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CHARACTERIZATIONS OF FRENET CURVES IN GALILEAN 3-SPACE

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Abstract. The aim of this paper is to prove that the distance function of every Frenet curve in G_3 satisfies a 4-th order differential equation. Also, we show that if α is a unit speed Frenet curve in G_3, then $<\alpha(s), T(s)>= s+c$ if and only if α is a rectifying curve. Finally, we obtain some characterizations of spherical curves and helices via the 4-th order differential equation (4).

Keywords: Galilean Space; rectifying curves; spherical curves; helices.

2010 AMS Subject Classification: 53A35, 51A05.

1. Introduction

Curves satisfying particular relationships with respect to their curvatures are of greater significance in the theoretical study of differential geometry and applications. The rectifying curves and the helices are two of these famous curves [1, 2, 3, 4, 5].

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A curve α is referred to as a curve of Frenet if $\kappa > 0$ and $\tau \neq 0$. The distance function d of α is denoted by the equation $d(s) = \| \alpha(s) \|$. The distance function d plays an important role in obtaining the characterization of rectifying curves in addition to curvature and torsion [2].

A helix is a curve whose tangent makes a constant angle with a fixed direction (axis) [6]. There is a well-known general helix classification, a curve is called general helix if and only if $\frac{\kappa}{\tau} = constant$ [6].

The geometry of Manifolds in the de-Sitter space S_1^2 [7], Minkowski space [8, 9] represented as a popular topics for many researchers.

Galilean space is the space of Galilean relativity. More details about Galilean space and pseudo-Galilean space can be seen in the following references [10, 11, 12, 13, 14, 15, 16]. The Galilean relativity geometry serves as a bridge to special relativity from the Euclidean geometry. The curve geometry in Euclidean space was developed along time ago. Mathematicians have started studying curves and surfaces in Galilean space in recent years [6, 17, 18, 19, 20, 21],[22].

In this article, we first prove that the distance function of each Frenet curve in G_3 satisfies a differential equation of 4-th order. Finally, we give some characterizations of rectifying curves, spherical curves, and helices as a consequence of this differential equation.

2. PRELIMINARIES

The Cayley-Klein space is equipped with a signature projective metric $(0,0,+,+)$ represented Galilean three-dimensional space G_3. The absolute of Galilean geometry is the ordered triple (w, f, I) where w is the ideal plane (absolute), the line f in w (absolute line) and I is elliptic point of involution $(0,0,x_2,x_3) \to (0,0,x_3,-x_2)$ [17].

Suppose that $\overrightarrow{x} = (x_1,x_2,x_3)$ and $\overrightarrow{y} = (y_1,y_2,y_3)$ are two vectors in G_3. Galilean scalar product can be written in G_3 as

$$< \overrightarrow{x}, \overrightarrow{y} >_{G_3} = \begin{cases} x_1 y_1 & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0; \\ x_2 y_2 + x_3 y_3 & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

Norm of the vector $\overrightarrow{x} = (x_1,x_2,x_3)$ is defined by $\| \overrightarrow{x} \|_{G_3} = \sqrt{< \overrightarrow{x}, \overrightarrow{x} >_{G_3}}$.

If $\overrightarrow{x}$ and $\overrightarrow{y}$ are vectors in Galilean space G_3, then the vector product of $\overrightarrow{x}$ and $\overrightarrow{y}$ is

$$\overrightarrow{x}\times\overrightarrow{y}=\begin{cases}\begin{vmatrix}0 & e_2 & e_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} \text{ if } x_1\neq 0 \text{ or } y_1\neq 0,\\ \begin{vmatrix}e_1 & e_2 & e_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} \text{ if } x_1=0 \text{ and } y_1=0.\end{cases}$$

