore, $\beta^{\underline{\nu_A}}(x) = \beta$

On the other hand, for each $\beta \in [b, 1]$, we have $\nu_A(x) = b < \beta$ and $\beta^{\underline{\nu_A}} = 0$.

Therefore, $\left(\left(\bigcap_{\alpha\in[0,1]} \alpha^{\underline{\mu_A}}\right), \left(\bigcap_{\gamma\in[0,1]} \gamma^{\underline{\eta_A}}\right), \left(\bigcup_{\beta\in[0,1]} \beta^{\underline{\nu_A}}\right)\right)$

$$= \left(\bigwedge_{\alpha\in[a,1]} \alpha^{\underline{\mu_A}}, \bigwedge_{\gamma\in[c,1]} \gamma^{\underline{\eta_A}}, \bigvee_{\beta\in[0,b]} \beta^{\underline{\nu_A}}\right)$$

$$= \left(\bigwedge_{\alpha\in[a,1]} \alpha, \bigwedge_{\gamma\in[c,1]} \gamma, \bigvee_{\beta\in[0,b]} \beta\right)$$

$$= (a, c, b)$$

$$= A$$

Example 3.10 (a): Let A be any picture fuzzy set in X, given by

$A = \{(x_1, 0.5,0.2,0.3), (x_2, 0.2,0.3,0.4), (x_3, 0.6,0.3,0.1), (x_4, 0.3, 0.3, 0.2)\}$.

Let us denote A for convenience as

$$A = \frac{(0.5,0.2,0.3)}{x_1} + \frac{(0.2,0.3,0.4)}{x_2} + \frac{(0.6,0.3,0.1)}{x_3} + \frac{(0.3,0.3,0.2)}{x_4}.$$

Then, we have four distinct lower (α, γ, β)-cuts, which are defined by the following characteristic functions (viewed here as special membership functions):

$$A_{(0.5,0.2,0.3)} = \frac{(1,1,1)}{x_1} + \frac{(1,0,1)}{x_2} + \frac{(0,0,0)}{x_3} + \frac{(1,0,0)}{x_4},$$

$$A_{(0.2,0.3,0.4)} = \frac{(0,1,0)}{x_1} + \frac{(1,1,1)}{x_2} + \frac{(0,1,0)}{x_3} + \frac{(0,1\ 0)}{x_4},$$

$$A_{(0.6,0.3,0.1)} = \frac{(1,1,1)}{x_1} + \frac{(1,1,1)}{x_2} + \frac{(1,1,1)}{x_3} + \frac{(1,1,1)}{x_4},$$

$$A_{(0.3,0.3,0.2)} = \frac{(0,1,1)}{x_1} + \frac{(1,1,1)}{x_2} + \frac{(0,1,0)}{x_3} + \frac{(1,1,1)}{x_4}.$$

We now convert each of the lower (α, γ, β)-cuts to a special picture fuzzy set $(\alpha, \gamma, \beta)A$, defined for each $x \in X = \{x_1, x_2, x_3, x_4\}$ as follows:

$(\alpha, \gamma, \beta)A = (\alpha, \gamma, \beta)A_{(\alpha,\gamma,\beta)} = \{(x, \alpha^{\underline{\mu_A}}, \gamma^{\underline{\eta_A}}, \beta^{\underline{\nu_A}}) : x \in X\}$.

We obtain

$$(0.5,0.2,0.3)A = \frac{(0.5,0.2,0.3)}{x_1} + \frac{(0.5,0,0.3)}{x_2} + \frac{(0,0,0)}{x_3} + \frac{(0.5,0,0)}{x_4} \quad (3.1)$$

$$(0.2,0.3,0.4)A = \frac{(0,0.3,0)}{x_1} + \frac{(0.2,0.3,0.4)}{x_2} + \frac{(0,0.3,0)}{x_3} + \frac{(0,0.3,0)}{x_4} \quad (3.2)$$

$$(0.6,0.3,0.1)A = \frac{(0.6,0.3,0.1)}{x_1} + \frac{(0.6,0.3,0.1)}{x_2} + \frac{(0.6,0.3,0.1)}{x_3} + \frac{(0.6,0.3,0.1)}{x_4} \quad (3.3)$$

$$(0.3,0.3,0.2)A = \frac{(0,0.3,0.2)}{x_1} + \frac{(0.3,0.3,0.2)}{x_2} + \frac{(0,0.3,0)}{x_3} + \frac{(0.3,0.3,0.2)}{x_4} \quad (3.4)$$

