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FIXED POINT THEOREM FOR SIX SELF MAPPINGS INVOLVING CUBIC TERMS OF $\mathscr{M}(x,y,t)$ IN FUZZY METRIC SPACE

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Abstract. In this study, we first describe the generalized ψ-weak contraction condition, which involves cubic and quadratic terms of $\mathscr{M}(x,y,t)$, and then show common fixed-point theorems using weakly compatible for six self-mappings in fuzzy metric space.

Keywords: ψ-weak contraction; weakly compatible mappings; fuzzy metric space.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Probability theory has been exploring a sort of uncertainty in the occurrence of an event since the 16th century. When the uncertainty is due to fuzziness rather than randomness, as it is typically in the measurement of an ordinary length, the concept of a fuzzy metric space looks to be more suitable.

Zadeh [22] suggested the concept of fuzzy sets, which gives a precise natural framework for mathematical modelling of real-world situations distinguished by ambiguous situations attributable to non-probabilistic aspects. The fuzzification of those fields of mathematics that are based

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on set theory, which resulted in a new branch of mathematics named ‘Fuzzy Mathematics,’ is a significant developmental path for fuzzy set theory.

The advent of the probabilistic metric space was influenced by the fact that the distance between two points is generally inexact rather than a single figure. Deng [1], Erceg [2], Kaleva and Seikkala [9], and Kramosil and Michalek [8] have all introduced fuzzy metric space in different methods. The fuzzy form of the Banach contraction principle was discovered by Grabiec [3], who was followed by Kramosil and Michalek [8].

Definition 1.1 ([17])**.** Let $\mathscr{A}$ and $\mathscr{B}$ be two self-mappings in a fuzzy metric space $(\mathfrak{W}, \mathscr{M}, *)$. The mappings $\mathscr{A}$ and $\mathscr{B}$ are said to be compatible if $\lim_{n\to\infty} \mathscr{M}(\mathscr{A}\mathscr{B}x_n, \mathscr{B}\mathscr{A}x_n, t) = 1$, for all $t > 0$, whenever $\{x_n\}$ is a sequence in $\mathfrak{W}$ such that $\lim_{n\to\infty} \mathscr{A}x_n = \lim_{n\to\infty} \mathscr{B}x_n = u$ for some u in $\mathfrak{W}$.

Definition 1.2 ([17])**.** Two mappings $\mathscr{A}$ and $\mathscr{B}$ are said to be weakly compatible if they commute at their coincidence points.

In 1996, Jungck [4] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but converse may not be true.

2. Preliminaries

Definition 2.1 ([3])**.** The triplet $(\mathfrak{W}, \mathscr{M}, *)$ is a fuzzy metric space if $\mathfrak{W}$ is arbitrary set, $*$ is a continuous t-norm, $\mathscr{M}$ is a fuzzy set in $\mathfrak{W}^2 \times [0, \infty)$ satisfying the following conditions:

(i) $\mathscr{M}(x, y, 0) = 0$, $\mathscr{M}(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(ii) $\mathscr{M}(x, y, t) = \mathscr{M}(y, x, t)$, $(\mathscr{M}(x, y, t) * \mathscr{M}(y, z, s)) \leq \mathscr{M}(x, z, t + s)$, $\mathscr{M}(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous for all $x, y, z \in \mathfrak{V}$ and $s, t > 0$,

(iii) $\lim_{t\to\infty} \mathscr{M}(x, y, t) = 1$, for all x, y in $\mathfrak{V}$.$\mathscr{M}(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.2 ([13])**.** Let $(\mathfrak{W}, \mathscr{M}, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $\mathfrak{W}$ is said to be:

(i) Converge to $x \in \mathfrak{W}$ if $\lim_{n\to\infty} \mathscr{M}(x_n, x, t) = 1$ fo each $t > 0$.

(ii) Cauchy sequence if $\lim_{n\to\infty} \mathscr{M}(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

(iii) Complete if every Cauchy sequence in $\mathfrak{W}$ is convergent in $\mathfrak{W}$.

Proposition 2.1 ([7])**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be compatible mappings of a fuzzy metric space $(\mathfrak{W}, \mathscr{M}, *)$ into itself, If*

$$\mathscr{A}t = \mathscr{B}t \text{ for some } t \text{ in } \mathfrak{W}, \text{ then } \mathscr{A}\mathscr{B}t = \mathscr{A}\mathscr{A}t = \mathscr{B}\mathscr{B}t = \mathscr{B}\mathscr{A}t.$$

Proposition 2.2 ([7])**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be compatible mappings of a fuzzy metric space $(\mathfrak{W}, \mathscr{M}, *)$ into itself.*