Suppose that $\alpha : I\to G_3$ is a unit speed curve in Galilean 3-space G_3 with Frenet-apparatus $\{\kappa,\tau,T,N,B\}$, where the curvature, torsion, unit tangent, unit principal normal and unit binormal of α are denoted in Galilean 3-space by κ, τ, T, N and B [12]. For the curve $\alpha(s)=(s,y(s),z(s))$, we have [6]

$$\kappa(s)=\parallel \alpha^{''}(s)\parallel_{G_3}=\sqrt{y''^2(s)+z''^2(s)}, \tag{1}$$

$$\tau(s)=\frac{det(\alpha^{'}(s),\alpha^{''}(s),\alpha^{'''}(s))}{\kappa^2(s)}, \tag{2}$$

$$T(s)=\alpha^{'}(s)=(1,y^{'}(s),z^{'}(s)),$$
$$N(s)=\frac{\alpha^{''}(s)}{\kappa(s)}=\frac{1}{\kappa(s)}(0,y^{''}(s),z^{''}(s)),$$
$$B(s)=\frac{1}{\kappa(s)}(0,-z^{''}(s),y^{''}(s)).$$

In addition, Frenet formulae can be written as

$$\frac{d}{ds}\begin{bmatrix}T(s)\\ N(s)\\ B(s)\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ 0 & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}T(s)\\ N(s)\\ B(s)\end{bmatrix}.$$

3. Frenet Curves in Galilean 3-Space

In this section we will define the Frenet curve and will prove that if α is a Frenet curve in G_3, then the distance function for α satisfies a 4-th order differential equation.

Definition 1. In Galilean 3-space G_3, a curve α is referred to as a Frenet curve if $\kappa>0$ and $\tau\neq 0$.

Definition 2. For a curve $\alpha : I \to G_3$, $I \subset \mathbb{R}$, parametrized by the invariant parameter s, the distance function $d(s)$ is defined by $d(s) =< \alpha(s), \alpha(s) >^{\frac{1}{2}}$. Moreover, we can define the function $h(s) =< \alpha(s), T(s) >$.

We can easily prove the next proposition.

Proposition 1. *Each unit speed Frenet curve $\alpha = \alpha(s)$ in G_3 satisfies the following differential equations*

(1) $h^{'''}(s) =< \alpha(s), T^{'''}(s) >$.
(2) $h^{'''}(s) = (2\kappa'\tau + \kappa\tau') < \alpha(s), B(s) > +(\kappa'' - \kappa\tau^2) < \alpha(s), N(s) >$.

Lemma 1. *Let $\alpha : I \to G_3$ be Frenet unit speed curve. We can easily prove the following statements.*

$$\begin{cases} < \alpha(s), T(s) >^{'} &= 1 + \kappa < \alpha(s), N(s) >, \\ < \alpha(s), N(s) >^{'} &= \tau < \alpha(s), B(s) >, \\ < \alpha(s), B(s) >^{'} &= -\tau < \alpha(s), N(s) > . \end{cases} \tag{3}$$

Theorem 1. *If $\alpha(s)$ is Frenet curve of unit speed in G_3, then $\alpha(s)$ satisfies the 4-th order differential equation of the form*

$$(\rho\ \sigma)h^{'''} + ((\sigma\rho)^{'} + \sigma\rho^{'})h^{''} + ((\sigma\rho^{'})^{'} + \frac{\rho}{\sigma})h^{'} = (\sigma\rho^{'})^{'} + \frac{\rho}{\sigma}. \tag{4}$$

where $\rho = \kappa^{-1}$, $\sigma = \tau^{-1}$, $h(s) = d(s)d^{'}(s)$ and $' = \frac{d}{ds}$.

Proof. By differentiating $h(s) =< \alpha(s), T(s) >$ and using (3), we get

$$\rho(s)\ (h^{'}(s) - 1) =< \alpha(s), N(s) > . \tag{5}$$

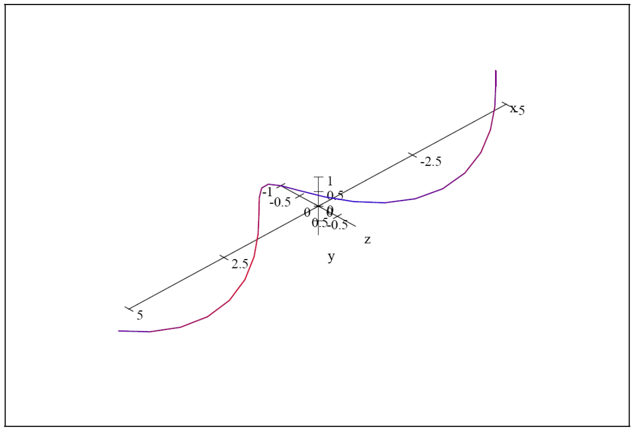
By differentiating (5), we obtain

$$\sigma(s)\ \rho(s)\ h^{''}(s) + \sigma(s)\ \rho^{'}(s)(h^{'}(s) - 1) =< \alpha(s), B(s) > . \tag{6}$$