Using the equations (3.1), (3.2), (3.3) and (3.4), we have

$$A = \left(\bigcap_{\alpha\in[0,1]} \alpha^{\underline{\mu_A}}, \bigcap_{\gamma\in[0,1]} \gamma^{\underline{\eta_A}}, \bigcup_{\beta\in[0,1]} \beta^{\underline{\nu_A}}\right).$$

Theorem 3.11: (Second minimal decomposition theorem):

Let X be a non-empty set. For a picture fuzzy set

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

$$A = \left(\bigcap_{\alpha\in[0,1]} \alpha +^{\underline{\mu_A}}, \bigcap_{\gamma\in[0,1]} \gamma +^{\underline{\eta_A}}, \bigcup_{\beta\in[0,1]} \beta +^{\underline{\nu_A}}\right) .$$

Proof: The proof is similar to the theorem 3.10 is neglected here.

Theorem 3.12: (Third minimal decomposition theorem):

For every

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))\} \in PFS(X), \text{ then}$$

$$A = \left(\bigcap_{\alpha\in\wedge(A_+)} \alpha^{\underline{\mu_A}}, \bigcap_{\gamma\in\wedge(A_\pm)} \gamma^{\underline{\eta_A}}, \bigcup_{\beta\in\wedge(A_-)} \beta^{\underline{\nu_A}}\right).$$

Proof: The proof is similar to the decomposition theorems.

4. Minimal Extension Principle and Arithmetic Operations of Picture Fuzzy Sets

In this section, some properties of minimal extension principle for picture fuzzy sets are explored by using the lower (α, γ, β)-cut and strong lower (α, γ, β)-cut of picture fuzzy sets. We also describe some arithmetic operations of picture fuzzy sets by using the minimal extension principle.

Definition 4.1: [7] Let X and Y be two non-empty sets and $\bar{f}: X \to Y$ be a mapping. Two mappings can be induced by $\bar{f}$ as the following:

$$\bar{f}: PFS(X) \to PFS(Y) \quad \text{and} \quad \bar{f}^{-1}: PFS(Y) \to PFS(X)$$

which are defined by

$$\bar{f}(A)(y) = \left(\mu_{\bar{f}(A)}(y), \eta_{\bar{f}(A)}(y), \nu_{\bar{f}(A)}(y)\right), \text{ where } A \in PFS(X)$$

and

$$\mu_{\bar{f}(A)}(y) = \begin{cases} \vee \left\{\mu_A(x): x \in \bar{f}^{-1}(y)\right\} & ; \ \bar{f}^{-1}(y) \neq \phi \\ 0 & ; \ \text{Otherwise} \end{cases}$$

$$\eta_{\bar{f}(A)}(y) = \begin{cases} \wedge \left\{\eta_A(x): x \in \bar{f}^{-1}(y)\right\} & ; \ \bar{f}^{-1}(y) \neq \phi \\ 0 & ; \ \text{Otherwise} \end{cases}$$

$$\nu_{\bar{f}(A)}(y) = \begin{cases} \wedge \left\{\nu_A(x): x \in \bar{f}^{-1}(y)\right\} & ; \ \bar{f}^{-1}(y) \neq \phi \\ 0 & ; \ \text{Otherwise} \end{cases}.$$

and

$$\bar{f}^{-1}(B)(x) = \left(\mu_{\bar{f}^{-1}(B)}(x), \eta_{\bar{f}^{-1}(B)}(x), \nu_{\bar{f}^{-1}(B)}(x)\right), \text{ where } B \in PFS(Y)$$

and

$$\mu_{\bar{f}^{-1}(B)}(x) = \mu_B\left(\bar{f}(x)\right)$$

$$\eta_{\bar{f}^{-1}(B)}(x) = \eta_B\left(\bar{f}(x)\right)$$

$$\nu_{\bar{f}^{-1}(B)}(x) = \nu_B\left(\bar{f}(x)\right)$$

Definition 4.2: Let X and Y be two non-empty sets and $\underline{f}: X \to Y$ be a mapping. Two mappings can be induced by $\underline{f}$ as the following:

$$\underline{f}: PFS(X) \to PFS(Y) \quad \text{and} \quad \underline{f}^{-1}: PFS(Y) \to PFS(X)$$