Suppose that $\lim_n \mathscr{A}x_n = \lim_n \mathscr{B}x_n = t$ for some t in $\mathfrak{W}$. Then the following holds:

(i) $\lim_n \mathscr{B}\mathscr{A}x_n = \mathscr{A}t$ *if $\mathscr{A}$ is continuous at t;*

(ii) $\lim_n \mathscr{A}\mathscr{B}x_n = \mathscr{B}t$ *if $\mathscr{B}$ is continuous at t;*

(iii) $\mathscr{A}\mathscr{B}t = \mathscr{B}\mathscr{A}t$ *and $\mathscr{A}t = \mathscr{B}t$ if $\mathscr{A}$ and $\mathscr{B}$ are continuous at t.*

Lemma 2.1 ([17])**.** *Let $(\mathfrak{W}, \mathscr{M}, *)$ be a fuzzy metric space. If there exists $q \in (0,1)$ such that $\mathscr{M}(x,y,qt) \geq \mathscr{M}(x,y,t)$ for all $x,y \in \mathfrak{B}$, and $t > 0$, then $x = y$.*

Lemma 2.2 ([17])**.** *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(\mathfrak{W}, \mathscr{M}, *)$. If there exists $q \in (0,1)$ such that $\mathscr{M}(y_{n+2}, y_{n+1}, qt) \geq \mathscr{M}(y_{n+1}, y_n, t)$, $t > 0$, $n \in N$, then $y_n$is a Cauchy sequence in $\mathfrak{W}$.*

Lemma 2.3 ([20])**.** *Let $(\mathfrak{W}, \mathscr{M}, *)$ be a fuzzy metric space. If there is a sequence $\{x_n\}$ in X, such that for every $n \in N$.*

$$\mathscr{M}(x_n, x_{n+1}, t) \geq \mathscr{M}(x_0, x_1, k^n t)$$

for every $k > 1$, then the sequence is a Cauchy sequence.

3. Main Results

3.1. A class of Implicit Relation. Let Ψ be set of all continuous functions $\psi : [0,1]^4 \to [0,1]$ increasing in any coordinate and $\psi(t,t,t,t) > t$.

Theorem 3.1. *Let $(\mathfrak{W}, \mathscr{M}, *)$ be a complete fuzzy metric space. Let $\mathscr{N}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{W}$ are six self-mappings of a complete fuzzy metric space $(\mathfrak{W}, \mathscr{M}, *)$ into itself satisfying*

(C1) $\mathscr{T}(\mathfrak{W}) \subseteq \mathscr{N}\mathscr{P}(\mathfrak{W})$, $\mathscr{W}(\mathfrak{W}) \subseteq \mathscr{Q}\mathscr{S}(\mathfrak{W})$,

(C2) $\mathscr{Q}\mathscr{S} = \mathscr{S}\mathscr{Q}$, $\mathscr{N}\mathscr{P} = \mathscr{P}\mathscr{N}$, $\mathscr{T}\mathscr{S} = \mathscr{S}\mathscr{T}$, $\mathscr{W}\mathscr{P} = \mathscr{P}\mathscr{W}$,

(C3) *One of $\mathscr{N}\mathscr{P}(\mathfrak{W})$, $\mathscr{W}(\mathfrak{W})$, $\mathscr{Q}\mathscr{S}(\mathfrak{W})$ or $\mathscr{T}(\mathfrak{W})$ is complete,* (C4) *The pair* $(\mathscr{T},\mathscr{Q}\mathscr{S})$ *and* $(\mathscr{W},\mathscr{N}\mathscr{P})$ *are weakly compatible,*

(C5) $$\mathscr{M}^3(\mathscr{T}u,\mathscr{W}v,t)\geq\Psi\left\{\begin{array}{c}\mathscr{M}^2(\mathscr{Q}\mathscr{S}u,\mathscr{T}u,kt)\mathscr{M}(\mathscr{W}v,\mathscr{N}\mathscr{P}v,kt)\\ \mathscr{M}(\mathscr{Q}\mathscr{S}u,\mathscr{T}u,kt)\mathscr{M}^2(\mathscr{W}v,\mathscr{N}\mathscr{P}v,kt),\\ \mathscr{M}(\mathscr{Q}\mathscr{S}u,\mathscr{T}u,kt)\mathscr{M}(\mathscr{T}u,\mathscr{W}v,kt)\mathscr{M}(\mathscr{W}v,\mathscr{N}\mathscr{P}v,kt),\\ \mathscr{M}(\mathscr{W}v,\mathscr{N}\mathscr{P}v,kt)\mathscr{M}(\mathscr{Q}\mathscr{S}u,\mathscr{N}\mathscr{P}v,kt)\mathscr{M}(\mathscr{Q}\mathscr{S}u,\mathscr{T}u,kt)\end{array}\right\}$$

for all $u,v\in\mathfrak{W}$, $k>1$ *and* $\psi\in\Psi$.