After differentiating (6), and considering (5), we get

$(\sigma\ \rho)\ h^{'''} + ((\sigma\ \rho)^{'} + \sigma\rho^{'})h^{''} + ((\sigma\ \rho^{'})^{'} + \frac{\rho}{\sigma})h^{'} = (\sigma\ \rho^{'})^{'} + \frac{\rho}{\sigma}.$ □

Example 1. Suppose that $\alpha(s) = (s, -\cos(s), \sin(s))$.



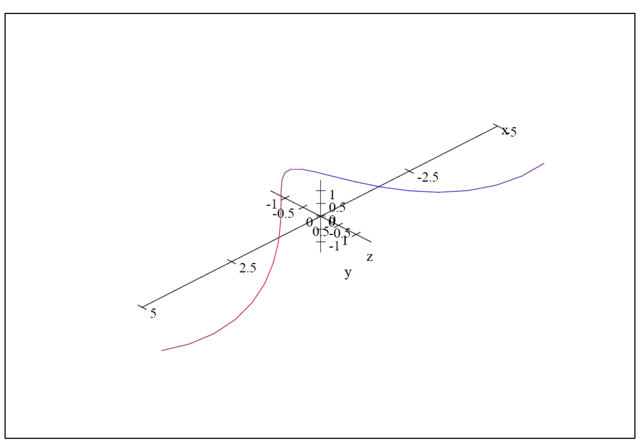
$$\alpha(s) = (s, -\cos(s), \sin(s)).$$

By simple computation, we get the Frenet vectors of $\alpha(s)$ as :

$$\begin{aligned} T &= (1, sins, coss), \\ N &= (0, coss, -sins), \\ B &= (0, sins, coss). \end{aligned}$$

respectively. The curvature of $\alpha(s)$ is $\kappa = 1$, and the torsion of $\alpha(s)$ is $\tau = -1$. Finally, $h'(s) = 1$ and hence equation (4) is satisfied.

Example 2. Consider the curve $\beta(s) = (s, \sin(\frac{s}{\sqrt{2}}) - \cos(\frac{s}{\sqrt{2}}), \sin(\frac{s}{\sqrt{2}}) + \cos(\frac{s}{\sqrt{2}}))$.



$$\beta(s) = (s, \sin(\frac{s}{\sqrt{2}}) - \cos(\frac{s}{\sqrt{2}}), \sin(\frac{s}{\sqrt{2}}) + \cos(\frac{s}{\sqrt{2}})).$$

By calculations, we have the distance function of β is $d(s) = s$. The tangent, normal, and binormal vectors of β are

$$\begin{aligned} T &= (1, \frac{1}{\sqrt{2}} cos(\frac{s}{\sqrt{2}}) + \frac{1}{\sqrt{2}} sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}} cos(\frac{s}{\sqrt{2}}) - \frac{1}{\sqrt{2}} sin(\frac{s}{\sqrt{2}})), \\ N &= \sqrt{2}(0, -\frac{1}{2} sin(\frac{s}{\sqrt{2}}) + \frac{1}{2} cos(\frac{s}{\sqrt{2}}), -\frac{1}{2} sin(\frac{s}{\sqrt{2}}) - \frac{1}{2} cos(\frac{s}{\sqrt{2}})), \\ B &= \sqrt{2}(0, \frac{1}{2} sin(\frac{s}{\sqrt{2}}) + \frac{1}{2} cos(\frac{s}{\sqrt{2}}), -\frac{1}{2} sin(\frac{s}{\sqrt{2}}) + \frac{1}{2} cos(\frac{s}{\sqrt{2}})). \end{aligned}$$

Moreover, the curvature of $\beta(s)$ is $\kappa = \frac{1}{\sqrt{2}}$, the torsion of $\beta(s)$ is $\tau = \frac{-1}{2\sqrt{2}}$ and $h'(s) = 1$ which satisfies equation (4).

