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# NUMERICAL SIMULATION OF CONTACT PROBLEM WITH POTENTIAL KERNEL

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**Abstract.** In this paper, the contact problem in integral form, which describes the contact potential force between two rigid surfaces under certain conditions with respect to pressure and moment, is considered. A numerical simulation for the solution of the contact problem is presented. The technique depends on the properties of some orthogonal polynomials. The optimal simulation of potential function and the estimated error are calculated using Maple programming. Also, the potential function in some special cases are plotted.

**Keywords:** cntact problem; Banach fixed point; linear system; orthogonal polynomials.

**2020 AMS Subject Classification:** 45B05, 45E10.

## 1. Introduction

The theory of contact problems have closed contacts with many different areas of the mathematical sciences. Many of the problems of continuous media, fluid dynamics, viscoelasticity, biology, and medicine represent the important shapes of contact problems; see [1], [2], [3], [4], [5],and [6]. The solution to these kinds of problems were the larg intested ariea of

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the authors's extensive research . In [41- [7]] , many different numerical methods are employed to obtain the solution of the Fredholm integral equation (**FIE**) with many kinds of kernels [7], [8], [7], [9], [10], [11], [12]

In [7] , Abdou discussed the solution of an integral equation (**IE**) of the first kind in one, two, and three dimensions. Abdou in [8] applied the regular and singular asymptotic methods in one, two, and three dimensions to obtain the solution of **IE**.

Abdou [7], discussed the spectral relationships that have many important applications in astrophysics for the **IE** of the first kind, when the kernel takes asingulor form. Abdoi and Nasr in [9] used the Chebyshev polynomial to obtain the solution of **IE** when the kernal takes a logarthmi form.

The relation between the contact problem and the integral equation in three dimensions were obtained by Abdou and Moustafa in [10].Abdou and Salama obtained in [11], the spectral relationships for the **IE** of the first kind. Bukhari, in [12], solves some problems of **IE** with a singular kernel in fluid dynamics by using the Toeplitz matrix and the product Nystrom method. Alharbi in [13] discussed the solution of an integral equation in two dimensions using spectral relationships. The numerical solutions of contact problems of integro-differential types with smooth and singular kernels are discussed in [8].

In this work, the existence of a unique solution of **FIE** with a potential kernel is proved by using Banach fixed points on a free surface, which is reduced to a linear system of **FIE** that enhances a unique solution under specific certiae condition.Then, the relation between the integral system and the Jacoby polynomials and Lamma functions equation is used to obtain an ifinite linear system of algabric. The optimal sumulation of contact problem errors is computed with a lower error rate.

## 2. Formulation of the Contact Problem

Consider the **FIE** of the second kind,

$$\phi(x,y) = f(x,y) + \lambda \int_{\Omega} k(x-\xi, y-\eta)\phi(\xi,\eta)d\xi d\eta \tag{2.1}$$

under the certain conditions:

$$
\begin{aligned}
&\int_{\Omega} \phi(x,y)\,dx\,dy = M \\
&\int_{\Omega} xy\,\phi(x,y)\,dx\,dy = N
\end{aligned}
\tag{2.2}
$$

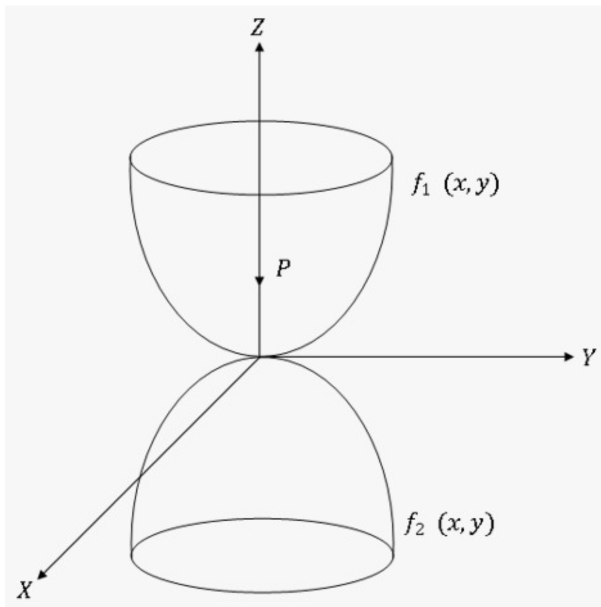


**Figure 1**

where, $\Omega$ is the domain of integration. The equation (2.1) is investigate from the contact problem of two rigid surfases having two elastic materials occupying the domain $\Omega$ . where $f(x.y) \in L_2(\Omega)$ and difined as :$f(x,y) = f_1(x,y) - f_2(x,y) - \gamma$ describing the two surfaces as shown in figure (1).