Then $\mathscr{N}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{S}$, $\mathscr{T}$ *and* $\mathscr{W}$ *have a unique common fixed point in* $\mathfrak{W}$.

Proof. Let $x_0\in\mathfrak{W}$ be an arbitrary point. From (C_1) we can find a point x_1 such that $\mathscr{T}(x_0)=\mathscr{N}\mathscr{P}(x_1)=y_0$. For this point x_1 one can find a point $x_2\in\mathfrak{W}$ such that $\mathscr{W}(x_1)=\mathscr{Q}\mathscr{S}(x_2)=y_1$. Continuing in this way, one can construct a sequence $\{x_n\}$ such that $y_{2n}=\mathscr{T}(x_{2n})=\mathscr{N}\mathscr{P}(x_{2n+1})$,

$$y_{2n+1}=\mathscr{W}(x_{2n+1})=\mathscr{Q}\mathscr{S}(x_{2n+2}),\quad \text{for each } n\geq 0. \tag{3.1}$$

For brevity, we write $\alpha_m(t)=\mathscr{M}(y_m,y_{m+1},t)$.

First, we prove that $\{y_n\}$ is a Cauchy sequence

Case I: If n is even, taking $u=x_{2n}$ and $v=x_{2n+1}$ in (C_5), we get

$$\mathscr{M}^3(\mathscr{T}x_{2n},\mathscr{W}x_{2n+1},t)$$
$$\geq\psi\left\{\begin{array}{c}\mathscr{M}^2(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt),\\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}^2(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt),\\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}(\mathscr{T}x_{2n},\mathscr{W}x_{2n+1},kt)\mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt),\\ \mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{N}\mathscr{P}x_{2n+1},kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\end{array}\right\}$$

Using (3.1), we have

$$\mathscr{M}^3(y_{2n}\ ,y_{2n+1},t)\geq\psi\left\{\begin{array}{c}\mathscr{M}^2(y_{2n-1},y_{2n},kt)\mathscr{M}(y_{2n+1},y_{2n},kt),\\ \mathscr{M}(y_{2n-1},y_{2n},kt)\mathscr{M}^2(y_{2n+1},y_{2n},kt),\\ \mathscr{M}(y_{2n-1},y_{2n},kt)\mathscr{M}(y_{2n},y_{2n+1},kt)\mathscr{M}(y_{2n+1},y_{2n},kt),\\ \mathscr{M}(y_{2n+1},y_{2n},kt)\mathscr{M}(y_{2n-1},y_{2n},kt)\mathscr{M}(y_{2n-1},y_{2n},kt)\end{array}\right\}$$

On using $\alpha_{2n}(t) = \mathscr{M}(y_{2n}, y_{2n+1}, t)$ in the above inequality, we have

$$\alpha^3_{2n}(t) \geq \psi \left\{ \begin{array}{c} \alpha^2_{2n-1}(kt)\alpha_{2n}(kt), \alpha_{2n-1}(kt)\alpha^2_{2n}(kt), \\ \alpha_{2n-1}(kt)\alpha^2_{2n}(kt), \alpha_{2n}(kt)\alpha^2_{2n-1}(kt) \end{array} \right\}. \tag{3.2}$$

We claim that $\alpha_{2n}(kt) \geq \alpha_{2n-1}(kt)$.

If $\alpha_{2n}(kt) < \alpha_{2n-1}(kt)$, then (3.2) reduces to

$$\alpha^3_{2n}(t) \geq \psi\{\alpha^3_{2n}(kt), \alpha^3_{2n}(kt), \alpha^3_{2n}(kt), \alpha^3_{2n}(kt)\}.$$

Using property of ψ we get

$$\alpha^3_{2n}(t) > \alpha^3_{2n}(kt) \Longrightarrow \alpha_{2n}(t) > \alpha_{2n}(kt), \text{ a contradiction.}$$

Therefore $\alpha_{2n}(kt) \geq \alpha_{2n-1}(kt)$.

In a similar way, if n is odd, then we can obtain $\alpha_{2n+1}(kt) \geq \alpha_{2n}(kt)$.

It follows that the sequence $\{\alpha_n(t)\}$ is increasing in $[0,1]$, thus (3.2) reduces to

$$\alpha^3_{2n}(t) \geq \psi\{\alpha^3_{2n-1}(kt), \alpha^3_{2n-1}(kt), \alpha^3_{2n-1}(kt), \alpha^3_{2n-1}(kt)\}$$

Using property of ψ we get

$$\alpha^3_{2n}(t) > \alpha^3_{2n-1}(kt).$$

Thus we get $\alpha_{2n}(t) \geq \alpha_{2n-1}(kt)$.