4. Some Examples of Frenet Curves in G_3

In this section, we can easily use theorem 1 for useful characterizations of rectifying curves, spherical curves and helices in G_3.

4.1. Rectifying Curves in G_3.

In this subsection we will introduce rectifying curves in terms of the distance function and apply theorem 1 on it.

Definition 3. [5] Let α be a curve in Galilean 3-dimensional space G_3. If the position vector of α always lies in its rectifying plane, then α is called rectifying curve.

H. Oztekin [5] prove that if α is a Frenet unit speed curve in G_3, then $<\alpha(s), T(s)>= s + c$ if and only if α is a rectifying curve.

Now we prove the following corollary.

Corollary 1. *Every rectifying curve in G_3 satisfies the fourth order differential equation* (4).

Proof. Assume that α is a rectifying curve, then $d(s) = \sqrt{s^2 + ms + n}$ is the distance function, where $m \in \mathbb{R}$ and $n \in \mathbb{R} - \{0\}$. Therefore

$$d'(s) = \frac{2s + m}{2\sqrt{s^2 + ms + n}},$$

which gives

$$h(s) = s + c,$$

with $c = \frac{m}{2}$ which satisfies equation (4). □

4.2. Spherical Curves in G_3. In this subsection, we will consider the spherical curves in G_3. We also give some classifications of such curves according to theorem 1.

Definition 4. [5, 24] Galilean sphere of radius r and center m is defined by the relation

$$S^2(m,r) = \{\varphi - m \in G_3 :< \varphi - m, \varphi - m >= \pm r^2\}$$

Moreover, spherical curves are the special space curves that lies on the sphere [23]. So we will give the next proposition for Frenet unit speed curve to be a spherical curve in G_3.

Proposition 2. *Let $\alpha = \alpha(s)$ be a Frenet curve of unit speed in G_3. Subsequently, if $\alpha(s)$ is spherical curve, it fulfills the relation $(\sigma\,\rho^{'})^{'} + \frac{\rho}{\sigma} = 0$.*

Proof. Let $\alpha(s)$ be a spherical curve that lies on a sphere of radius a. We consider the sphere is centered at the origin without loss of generality, so the distance function $d(s) = a$ which gives $h(s) = d(s)d^{'}(s) = 0$. By substituting in the equation (4), we have $(\sigma\,\rho^{'})^{'} + \frac{\rho}{\sigma} = 0$. □

Example 3. Consider the spherical curve $\alpha(s) = (s, y(s), z(s))$ with curvature $\kappa(s) = \frac{1}{s}$,$s > 0$ and substituting in the equation $(\sigma\,\rho^{'})^{'} + \frac{\rho}{\sigma} = 0$, we can easily obtain $\tau(s) = \frac{1}{\sqrt{c_1 - s^2}}$ where c_1 is constant. Substituting for $\kappa(s) = \frac{1}{s}$ and $\tau(s) = \frac{1}{\sqrt{c_1 - s^2}}$ in the equations (1) and (2) we will obtain the following differential equations

$$y'' + z'' = \frac{1}{s^2}, \tag{7}$$

$$y''z''' - y'''z'' = \frac{1}{s^2\sqrt{c_1 - s^2}}. \tag{8}$$

By integrating equation (7) twice with respect to s, which yields

$$y(s) = a_2 + a_1 s - \ln s - z(s). \tag{9}$$

Next we differentiate equation (7) with respect to s, which yields

$$z''' = -\frac{2}{s^3} - y'''. \tag{10}$$

By making use of (7) and (10) one gets from (8) that

$$y''\left(-\frac{2}{s^3}-y'''\right)-y'''\left(\frac{1}{s^2}-y''\right)=\frac{1}{s^2\sqrt{c_1-s^2}}\Rightarrow$$

(11) $$\frac{y'''}{s^2}+\frac{2}{s^3}y''=\frac{-1}{s^2\sqrt{c_1-s^2}}.$$

Writing $y''(s)=u(s)\Rightarrow y'''=u'(s)$ in (11) and multiplying both sides by s^2, we obtain