which are defined by

$$\underline{f}(A)(y) = \left(\mu_{\underline{f}(A)}(y), \eta_{\underline{f}(A)}(y), \nu_{\underline{f}(A)}(y)\right), \text{ where } A \in PFS(X)$$

and

$$\mu_{\underline{f}(A)}(y) = \begin{cases} \wedge \left\{\mu_A(x) : x \in \underline{f}^{-1}(y)\right\} & ;\ \underline{f}^{-1}(y) \neq \phi \\ 0 & ;\ \text{Otherwise} \end{cases}$$

$$\eta_{\underline{f}(A)}(y) = \begin{cases} \wedge \left\{\eta_A(x) : x \in \underline{f}^{-1}(y)\right\} & ;\ \underline{f}^{-1}(y) \neq \phi \\ 0 & ;\ \text{Otherwise} \end{cases}$$

$$\nu_{\underline{f}(A)}(y) = \begin{cases} \vee \left\{\nu_A(x) : x \in \underline{f}^{-1}(y)\right\} & ;\ \underline{f}^{-1}(y) \neq \phi \\ 0 & ;\ \text{Otherwise} \end{cases}.$$

and

$$\underline{f}^{-1}(B)(x) = \left(\mu_{\underline{f}^{-1}(B)}(x), \eta_{\underline{f}^{-1}(B)}(x), \nu_{\underline{f}^{-1}(B)}(x)\right), \text{ where } B \in PFS(Y)$$

and

$$\mu_{\underline{f}^{-1}(B)}(x) = \mu_B\left(\underline{f}(x)\right)$$

$$\eta_{\underline{f}^{-1}(B)}(x) = \eta_B\left(\underline{f}(x)\right)$$

$$\nu_{\underline{f}^{-1}(B)}(x) = \nu_B\left(\underline{f}(x)\right)$$

This above statement is called the ***minimal extension principle for picture fuzzy sets***.

Theorem 4.3: Let $\underline{f}: X \to Y$ and $A, B, A_i \in PFS(X)$, then induced mapping $\underline{f}$ satisfies that

1. $A \subseteq B$ implies $\underline{f}(A) \subseteq \underline{f}(B)$,
2. $\underline{f}(\cap_{i\in I} A_i) = \left(\cap_{i\in I}\left(\underline{f}(A_i)\right)\right)$,
3. $\underline{f}(\cup_{i\in I} A_i) \subseteq \left(\cup_{i\in I}\left(\underline{f}(A_i)\right)\right)$,
4. $\underline{f}\left(A_{(\alpha,\gamma,\beta)}\right) \subseteq \left(\underline{f}(A)\right)_{(\alpha,\gamma,\beta)}$,
5. $\underline{f}\left(A_{(\alpha,\gamma,\beta)+}\right) = \left(\underline{f}(A)\right)_{(\alpha,\gamma,\beta)+}$.

Proof: The proof 1 is trivial.

2. For each $\in Y$, if $\underline{f}^{-1}$ is not empty,

$$\underline{f}(\cap_{i\in I} A_i)(y) = \left(\mu_{\underline{f}(\cap_{i\in I} A_i)}(y), \eta_{\underline{f}(\cap_{i\in I} A_i)}(y), \nu_{\underline{f}(\cup_{i\in I} A_i)}(y)\right),$$

where

$$\mu_{\underline{f}(\cap_{i\in I}A_i)}(y) = \wedge_{x\in \underline{f}^{-1}(y)}\{\mu_{\cap_{i\in I}A_i}(x)\}$$

$$= \wedge_{x\in \underline{f}^{-1}(y)}\{\wedge_{i\in I}\{\mu_{A_i}(x)\}\} \text{ ; [From the definition 4.2]}$$

$$= \wedge_{i\in I}\left\{\wedge_{x\in \underline{f}^{-1}(y)}\{\mu_{A_i}(x)\}\right\}$$

$$= \wedge_{i\in I}\left\{\mu_{\underline{f}(A_i)}(y)\right\}$$

$\therefore \mu_{\underline{f}(\cap_{i\in I}A_i)}(y) = \mu_{\underline{f}(A_i)}(y)$.