Similarly for an odd integer $m = 2n+1$, we have $\alpha_{2n+1}(t) \geq \alpha_{2n}(kt)$.

Hence $\alpha_n(t) \geq \alpha_{n-1}(kt)$. That is,

$$\mathscr{M}(y_n, y_{n+1}, t) \geq \mathscr{M}(y_{n-1}, y_n, kt) \geq \ldots \geq \mathscr{M}(y_0, y_1, k^n t).$$

Hence by Lemma 2.3 $\{y_n\}$ is a Cauchy sequence in $\mathfrak{W}$.

Case 1: $\mathscr{N}\mathscr{P}(\mathfrak{W})$ is complete. In this case $\{y_{2n}\} = \{\mathscr{N}\mathscr{P}x_{2n+1}\}$ is a Cauchy sequence in $\mathscr{N}\mathscr{P}(\mathfrak{W})$, which is complete then the sequence $\{y_{2n}\}$ converges to some point $z \in \mathscr{N}\mathscr{P}(\mathfrak{W})$. Consequently, the subsequence's $\{\mathscr{T}x_{2n}\}$, $\{\mathscr{Q}\mathscr{S}x_{2n}\}$, $\{\mathscr{N}\mathscr{P}x_{2n+1}\}$ and $\{\mathscr{W}x_{2n+1}\}$ also converges to the same point z. As $z \in \mathscr{N}\mathscr{P}(\mathfrak{W})$, there exists $r \in \mathfrak{W}$ such that $z = \mathscr{N}\mathscr{P}r$.

Now we claim that $z = \mathscr{W}r$. For this putting $u = x_{2n}$ and $v = r$ in (C_5), we get

$$\mathscr{M}^3(\mathscr{T}x_{2n}, \mathscr{W}r, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}x_{2n}, \mathscr{T}x_{2n}, kt)\mathscr{M}(\mathscr{W}r, \mathscr{N}\mathscr{P}r, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n}, \mathscr{T}x_{2n}, kt)\mathscr{M}^2(\mathscr{W}r, \mathscr{N}\mathscr{P}r, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n}, \mathscr{T}x_{2n}, kt)\mathscr{M}(\mathscr{T}x_{2n}, \mathscr{W}r, kt)\mathscr{M}(\mathscr{W}r, \mathscr{N}\mathscr{P}r, kt), \\ \mathscr{M}(\mathscr{W}r, \mathscr{N}\mathscr{P}r, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n}, \mathscr{N}\mathscr{P}r, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n}, \mathscr{T}x_{2n}, kt) \end{array} \right\}$$

Taking limit $n \to \infty$ and using $z = \mathscr{N}\mathscr{P}r$ in above inequality we have,

$$\mathscr{M}^3(z, \mathscr{W}r, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(z, z, kt)\mathscr{M}(\mathscr{W}r, z, kt), \\ \mathscr{M}(z, z, kt)\mathscr{M}^2(\mathscr{W}r, z, kt), \\ \mathscr{M}(z, z, kt)\mathscr{M}(z, \mathscr{W}r, kt)\mathscr{M}(\mathscr{W}r, z, kt), \\ \mathscr{M}(\mathscr{W}r, z, kt)\mathscr{M}(z, z, kt)\mathscr{M}(z, z, kt) \end{array} \right\}$$

$$\mathscr{M}^3(z, \mathscr{W}r, t) \geq \psi \left\{ \begin{array}{c} 1.1.\mathscr{M}(\mathscr{W}r, z, kt), \\ 1.\mathscr{M}^2(\mathscr{W}r, z, kt), \\ 1.\mathscr{M}(z, \mathscr{W}r, kt)\mathscr{M}(\mathscr{W}r, z, kt), \\ \mathscr{M}(\mathscr{W}r, z, kt).1.1 \end{array} \right\}$$

Suppose $\mathscr{W}r \neq z$, then $\mathscr{M}(z, \mathscr{W}r, kt) < 1$, using this in above inequality we get

$$\mathscr{M}^3(z, \mathscr{W}r, t) \geq \psi\{\mathscr{M}^3(z, \mathscr{W}r, kt), \mathscr{M}^3(z, \mathscr{W}r, kt), \mathscr{M}^3(z, \mathscr{W}r, kt), \mathscr{M}^3(z, \mathscr{W}r, kt)\}$$

Using property of ψ we get

$$\begin{aligned} & \mathscr{M}^3(z, \mathscr{W}r, t) > \mathscr{M}^3(z, \mathscr{W}r, kt) \\ \Longrightarrow \quad & \mathscr{M}(z, \mathscr{W}r, t) > M(z, \mathscr{W}r, kt), \text{ a contradiction.} \end{aligned}$$

Hence $\mathscr{W}r = z$

Thus $\mathscr{W}r = z = \mathscr{N}\mathscr{P}r$. Since$(\mathscr{W}, \mathscr{N}\mathscr{P})$ are weakly compatible, so we have $\mathscr{W}z = \mathscr{N}\mathscr{P}z$.