(12) $$u'+\frac{2}{s}u=\frac{-1}{\sqrt{c_1-s^2}},$$

which is a linear first order differential equation, its integrating factor $\mu=\exp(\int\frac{2}{s}ds)=s^2$. Hence the solution of equation (12) is given by

$$s^2u(s)=\int\frac{-s^2}{\sqrt{c_1-s^2}}ds+\alpha_1\Rightarrow$$

(13) $$y''(s)=\frac{\sqrt{c_1-s^2}}{2s}-\frac{c_1}{2s^2}\tan^{-1}\left(\frac{s}{\sqrt{c_1-s^2}}\right)+\frac{\alpha_1}{s^2}.$$

By integrating equation (13), with respect to s, we get

$$y'(s)=\left(\frac{1}{2}+\frac{1}{s}\right)\frac{c_1}{2}\tan^{-1}\left(\frac{s}{\sqrt{c_1-s^2}}\right)+\frac{\sqrt{c_1}}{2}\ln\left(\frac{2c_1+2\sqrt{c_1-s^2}}{s}\right)+$$

(14) $$\frac{s}{4}\sqrt{c_1-s^2}-\frac{\alpha_1}{s}+\alpha_2.$$

Finally by integrating equation (14), we have

$$\begin{aligned} y(s) \quad = \quad & \frac{5c_1s^2+2c_1^2-s^4}{12\sqrt{c_1-s^2}}+\left(\frac{s}{2}+1+\ln s\right)\frac{c_1}{2}\tan^{-1}\left(\frac{s}{\sqrt{c_1-s^2}}\right)+ \\ & \frac{ic_1}{4}\mathrm{Li}_2\left(e^{-2\sinh^{-1}\left(\frac{is}{\sqrt{c_1}}\right)}\right)+\frac{\sqrt{c_1}}{2}s\ln\left(2s\left(\sqrt{c_1\left(c_1-s^2\right)}+c_1\right)\right)+ \end{aligned}$$

$$\frac{ic_1}{2}\ln s\left(\ln\left(\sqrt{1-\frac{s^2}{c_1}}+\frac{is}{\sqrt{c_1}}\right)\right)-\frac{ic_1}{4}\left(\sinh^{-1}\left(\frac{is}{\sqrt{c_1}}\right)\right)^2-$$

$$\frac{ic_1}{2}\left(\sinh^{-1}\left(\frac{is}{\sqrt{c_1}}\right)\right)\ln\left(1-e^{-2\sinh^{-1}\left(\frac{is}{\sqrt{c_1}}\right)}\right)-\alpha_1\ln s+\alpha_2 s+\alpha_3.$$

The function $\mathrm{Li}_a(z)$ is a polylogarithm function and it is defined by

$$\mathrm{Li}_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{a^n},\ |z| < 1.$$

Substituting y(s) in the equation (9), we will obtain $z(s)$.

4.3. Helices in G_3. In this subsection, the definitions of general helix and circular helix in G_3 will be introduced. The sufficient condition for the general helix to be Frenet curve will be considered.

Definition 5. [6, 24] Let α be a curve in Galilean 3-dimensional space G_3, and let $\{T,N,B\}$ be the Frenet frame along α. A curve such that $\frac{\kappa}{\tau}$ equals constant is called a general helix.

Definition 6. [6] Let α be a curve in Galilean 3-dimensional space G_3, and let $\{T,N,B\}$ be Frenet frame along α. If there are positive constants κ and τ along α, then α is referred to as a circular helix.

There exists some properties of general and circular helix which will be used in proving the next theorem [6].

(1) $T^{'''} - KT^{'} = 3\kappa^{'} N^{'}$.
(2) $T^{'''} - KT^{'} = 3\ \lambda\ \tau^{'}\ N^{'}$.
(3) $T^{'''} - \tilde{K}\ B^{'} = 3\ \kappa^{'} N^{'}$.
(4) $T^{'''} - \tilde{K}\ B^{'} = 3\ \lambda\ \tau^{'} N^{'}$.

where $K = \frac{\kappa^{''}}{\kappa} - \tau^2$, $\tilde{K} = \frac{-\kappa^{''}}{\tau} + \kappa\ \tau$, and $\lambda = \frac{\kappa}{\tau}$.