Similarly,

$$\therefore \eta_{\underline{f}(\cap_{i\in I}A_i)}(y) = \eta_{\cap_{i\in I}\underline{f}(A_i)}(y)$$

Again,

$$\nu_{\underline{f}(\cup_{i\in I}A_i)}(y) = \vee_{x\in \underline{f}^{-1}(y)}\{\nu_{\cup_{i\in I}A_i}(x)\}$$

$$= \vee_{x\in \underline{f}^{-1}(y)}\{\vee_{i\in I}\{\nu_{A_i}(x)\}\} \text{ ; [From the definition 4.2]}$$

$$= \vee_{i\in I}\left\{\vee_{x\in \underline{f}^{-1}(y)}\{\nu_{A_i}(x)\}\right\}$$

$$= \vee_{i\in I}\left\{\left\{\nu_{\underline{f}(A_i)}(y)\right\}\right\}$$

Therefore, $\nu_{\underline{f}(\cup_{i\in I}A_i)}(y) = \nu_{\cup_{i\in I}\underline{f}(A_i)}(y)$.

Hence, $\underline{f}(\cap_{i\in I} A_i) = \left(\cap_{i\in I}\left(\underline{f}(A_i)\right)\right)$.

3. Proof is similar to 2.

4. Let $y \in \underline{f}\left(A_{(\alpha,\gamma,\beta)}\right)$, then there exists $x \in A_{(\alpha,\gamma,\beta)}$ such that $\underline{f}(x) = y$ and

$$\mu_A(y) \le \alpha \,, \eta_A(y) \le \gamma \,, \nu_A(y) \ge \beta$$

$$\Rightarrow \quad \wedge\left\{\mu_A(x) : x \in \underline{f}^{-1}(x) \le \alpha\right\}$$

$$\wedge\left\{\eta_A(x) : x \in \underline{f}^{-1}(x) \le \gamma\right\}$$

$$\vee\left\{\nu_A(x) : x \in \underline{f}^{-1}(x) \ge \beta\right\}$$

i.e.,

$$\mu_{f(A)}(y) \leq \alpha\,, \eta_{f(A)}(y) \leq \gamma\,, \nu_{f(A)}(y) \geq \beta$$

$$\Rightarrow y \in \left(\underline{f}(A)\right)_{(\alpha,\gamma,\beta)}$$

$\therefore \underline{f}\left(A_{(\alpha,\gamma,\beta)}\right) \subseteq \left(\underline{f}(A)\right)_{(\alpha,\gamma,\beta)}$.

5. Let $y \in \underline{f}\left(A_{(\alpha,\gamma,\beta)+}\right)$, then there exists $x \in A_{(\alpha,\gamma,\beta)+}$ such that $\underline{f}(x) = y$ and

$\mu_{f(A)}(y) < \alpha\,, \eta_{f(A)}(y) < \gamma\,, \nu_{f(A)}(y) > \beta$

$\Leftrightarrow \bigvee_{x\in f^{-1}(y)} \mu_A(x) < \alpha,\ \bigwedge_{x\in f^{-1}(y)} \eta_A(x) < \gamma, \bigwedge_{x\in f^{-1}(y)} \nu_A(x) > \beta$

$\Leftrightarrow (\exists x_0 \in X)\left(y = f(x_0)\right)$ and $\mu_A(x_0) < \alpha\,, \eta_A(x_0) < \gamma\,, \nu_A(x_0) > \beta$

$\Leftrightarrow (\exists x_0 \in X)\left(y = f(x_0)\right)$ and $x_0 \in A_{(\alpha,\gamma,\beta)+}$

$\Leftrightarrow y \in f\left(A_{(\alpha,\gamma,\beta)+}\right)$.

Thus, $\underline{f}\left(A_{(\alpha,\gamma,\beta)+}\right) = \left(\underline{f}(A)\right)_{(\alpha,\gamma,\beta)+}$.