Next we will show that $\mathscr{P}z = z$, for this putting $u = x_{2n}$ and $v = \mathscr{P}r$ in (C_5), we get

$$\mathscr{M}^3(\mathscr{T}x_{2n}, \mathscr{W}\mathscr{P}r, t)$$

$$\geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}(\mathscr{W}\mathscr{P}r,\mathscr{N}\mathscr{P}\mathscr{P}r,kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}^2(\mathscr{W}\mathscr{P}r,\mathscr{N}\mathscr{P}\mathscr{P}r,kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}(\mathscr{T}x_{2n},\mathscr{W}\mathscr{P}r,kt)\mathscr{M}(\mathscr{W}\mathscr{P}r,\mathscr{N}\mathscr{P}\mathscr{P}r,kt), \\ \mathscr{M}(\mathscr{W}\mathscr{P}r,\mathscr{N}\mathscr{P}\mathscr{P}r,kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{N}\mathscr{P}\mathscr{P}r,kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt) \end{array} \right\}$$

From (C_2) $\mathscr{W}\mathscr{P}=\mathscr{P}\mathscr{W}$ and $\mathscr{N}\mathscr{P}=\mathscr{P}\mathscr{N}$ using in above inequality we get,

$$\mathscr{M}^3(\mathscr{T}x_{2n},\mathscr{P}\mathscr{W}r,t)$$

$$\geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}(\mathscr{P}\mathscr{W}r,\mathscr{P}\mathscr{N}\mathscr{P}r,kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}^2(\mathscr{P}\mathscr{W}r,\mathscr{P}\mathscr{N}\mathscr{P}r,kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt)\mathscr{M}(\mathscr{T}x_{2n},\mathscr{P}\mathscr{W}r,kt)\mathscr{M}(\mathscr{P}\mathscr{W}r,\mathscr{P}\mathscr{N}\mathscr{P}r,kt), \\ \mathscr{M}(\mathscr{P}\mathscr{W}r,\mathscr{P}\mathscr{N}\mathscr{P}r,kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{P}\mathscr{N}\mathscr{P}r,kt)\mathscr{M}(\mathscr{Q}\mathscr{S}x_{2n},\mathscr{T}x_{2n},kt) \end{array} \right\}$$

Taking limit $n \to \infty$ and using $\mathscr{W}r = z = \mathscr{N}\mathscr{P}r$ in above inequality we have,

$$\mathscr{M}^3(z,\mathscr{P}z,t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(z,z,kt)\mathscr{M}(\mathscr{P}z,\mathscr{P}z,kt), \\ \mathscr{M}(z,z,kt)\mathscr{M}^2(\mathscr{P}z,\mathscr{P}z,kt), \\ \mathscr{M}(z,z,kt)\mathscr{M}(z,\mathscr{P}z,kt)\mathscr{M}(\mathscr{P}z,\mathscr{P}z,kt), \\ \mathscr{M}(\mathscr{P}z,\mathscr{P}z,kt)\mathscr{M}(z,\mathscr{P}z,kt)\mathscr{M}(z,z,kt) \end{array} \right\}$$

Suppose $\mathscr{P}z \neq z$, then $\mathscr{M}(z,\mathscr{P}z,kt) < 1$, using this in above inequality we get

$$\mathscr{M}^3(z,\mathscr{P}z,t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^3(z,\mathscr{P}z,kt),\mathscr{M}^3(z,\mathscr{P}z,kt), \\ \mathscr{M}^3(z,\mathscr{P}z,kt),\mathscr{M}^3(z,\mathscr{P}z,kt) \end{array} \right\}$$

Using property of ψ we get

$$\mathscr{M}^3(z,\mathscr{P}z,t) > \mathscr{M}^3(z,\mathscr{P}z,kt)$$

$$\implies \quad \mathscr{M}(z,\mathscr{P}z,t) > M(z,\mathscr{P}z,kt), \text{ a contradiction.}$$

Hence $z = \mathscr{P}z$.

Thus $\mathscr{P}z = \mathscr{N}\mathscr{P}z = z \Rightarrow \mathscr{N}z = z$.

Thus $\mathscr{N}z = \mathscr{P}z = \mathscr{W}z = z$.

As $\mathscr{W}(\mathfrak{W}) \subseteq \mathscr{Q}\mathscr{S}(\mathfrak{W})$, there exist $m \in \mathfrak{W}$ such that $z = \mathscr{W}z = \mathscr{Q}\mathscr{S}m$.