If the upper surface is impressed onto the lower one by a constant force M, causes a rigid displacement $\gamma =$ constant. In absene of body forces and when the forces of friction in the domain of contact between the two surfaes are so small such to be neglected. In equation (2.1), the unknown function $\phi(x,y)$ represent the unknown normal stresses between the two surfaces. $\lambda$ is the coefficients bed of the compressible materials that depend on their geometry and physical properties.

Here:

(i) $k(x-\xi,y-\eta)$ is the kernel of integral equation and satisfies the discontinuouty condition (Fredholm condition) in the space $L_2(\Omega)$
$\left(\iint_\Omega \iint_\Omega k^2(x-\xi,y-\eta)d\xi d\eta\right)^{1/2} \leqslant c$, $c$ is constant.

(ii) $\phi(x,y)$ is the unknown function and satisfies Lipschitz conditions
$\|\phi(x_1,y_1)-\phi(x_2,y_2)\| \leqslant \|(x_1,y_1)-(x_2,y_2)\|$

(iii) $f(x,y)$ is the given function which called the free term and it is continuous in the space $L_2(\Omega)$ and its norm can be defined as:
$\|f(x,y)\| = \left(\int_\Omega f^2(x,y)dxdy\right)^{1/2} \leqslant H$ , H is constant.

## 3. Existence a Unique Solution of the Problem

**Theorem 3.1.** *The integral equation* (2.1) *under the conditions (i)-(iii) has a unique solution in the space* $L_2(\Omega)$ *which the constant* $\lambda$ *satisfy the condition.* $|\lambda| < \frac{1}{c}$

**Proof:**

To prove the existence of a unique solution of the integral equation (2.1) we must use Banh fixed point theorem. For this, we write the integral equation (2.1) in the form of integral operator as:

$$w\phi = k\phi \tag{3.1}$$

where

$$k\phi = \lambda \iint_\Omega k(x-\xi,y-\eta)\phi(\xi,\eta)d\xi d\eta \tag{3.2}$$

then we prove the normality and continuity of the integral operator (3.1).

(a) For the normality

$$\|w\phi\| \leq |\lambda| \iint_\Omega k(x-\xi,y-\eta)\,\phi(\xi,\eta)\,d\xi d\eta$$

By using cauchy-shwarz inequuality we have,

$$\|w\phi\| \leq |\lambda| \left\| \left(\iint_\Omega |k(x-\xi,y-\eta)|^2 d\xi d\eta\right)^{1/2} \left(\iint_\Omega \phi^2(\xi,\eta)\,d\xi d\eta\right)^{1/2} \right\|$$

then

$$\|w\phi\| \leq |\lambda|\, c\, \|\phi\|$$

Therefore, $\|w\phi\| \leq \beta \,\|\phi\|$ , $\beta = |\lambda|\, c < 1$. Hence, $w$ is a norm operator that lead directly the normality of equation (2.1) after using condition (ii).

(b) For discussion the continuty of integral operator $w$ we assume the two potential function $\phi_1(x,y)$ , $\phi_2(x,y)$ satisfies equation (2.1) , then

$$\|w(\phi_1-\phi_2)\| \leq |\lambda| \left\| \iint_\Omega k(x-\xi, y-\eta)(\phi_1(\xi,\eta)-\phi_2(\xi,\eta))\, d\xi d\eta \right\|$$

using Cachy-shwarz inequality we have,

$$\|w(\phi_1-\phi_2)\| \leq |\lambda| \left\| \left(\iint_\Omega |k(x-\xi, y-\eta)|^2 d\xi d\eta\right)^{1/2} \left(\iint_\Omega |\phi_1-\phi_2|^2\, d\xi d\eta\right)^{1/2} \right\|$$

by applying the conditions (i)-(ii) to obtain

$$\|w(\phi_1-\phi_2)\| \leq \beta\, \|\phi_1-\phi_2\|, \beta \text{is a constant} \tag{3.3}$$

So, $w$ is a continuous operator then $w$ is a contraction operator when $\beta < 1$ and equation (2.1) has a unique solution.