Theorem 4.4: Let $\underline{f}: X \to Y$ and $A, B\,, B_i \in PFS(Y)$, then induced mapping $\underline{f}^{-1}$ satisfies that

1. $A \subseteq B$ implies $\underline{f}^{-1}(A) \subseteq \underline{f}^{-1}(B)$,
2. $\underline{f}^{-1}(B^c) = \left(\underline{f}^{-1}(B)\right)^c$,
3. $\underline{f}^{-1}\left(\cup_{i\in I}\,(B_i)\right) = \left(\cup_{i\in I}\,\underline{f}^{-1}(B_i)\right)$,
4. $\underline{f}^{-1}\left(\cap_{i\in I}\,(B_i)\right) = \left(\cap_{i\in I}\,\underline{f}^{-1}(B_i)\right)$,
5. $\underline{f}^{-1}\left(B_{(\alpha,\gamma,\beta)}\right) = \left(\underline{f}^{-1}(B)\right)_{(\alpha,\gamma,\beta)}$,
6. $\underline{f}^{-1}\left(B_{(\alpha,\gamma,\beta)+}\right) = \left(\underline{f}^{-1}(B)\right)_{(\alpha,\gamma,\beta)+}$.

Proof: The proofs of 1 and 2 are trivial.

3. For all $x \in X$, we have

$$\underline{f}^{-1}(\cup_{i\in I}\, B_i)(x) = \left(\mu_{\underline{f}^{-1}(\cup_{i\in I}B_i)}(x), \eta_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x), \nu_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x)\right)$$

where

$$\mu_{\underline{f}^{-1}(\cup_{i\in I}B_i)}(x) = \mu_{\cup_{i\in I}B_i}\left(\underline{f}(x)\right)$$

$$= \vee_{i\in I}\left\{\mu_{B_i}\left(\underline{f}(x)\right)\right\}$$

$$= \vee_{i\in I}\left\{\mu_{\underline{f}^{-1}(B_i)}(x)\right\}$$

$$\therefore \mu_{\underline{f}^{-1}{}_{(\cup_{i\in I}B_i)}}(x) = \mu_{\cup_{i\in I}\underline{f}^{-1}(B_i)}(x)$$

Again,

$$\eta_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x) = \eta_{\cap_{i\in I}B_i}\left(\bar{f}(x)\right)$$

$$= \wedge_{i\in I}\left\{\eta_{B_i}\left(\bar{f}(x)\right)\right\}$$

$$= \wedge_{i\in I}\left\{\eta_{\underline{f}^{-1}(B_i)}(x)\right\}$$

$$\therefore \eta_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x) = \eta_{\cap_{i\in I}\underline{f}^{-1}(B_i)}(x).$$

Similarly, we can prove that

$$\nu_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x) = \nu_{\cap_{i\in I}\underline{f}^{-1}(B_i)}(x).$$

Hence, $\underline{f}^{-1}(\cup_{i\in I} B_i) = \left(\cup_{i\in I} \underline{f}^{-1}(B_i)\right)$.

4. For all $x \in X$, we have

$$\underline{f}^{-1}\left(\cap_{i\in I} (B_i)\right)(x) = \left(\mu_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x), \eta_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x), \nu_{\underline{f}^{-1}(\cup_{i\in I}B_i)}(x)\right)$$

where

$$\mu_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x) = \mu_{\cap_{i\in I}B_i}\left(\underline{f}(x)\right)$$

$$= \wedge_{i\in I}\left\{\mu_{B_i}\left(\underline{f}(x)\right)\right\}$$

$$= \wedge_{i\in I}\left\{\mu_{\underline{f}^{-1}(B_i)}(x)\right\}.$$

$$\therefore \mu_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x) = \mu_{\cap_{i\in I}\underline{f}^{-1}(B_i)}(x).$$

Similarly, we can prove that

$$\eta_{\underline{f}^{-1}(\cap_{i\in I}B_i)}(x) = \eta_{\cap_{i\in I}\underline{f}^{-1}(B_i)}(x).$$

Again,

$$\nu_{\underline{f}^{-1}(\cup_{i\in I}B_i)}(x) = \nu_{\cup_{i\in I}B_i}\left(\underline{f}(x)\right)$$

$$= \vee_{i\in I}\left\{\nu_{B_i}\left(\underline{f}(x)\right)\right\}$$

$$= \vee_{i\in I}\left\{\nu_{\underline{f}^{-1}(B_i)}(x)\right\}$$

$$\therefore \nu_{\bar{f}^{-1}(\cup_{i\in I}B_i)}(x) = \nu_{\cup_{i\in I}\underline{f}^{-1}(B_i)}(x)$$

Hence, $\underline{f}^{-1}(\cap_{i\in I} B_i) = \left(\cap_{i\in I} \underline{f}^{-1}(B_i)\right)$.