Next, we will show that $\mathscr{T}m = z$, for this putting $u = m$ and $v = x_{2n+1}$ in (C_5), we have

$$\mathscr{M}^3(\mathscr{T}m, \mathscr{W}x_{2n+1}, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}m, \mathscr{T}m, kt)\mathscr{M}(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}m, \mathscr{T}m, kt)\mathscr{M}^2(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}m, \mathscr{T}m, kt)\mathscr{M}(\mathscr{T}m, \mathscr{W}x_{2n+1}, kt)\mathscr{M}(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt), \\ \mathscr{M}(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}m, \mathscr{N}\mathscr{P}x_{2n+1}, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}m, \mathscr{T}m, kt) \end{array} \right\}$$

Taking limit $n \to \infty$ and using $z = \mathscr{W}z = \mathscr{Q}\mathscr{S}m$in above inequality we have,

$$\mathscr{M}^3(\mathscr{T}m, z, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(z, \mathscr{T}m, kt)\mathscr{M}(z, z, kt), \\ \mathscr{M}(z, \mathscr{T}m, kt)\mathscr{M}^2(z, z, kt), \\ \mathscr{M}(z, \mathscr{T}m, kt)\mathscr{M}(\mathscr{T}m, z, kt)\mathscr{M}(z, z, kt), \\ \mathscr{M}(z, z, kt)\mathscr{M}(z, z, kt)\mathscr{M}(z, \mathscr{T}m, kt) \end{array} \right\}$$

Suppose $\mathscr{T}m \neq z$, then $\mathscr{M}(\mathscr{T}m, z, kt) < 1$, using this in above inequality we get

$$\mathscr{M}^3(\mathscr{T}m, z, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^3(\mathscr{T}m, z, kt), \mathscr{M}^3(\mathscr{T}m, z, kt), \\ \mathscr{M}^3(\mathscr{T}m, z, kt), \mathscr{M}^3(\mathscr{T}m, z, kt) \end{array} \right\}$$

Using property of ψ we get

$$\begin{aligned} & \mathscr{M}^3(\mathscr{T}m, z, t) > \mathscr{M}^3(\mathscr{T}m, z, kt) \\ \implies \quad & \mathscr{M}(\mathscr{T}m, z, t) > M(\mathscr{T}m, z, kt), \text{ a contradiction.} \end{aligned}$$

Hence $\mathscr{T}m = z$.

Since $(\mathscr{T}, \mathscr{Q}\mathscr{S})$ are weakly compatible compatible,so $\mathscr{T}$ and $\mathscr{Q}\mathscr{S}$ commute their co-incidence point m, then we have $\mathscr{T}z = \mathscr{Q}\mathscr{S}z$.

Next we will show that $\mathscr{T}z = z$, for this putting $u = z$ and $v = x_{2n+1}$ in (C_5), we have

$$\mathscr{M}^3(\mathscr{T}z, \mathscr{W}x_{2n+1}, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt)\mathscr{M}(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt)\mathscr{M}^2(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt)\mathscr{M}(\mathscr{T}z, \mathscr{W}x_{2n+1}, kt)\mathscr{M}(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt), \\ \mathscr{M}(\mathscr{W}x_{2n+1}, \mathscr{N}\mathscr{P}x_{2n+1}, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{N}\mathscr{P}x_{2n+1}, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt) \end{array} \right\}$$

Taking limit $n \to \infty$ and using $\mathscr{T}z = \mathscr{Q}\mathscr{S}z$ in above inequality we have,

$$\mathscr{M}^3(\mathscr{T}z,z,t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{T}z,\mathscr{T}z,kt)\mathscr{M}(z,z,kt), \\ \mathscr{M}(\mathscr{T}z,\mathscr{T}z,kt)\mathscr{M}^2(z,z,kt), \\ \mathscr{M}(\mathscr{T}z,\mathscr{T}z,kt)\mathscr{M}(\mathscr{T}z,z,kt)\mathscr{M}(z,z,kt), \\ \mathscr{M}(z,z,kt)\mathscr{M}(\mathscr{T}z,z,kt)\mathscr{M}(\mathscr{T}z,\mathscr{T}z,kt) \end{array} \right\}$$

Suppose $\mathscr{T}z \neq z$, then $\mathscr{M}(\mathscr{T}z,z,kt) < 1$, using this in above inequality we get

$$\mathscr{M}^3(\mathscr{T}z,z,t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^3(\mathscr{T}z,z,kt),\mathscr{M}^3(\mathscr{T}z,z,kt), \\ \mathscr{M}^3(\mathscr{T}z,z,kt),\mathscr{M}^3(\mathscr{T}z,z,kt) \end{array} \right\}$$

Using property of ψ we get

$$\begin{aligned} & \mathscr{M}^3(\mathscr{T}z,z,t) > \mathscr{M}^3(\mathscr{T}z,z,kt) \\ \implies \quad & \mathscr{M}(\mathscr{T}z,z,t) > M(\mathscr{T}z,z,kt), \text{ a contradiction.} \end{aligned}$$

Hence $\mathscr{T}z = z$.