## 4. LINEAR INFINITE SYSTEMS

The solution of (2.1) depends on the kernel and the surface $f_m(u)$ when the initial and the tangent points of the surface are in contact with the origin 0 , we can expand $f_m(u)$ in Maclaurin expansion near $u_0 = 0$ as:

$$f_m(u) \cong f_m(0) + f'(0)u + \frac{f_m''(0)}{2!}u^2 + \frac{f_m'''(0)}{3!}u^3 + \ldots + \frac{f_m^n(0)}{n!}u^n + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}u^k \tag{4.1}$$

which gives the degree of displacements of the surface for any degree. In general we write,

$$f_m(u) = A_{2m}u^{2m}, \qquad A_{2m} = \frac{f_n^{(2m)}(0)}{n!}, \qquad A_2 = \frac{f_m''(0)}{2!} \tag{4.2}$$

where $m$ is the order harmonic of the contact problem. Hence the function $g_m(u)$ taken the form:

$$g_m(u) = (A_{2m}u^{2m})\sqrt{u} \tag{4.3}$$

The last equation (4.3) represents a polynomial of degree $(2m+1)/2$ and the solution of equation (2.1) depend on the kernel and equation (4.3). so, rewrite equation (4.3) to take the following form:

$$P_m(u) - \int_0^a k_m(u,v) P_m(v) dv = u^{3m+1/2} \tag{4.4}$$

The integral equation (4.4) is reduced to

$$P_m(r) - \lambda \int_0^a k_m(r,\rho)\, P_m(\rho)\, \rho\, d\rho = f_m(r) \tag{4.5}$$

Using the following notation in equation (4.5)

$r = au, \qquad \rho = av, \qquad \psi_m(r) = \sqrt{r}\, P_m(r), \qquad g_m(r) = \sqrt{r} f_m(r)$

to obtain,

$$\psi_m(u) - \lambda \int_0^1 k_m(u,v)\, \psi_m(v)\, dv = g_m(u) \tag{4.6}$$

The formula (4.6) has a unique solution under the condition,

$$|\lambda| \leq \frac{1}{\left[\int_0^1 \int_0^1 |k_m(u,v)|^2\, du\, dv\right]^{1/2}} \tag{4.7}$$

To solve equation (4.6), we are write the kernel in the form, see [1,2,3]

$$K_m(u,v) = \frac{c}{\sqrt{2}} (uv)^{m+1/2} \sum_{j=0}^{\infty} \frac{\Gamma^2(j+m+1/4)\, P_j^m(u)\, P_j^m(v)}{\Gamma^2(j+1+m)(2j+m-(2/u))^{-1}} \tag{4.8}$$

where

$$P_j^m(u) = P_j^{(m-1/u)}(1-2u^2) \tag{4.9}$$

Here $P_j^{(m,n)}(x)$ is the Jacobi polynomial. Hence the solution of equation (4.4) with the kernel (4.8) is equivalent to the solution of the linear system:

$$X_i - c \sum_{j=0}^{\infty} A_i\, B_{ij}\, X_j = f_i \tag{4.10}$$

where:

$$f_i = (2i+m+3/u)^{1/4} \int_0^1 u^{3m+3/2}\, P_i^m(u)\, du$$

$$A_j = \frac{1}{\sqrt{2}} \frac{\Gamma^2(j+m+3/u)(2j+m+3/u)^{1/4}}{\Gamma^2(j+m+1)} \tag{4.11}$$

and,

$$B_{ij} = (2j+m+3/u)(2i+m+3/u) \int_0^1 u^{2m+1} P_i^m(u)\, P_j^m(u)\, du$$

The infinte linear system (4.10) is solvable under the condition

$$\sum_{j=0}^{\infty}(A_j\, B_i j) < \frac{1}{c} \tag{4.12}$$

## 5. Numerical Results

**Application(1) :**

In this application, we discuss the solution when $m = 0.1, m = 1.1, m = 2.1, m = 3.1$, see figure 2.