5. For all $x \in X$, we have,

$$x \in \underline{f}^{-1}\left(B^{(\alpha,\gamma,\beta)}\right)$$

$$\Leftrightarrow \left\{x \in X: \mu_{\underline{f}^{-1}(B)}(x) \leq \alpha, \eta_{\underline{f}^{-1}(B)}(x) \leq \gamma, \nu_{\underline{f}^{-1}(B)}(x) \geq \beta\right\}$$

$$\Leftrightarrow \left\{x \in X: \mu_B\left(\underline{f}(x)\right) \leq \alpha, \eta_B\left(\underline{f}(x)\right) \leq \gamma, \nu_B\left(\underline{f}(x)\right) \geq \beta\right\}$$

$$\Leftrightarrow \left\{x \in X: \underline{f}(x) \in B^{(\alpha,\gamma,\beta)}\right\}.$$

Therefore, $x \in \underline{f}^{-1}\left(B^{(\alpha,\gamma,\beta)}\right)$.

Thus, $\underline{f}^{-1}\left(B^{(\alpha,\gamma,\beta)}\right) = \left(\underline{f}^{-1}(B)\right)^{(\alpha,\gamma,\beta)}$.

6. Proof is similar to 5.

Here, arithmetic operations for picture fuzzy sets by average extension principle are described.

Definition 4.5: Let $A, B \in PFS(X)$. Then $A * B$ (where $* \in (+, -, ., /)$) is defined by

$$A * B = \{z, \mu_{A*B}(z), \eta_{A*B}(z), \nu_{A*B}(z)\},$$

where

$$\mu_{A*B}(z) = \wedge_{z=x*y}[\mu_A(x) \vee \mu_B(y)],$$

$$\eta_{A*B}(z) = \wedge_{z=x*y}[\eta_A(x) \wedge \eta_B(y)],$$

and $$\nu_{A*B}(z) = \vee_{z=x*y}[\nu_A(x) \wedge \nu_B(y)].$$

Definition 4.6: (Addition operation)

Let $A, B \in PFS(X)$, then

$$A + B = \{z, \mu_{A+B}(z), \eta_{A+B}(z), \nu_{A+B}(z)\},$$

where

$$\mu_{A+B}(z) = \wedge_{z=x+y}[\mu_A(x) \vee \mu_B(y)],$$

$$\eta_{A+B}(z) = \wedge_{z=x+y}[\eta_A(x) \wedge \eta_B(y)],$$

and $$\nu_{A+B}(z) = \vee_{z=x+y}[\nu_A(x) \wedge \nu_B(y)].$$

Example 4.6(a): Let $A, B \in PFS(X)$, where

$$A = \{(1, 0.5, 0.3, 0.2), (2, 0.4, 0.3, 0.2)\}$$

and

$$B = \{(2, 0.5, 0.2, 0.1), (3, 0.2, 0.1, 0.4), (4, 0.6, 0.1, 0.2)\}.$$

Therefore,

$$A + B = \{(1+2, max(0.5,0.5), min(0.3, 0.2), min(0.2, 0.1)),$$

$$(1+3, max(0.5,0.2), min(0.3, 0.1), min(0.2, 0.4)),$$

$$(1+4, max(0.5,0.6), min(0.3, 0.1), min(0.2, 0.2)),$$

$$(2+2, max(0.4,0.5), min(0.3, 0.2), min(0.2, 0.1)),$$

$$(2+3, max(0.4,0.2), min(0.3, 0.1), min(0.2, 0.4)),$$

$$(2+4, max(0.4,0.6), min(0.3, 0.1), min(0.2, 0.2))\}$$

$$= \left\{\begin{matrix} (3, 0.5,0.2,0.1), (4, 0.5,0.1,0.2), (5, 0.6,0.1,0.2), (4, 0.5, 0.2, 0.1), \\ (5, 0.4, 0.1, 0.2), (6, 0.6, 0.1, 0.2) \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (3, 0.5, 0.2, 0.1), (4, min(0.5,0.5), min(0.1,0.2), max(0.2,0.1)), \\ (5, min(0.6,0.4), min(0.1,0.1), max(0.2,0.2)), (6, 0.4,0.1,0.2) \end{matrix}\right\}$$

$$= \{(3, 0.5, 0.2, 0.1), (4, 0.5, 0.1, 0.2), (5, 0.4, 0.1, 0.2), (6, 0.4, 0.1, 0.2)\}.$$

Definition 4.7: (Subtraction operation)