Thus $\mathscr{T}z = \mathscr{Q}\mathscr{S}z = z$.

Next we will show that $\mathscr{S}z = z$, for this putting $u = \mathscr{S}z$ and $v = x_{2n+1}$ in (C_5), we have

$$\begin{aligned} & \mathscr{M}^3(\mathscr{T}\mathscr{S}z,\mathscr{W}x_{2n+1},t) \\ & \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}\mathscr{S}z,\mathscr{T}\mathscr{S}z,kt)\mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}\mathscr{S}z,\mathscr{T}\mathscr{S}z,kt)\mathscr{M}^2(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}\mathscr{S}z,\mathscr{T}\mathscr{S}z,kt)\mathscr{M}(\mathscr{T}\mathscr{S}z,\mathscr{W}x_{2n+1},kt)\mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt), \\ \mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt)\mathscr{M}(\mathscr{Q}\mathscr{S}\mathscr{S}z,\mathscr{N}\mathscr{P}x_{2n+1},kt)\mathscr{M}(\mathscr{Q}\mathscr{S}\mathscr{S}z,\mathscr{T}\mathscr{S}z,kt) \end{array} \right\} \end{aligned}$$

From (C_2) $\mathscr{Q}\mathscr{S} = \mathscr{S}\mathscr{Q}$ and $\mathscr{T}\mathscr{S} = \mathscr{S}\mathscr{T}$ using in above inequality we have,

$$\begin{aligned} & \mathscr{M}^3(\mathscr{S}\mathscr{T}z,\mathscr{W}x_{2n+1},t) \\ & \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{S}\mathscr{Q}\mathscr{S}z,\mathscr{S}\mathscr{T}z,kt)\mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt), \\ \mathscr{M}(\mathscr{S}\mathscr{Q}\mathscr{S}z,\mathscr{S}\mathscr{T}z,kt)\mathscr{M}^2(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt), \\ \mathscr{M}(\mathscr{S}\mathscr{Q}\mathscr{S}z,\mathscr{S}\mathscr{T}z,kt)\mathscr{M}(\mathscr{S}\mathscr{T}z,\mathscr{W}x_{2n+1},kt)\mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt), \\ \mathscr{M}(\mathscr{W}x_{2n+1},\mathscr{N}\mathscr{P}x_{2n+1},kt)\mathscr{M}(\mathscr{S}\mathscr{Q}\mathscr{S}z,\mathscr{N}\mathscr{P}x_{2n+1},kt)\mathscr{M}(\mathscr{S}\mathscr{Q}\mathscr{S}z,\mathscr{S}\mathscr{T}z,kt) \end{array} \right\} \end{aligned}$$

Taking limit $n \to \infty$ and using $\mathscr{T}z = \mathscr{Q}\mathscr{S}z = z$ in above inequality we have,

$$\mathscr{M}^3(\mathscr{S}z, z, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{S}z, \mathscr{S}z, kt)\mathscr{M}(z, z, kt), \\ \mathscr{M}(\mathscr{S}z, \mathscr{S}z, kt)\mathscr{M}^2(z, z, kt), \\ \mathscr{M}(\mathscr{S}z, \mathscr{S}z, kt)\mathscr{M}(\mathscr{S}z, z, kt)\mathscr{M}(z, z, kt), \\ \mathscr{M}(z, z, kt)\mathscr{M}(\mathscr{S}z, z, kt)\mathscr{M}(\mathscr{S}z, \mathscr{S}z, kt) \end{array} \right\}$$

Suppose $\mathscr{S}z \neq z$, then $\mathscr{M}(\mathscr{S}z, z, kt) < 1$, using this in above inequality we get

$$\mathscr{M}^3(\mathscr{S}z, z, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^3(\mathscr{S}z, z, kt), \mathscr{M}^3(\mathscr{S}z, z, kt), \\ \mathscr{M}^3(\mathscr{S}z, z, kt), \mathscr{M}^3(\mathscr{S}z, z, kt) \end{array} \right\}$$

Using property of ψ we get

$$\mathscr{M}^3(\mathscr{S}z, z, t) > \mathscr{M}^3(\mathscr{S}z, z, kt)$$

$$\implies \quad \mathscr{M}(\mathscr{S}z, z, t) > M(\mathscr{S}z, z, kt), \text{ a contradiction.}$$

Hence $\mathscr{S}z = z$. Then $z = \mathscr{Q}\mathscr{S}z = \mathscr{Q}z$. Therefore $z = \mathscr{S}z = \mathscr{Q}z = \mathscr{T}z$.