If the curve is a circular helix, then

(1) $T^{'''} = -\ \tau^2\ T^{'}$.
(2) $T^{'''} = \kappa\ \tau\ B^{'}$.

Theorem 2. *Let $\alpha : I \to G_3$ be a unit speed Frenet curve in G_3. If α is a circular helix, then $h(s)$ satisfies the following equation*

$$h(s) = -c_1\ \sigma\ cos(\tau\ s) + c_2\ \sigma\ sin(\tau\ s) + s + c.$$

where $\sigma = \frac{1}{\tau}$, c_1, c_2 and c_3 are constants.

Proof. Assume that α is a circular helix. Then we have $\rho^{'}=0$ and $\sigma^{'}=0$. Writing them in equation (4), we obtain

$$\begin{aligned} \sigma\,\rho\, h^{'''}(s)+\tfrac{\rho}{\sigma}\, h^{'}(s)-\ \tfrac{\rho}{\sigma} &= 0,\\ h^{'''}(s)+\tfrac{1}{\sigma^2}\, h^{'}(s)-\tfrac{1}{\sigma^2} &= 0, \end{aligned}$$

$$h^{'''}(s)+\tau^2\, h^{'}(s)-\ \tau^2=0. \tag{15}$$

Let $h^{'}(s)=y(s)$, then $h^{''}(s)=y^{'}(s)$ and $h^{'''}(s)=y^{''}(s)$. Writing them in the equation (15), we obtain the following non homogeneous differential equation

$$y^{''}(s)+\tau^2\, y(s)=\tau^2. \tag{16}$$

Solving the homogeneous differential equation $y^{''}(s)+\tau^2\, y(s)=0$, we obtain

$$y_c=c_1\ sin(\tau\ s)+c_2\ cos(\tau\ s), \tag{17}$$

where c_1 and c_2 are constants. The particular solution of the non homogeneous equation is $y_p=A$, which gives $y_p^{'}=0$ and $y_p^{''}=0$. Substituting in equation (16), we get $\tau^2\ A=\tau^2$, that implies $A=1$. Now, the general solution of the given non homogeneous differential equation is given by

$$y=c_1\ sin(\tau\ s)+c_2\ cos(\tau\ s)+1.$$

Moreover,

$$\begin{aligned} h^{'}(s) &= c_1\ sin(\tau\ s)+c_2\ cos(\tau\ s)+1,\\ h(s) &= \frac{-c_1}{\tau}\ cos(\tau\ s)+\frac{c_2}{\tau}\ sin(\tau\ s)+s+c_3, \end{aligned}$$

and hence,

$$h(s)=-c_1\ \sigma\ cos(\tau\ s)+c_2\ \sigma\ sin(\tau\ s)+s+c_3.$$

□

Corollary 2. *Let α be Frenet curve in G_3. If α is a general helix, then*

(1) $h^{'''}(s)=<\alpha(s),\ KT^{'}+3\ \kappa^{'}\ N^{'}(s)>$; *where* $K=\frac{\kappa^{''}}{\kappa}-\tau^2$.

(2) $h^{'''}(s)=<\alpha(s),\tilde{K}B'+3\lambda\,\tau' N^{'}>$; *where* $\tilde{K}=-\frac{\kappa^{''}}{\tau}+k\tau$, *and* $\lambda=\frac{k}{\tau}=const.$

Proof. By direct substitution from the properties of the general helix, we obtain the two relations. □

Moreover, it is clear to prove the following corollary.

Corollary 3. *Let α be a Frenet curve in G_3 and α be a circular helix. Then*

(1) $h^{'''}(s) =< \alpha(s),\ -\tau^2\, T^{'} >$, *and*

(2) $h^{'''}(s) =< \alpha(s),\ \kappa\, \tau\, B^{'} >$.

5. Conclusion

In this study, Frenet curve in Galilean 3-dimensional space G_3 is defined. It is proved that the distance function $d(s)$ of each Frenet curve in G_3 satisfies a 4-th order differential equation. Also, rectifying curves, spherical curves, and helices are given. Moreover, some useful characterizations of such curves via this 4-th order differential equation is obtained.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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