Let $A, B \in PFS(X)$, then

$$A - B = \{z, \mu_{A-B}(z), \eta_{A-B}(z), \nu_{A-B}(z)\},$$

where

$$\mu_{A-B}(z) = \wedge_{z=x-y}[\mu_A(x) \vee \mu_B(y)],$$

$$\eta_{A-B}(z) = \wedge_{z=x-y}[\eta_A(x) \wedge \eta_B(y)],$$

and $$\nu_{A-B}(z) = \vee_{z=x-y}[\nu_A(x) \wedge \nu_B(y)].$$

Example 4.7 (a): Let $A, B \in PFS(X)$, where

$$A = \{(1, 0.5, 0.3, 0.2), (2, 0.4, 0.3, 0.2)\}$$

and

$$B = \{(2, 0.5, 0.2, 0.1), (3, 0.2, 0.1, 0.4), (4, 0.6, 0.1, 0.2)\}.$$

Therefore,

$$A + B = \{\big(1-2, max(0.5,0.5), min(0.3, 0.2), min(0.2, 0.1)\big),$$

$$\big(1-3, max(0.5,0.2), min(0.3, 0.1), min(0.2, 0.4)\big),$$

$$\big(1-4, max(0.5,0.6), min(0.3, 0.1), min(0.2, 0.2)\big),$$

$$\big(2-2, max(0.4,0.5), min(0.3, 0.2), min(0.2, 0.1)\big),$$

$$\big(2-3, max(0.4,0.2), min(0.3, 0.1), min(0.2, 0.4)\big),$$

$$\big(2-4, max(0.4,0.6), min(0.3, 0.1), min(0.2, 0.2)\big)\}$$

$$= \left\{\begin{matrix}(-1, 0.5,0.2,0.1), (-2, 0.5,0.1,0.2), (-3, 0.6,0.1,0.2), (0, 0.5, 0.2, 0.1), \\ (-1, 0.4, 0.1, 0.2), (-2, 0.6, 0.1, 0.2)\end{matrix}\right\}$$

$$= \left\{\begin{matrix}(-1, \min(0.5, 0.4), \min(0.2, 0.1), \max(0.1, 0.2)), \\ (-2, \min(0.5, 0.6), \min(0.1, 0.1), \max(0.2, 0.2)), \\ (-3, 0.6,0.1,0.2), (0, 0.5, 0.2, 0.1)\end{matrix}\right\}$$

$$= \{(-1, 0.4,0.1,0.2), (-2, 0.5,0.1,0.2), (-3, 0.6,0.1,0.2), (0, 0.5, 0.2, 0.1)\}.$$

Definition 4.8: (Multiplication operation) Let $A, B \in PFS(X)$, then

$$A \times B = \{z, \mu_{A\times B}(z), \eta_{A\times B}(z), \nu_{A\times B}(z)\},$$

where

$$\mu_{A\times B}(z) = \wedge_{z=x\times y}[\mu_A(x) \vee \mu_B(y)],$$

$$\eta_{A\times B}(z) = \wedge_{z=x\times y}[\eta_A(x) \wedge \eta_B(y)],$$

and $$\nu_{A\times B}(z) = \vee_{z=x\times y}[\nu_A(x) \wedge \nu_B(y)].$$

Example 4.8(a): Let $A, B \in PFS(X)$, where

$$A = \{(1, 0.5, 0.3, 0.2), (2, 0.4, 0.3, 0.2)\}$$

and

$$B = \{(2, 0.5, 0.2, 0.1), (3, 0.2, 0.1, 0.4), (4, 0.6, 0.1, 0.2)\}.$$

Therefore,

$$A \times B = \{(1 \times 2, max(0.5,0.5), min(0.3, 0.2), min(0.2, 0.1)),$$

$$(1 \times 3, max(0.5,0.2), min(0.3, 0.1), min(0.2, 0.4)),$$

$$(1 \times 4, max(0.5,0.6), min(0.3, 0.1), min(0.2, 0.2)),$$

$$(2 \times 2, max(0.4,0.5), min(0.3, 0.2), min(0.2, 0.1)),$$

$$(2 \times 3, max(0.4,0.2), min(0.3, 0.1), min(0.2, 0.4)),$$

$$(2 \times 4, max(0.4,0.6), min(0.3, 0.1), min(0.2, 0.2))\}$$

$$= \left\{ \begin{matrix} (2, 0.5,0.2,0.1), (3, 0.5,0.1,0.2), (4, 0.6,0.1,0.2), (4, 0.5,0.2,0.1), \\ (6, 0.4, 0.1, 0.2), (8, 0.6, 0.1, 0.2) \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} (2, 0.5,0.2,0.1), (3, 0.5,0.1,0.2), (4, min(0.6,0.5), min(0.1,0.2), max(0.2,0.1)), \\ (6, 0.4, 0.1, 0.2), (8, 0.6, 0.1, 0.2) \end{matrix} \right\}$$