Hence in all we have $z = \mathscr{S}z = \mathscr{Q}z = \mathscr{T}z = \mathscr{P}z = \mathscr{N}z = \mathscr{W}z$.

Case 2: When $\mathscr{T}(\mathfrak{W})$ is complete follows from above case as $\mathscr{T}(\mathfrak{W}) \subseteq \mathscr{N}\mathscr{P}(\mathfrak{W})$.

Case 3: When $\mathscr{Q}\mathscr{S}(\mathfrak{W})$ is complete. This case follows by symmetry. As $\mathscr{Q}(\mathfrak{W}) \subseteq \mathscr{Q}\mathscr{S}(\mathfrak{W})$, therefore the result also holds when $\mathscr{Q}(\mathfrak{W})$ is complete.

Uniqueness: Suppose that $z \neq m$ are two common fixed points of $\mathscr{N}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{W}$. On putting $u = z$, $v = m$ in (C_5), we have

$$\mathscr{M}^3(\mathscr{T}z, \mathscr{W}m, t) \geq \psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt)\mathscr{M}(\mathscr{W}m, \mathscr{N}\mathscr{P}m, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt)\mathscr{M}^2(\mathscr{W}m, \mathscr{N}\mathscr{P}m, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt)\mathscr{M}(\mathscr{T}z, \mathscr{W}m, kt)\mathscr{M}(\mathscr{W}m, \mathscr{N}\mathscr{P}m, kt), \\ \mathscr{M}(\mathscr{W}m, \mathscr{N}\mathscr{P}m, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{N}\mathscr{P}m, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}z, \mathscr{T}z, kt) \end{array} \right\}$$

If $z \neq m$ then $\mathscr{M}(z, m, kt) < 1$, using in above inequality and on simplification we get

$$\mathscr{M}(z, m, t) > M(z, m, kt), \text{ a contradiction}$$

Hence $z = m$.

Hence z be a unique common fixed point of $\mathscr{N}$, $\mathscr{Q}$, $\mathscr{P}$, $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{W}$.

4. APPLICATION

In 2002 Branciari obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality .

Now we prove the following theorem as an application of Theorem 3.1.

Theorem 4.1. *Let $\mathscr{N}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{W}$ be six self-mappings of a complete fuzzy metric space $(\mathfrak{W}, \mathscr{M}, *)$ satisfying the conditions (C_1), (C_2), (C_3), (C_4) and the following condition:*

$$\int_0^{\mathscr{M}^3(x,y,t)} \psi(w)dw \geq \int_0^{o(u,v)} \psi(w)dw$$

$$o(u,v) = \Psi \left\{ \begin{array}{c} \mathscr{M}^2(\mathscr{Q}\mathscr{S}u, \mathscr{T}u, kt)\mathscr{M}(\mathscr{W}v, \mathscr{N}\mathscr{P}v, kt) \\ \mathscr{M}(\mathscr{Q}\mathscr{S}u, \mathscr{T}u, kt)\mathscr{M}^2(\mathscr{W}v, \mathscr{N}\mathscr{P}v, kt), \\ \mathscr{M}(\mathscr{Q}\mathscr{S}u, \mathscr{T}u, kt)\mathscr{M}(\mathscr{T}u, \mathscr{W}v, kt)\mathscr{M}(\mathscr{W}v, \mathscr{N}\mathscr{P}v, kt), \\ \mathscr{M}(\mathscr{W}v, \mathscr{N}\mathscr{P}v, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}u, \mathscr{N}\mathscr{P}v, kt)\mathscr{M}(\mathscr{Q}\mathscr{S}u, \mathscr{T}u, kt) \end{array} \right\}$$

for all $u, v \in \mathfrak{W}$, where $\psi : [0,1]^4 \to [0,1]$ is increasing in any coordinate and $\psi(t,t,t,t) > t$ for every $t \in [0,1)$, where $\psi : [0,1]^4 \to [0,1]$ is a "Lebesgue-integrable function" which is summable, nonnegative, and such that, for each $\varepsilon > 0$, $\int_0^{\varepsilon} \psi(w)dw > 0$. Then $\mathscr{N}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{W}$ have a unique common fixed point in $\mathfrak{W}$.

Proof. The proof of the theorem follows on the same lines of the proof of Theorem 3.1. □

5. CONCLUSION

We show common fixed-point theorem for six self-mappings in fuzzy metric space that contains cubic and quadratic terms of the distance function $\mathscr{M}(x,y,t)$ in this study.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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