$$= \{(2, 0.5,0.2,0.1), (3, 0.5,0.1,0.2), (4, 0.5,0.1,0.2), (6, 0.4, 0.1, 0.2), (8, 0.6, 0.1, 0.2)\}.$$

Definition 4.9: (Division operation) Let $A, B \in PFS(X)$, then

$$A/B = \{z, \mu_{A/B}(z), \eta_{A/B}(z), \nu_{A/B}(z)\}$$

Where

$$\mu_{A/B}(z) = \wedge_{z=x/y}[\mu_A(x) \vee \mu_B(y)],$$

$$\eta_{A/B}(z) = \wedge_{z=x/y}[\eta_A(x) \wedge \eta_B(y)],$$

and $$\nu_{A/B}(z) = \vee_{z=x/y}[\nu_A(x) \wedge \nu_B(y)].$$

Example 4.9 (a): Let $A, B \in PFS(X)$, where

$$A = \{(1, 0.5, 0.3, 0.2), (2, 0.4, 0.3, 0.2)\}$$

and

$$B = \{(2, 0.5, 0.2, 0.1), (3, 0.2, 0.1, 0.4), (4, 0.6, 0.1, 0.2)\}.$$

Therefore,

$$A/B = \{(1/2, max(0.5,0.5), min(0.3, 0.2), min(0.2, 0.1)),$$

$$\big(1/3, max(0.5,0.2), min(0.3, 0.1), min(0.2, 0.4)\big),$$

$$\big(1/4, max(0.5,0.6), min(0.3, 0.1), min(0.2, 0.2)\big),$$

$$\big(2/2, max(0.4,0.5), min(0.3, 0.2), min(0.2, 0.1)\big),$$

$$\big(2/3, max(0.4,0.2), min(0.3, 0.1), min(0.2, 0.4)\big),$$

$$\big(2/4, max(0.4,0.6), min(0.3, 0.1), min(0.2, 0.2)\big)\}$$

$$= \begin{Bmatrix} \left(\frac{1}{2}, 0.5,0.2,0.1\right), \left(\frac{1}{3}, 0.5,0.1,0.2\right), \left(\frac{1}{4}, 0.6,0.1,0.2\right), \\ (1, 0.5,0.2,0.1), (2/3, 0.4, 0.1, 0.2), (1/2, 0.6, 0.1, 0.2) \end{Bmatrix}$$

$$= \begin{Bmatrix} \big(1/2, min(0.5,0.6), min(0.2,0.1), max(0.1,0.2)\big), \\ (1/3, 0.5,0.1,0.2), (1/4, 0.6,0.1,0.2), (1, 0.5,0.2,0.1), (2/3, 0.4, 0.1, 0.2) \end{Bmatrix}$$

$$= \begin{Bmatrix} \left(\frac{1}{2}, 0.5,0.1,0.2\right), \left(\frac{1}{3}, 0.5,0.1,0.2\right), \left(\frac{1}{4}, 0.6,0.1,0.2\right), \\ (1, 0.5,0.2,0.1), (2/3, 0.4, 0.1, 0.2) \end{Bmatrix}$$

5. CONCLUSIONS

This works concentrates on developing some structural properties of picture fuzzy sets. This study extended the works of Cuong and Kreinovich [3, 4] and Dutta and Ganju [7] in some aspects. Here we have defined the lower (α, γ, β)-cut and strong lower (α, γ, β)-cut for picture fuzzy sets and explored some properties of them. The concept of minimal decomposition theorems by using the lower (α, γ, β)-cut and strong lower (α, γ, β)-cut and level set are introduced. Some properties of the minimal extension principle for picture fuzzy sets are described. Arithmetic operations of picture fuzzy sets are also illustrated by the minimal extension principle.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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