| $x$ | $\phi_{(0.1)}(x)$ | $\phi_{(1.1)}(x)$ | $\phi_{(2.1)}(x)$ | $\phi_{(3.1)}(x)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.02355491 | -0.096040408 | -0.460864946 | -1.881035834 |
| 0.2 | 0.070945328 | -0.061099933 | -0.565180665 | -2.698056093 |
| 0.3 | 0.133130791 | -0.061097864 | -0.430057873 | -2.744258047 |
| 0.4 | 0.206921072 | -0.040800863 | -0.143295052 | -2.240550375 |
| 0.5 | 0.290466121 | 0.002424569 | 0.220891753 | -1.368553023 |
| 0.6 | 0.382482701 | 0.072515562 | 0.591412264 | -0.294248409 |
| 0.7 | 0.481954183 | 0.175606053 | 0.887671122 | 0.805424896 |
| 0.8 | 0.587931387 | 0.322699522 | 0.994096396 | 1.69256361 |
| 0.9 | 0.69918895 | 0.539817922 | 0.672136328 | 1.90069542 |
| 1 | 0.807995006 | 1.066293998 | -2.121792378 | -2.444723197 |

**Application(2) :**

In this application, we discuss the solution when $m = 0.5, m = 1.5, m = 2.5, m = 3.5$, see figure 3

| $x$ | $\phi_{(0.5)}(x)$ | $\phi_{(1.5)}(x)$ | $\phi_{(2.5)}(x)$ | $\phi_{(3.5)}(x)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.00858467 | 0.073771508 | -1.008563 | -2.945441257 |
| 0.2 | 0.035079639 | 0.199775635 | -1.376159412 | -4.347537315 |
| 0.3 | 0.070550641 | 0.341387521 | -1.292507933 | -4.613037336 |
| 0.4 | 0.111832889 | 0.472315021 | -0.900550453 | -4.047855322 |
| 0.5 | 0.156996772 | 0.568878012 | -0.31932275 | -2.901067046 |
| 0.6 | 0.204525642 | 0.605913424 | 0.340018889 | -1.396661216 |
| 0.7 | 0.252905337 | 0.551329324 | 0.955512494 | 0.231858866 |
| 0.8 | 0.300128402 | 0.354041088 | 1.35799347 | 1.678367092 |
| 0.9 | 0.342224623 | -0.099306785 | 1.207197158 | 2.355537501 |
| 1 | 0.345852459 | -1.819072528 | -2.36770855 | -2.54797181 |

**Application(3):**

In this application, we discuss the solution when $m = -0.1, m = -0.3, m = -0.5, m = -0.7$, see figure 4.

| $x$ | $\phi_{(-0.1)}(x)$ | $\phi_{(-0.3)}(x)$ | $\phi_{(-0.5)}(x)$ | $\phi_{(-0.7)}(x)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.041938292 | 0.066304542 | 0.082092951 | 0.073435514 |
| 0.2 | 0.112826359 | 0.167229384 | 0.201934187 | 0.182395245 |
| 0.3 | 0.203356005 | 0.293090559 | 0.349594947 | 0.317082385 |
| 0.4 | 0.310149922 | 0.440225021 | 0.521216074 | 0.473734201 |
| 0.5 | 0.431251081 | 0.606469841 | 0.714477045 | 0.650093189 |
| 0.6 | 0.565363981 | 0.790406101 | 0.927834513 | 0.844641837 |
| 0.7 | 0.711599772 | 0.991128371 | 1.160291874 | 1.056357294 |
| 0.8 | 0.869409888 | 1.20826027 | 1.411425972 | 1.284692378 |
| 0.9 | 1.03875853 | 1.442456216 | 1.681943547 | 1.529952337 |
| 1 | 1.224828513 | 1.706619992 | 1.986024392 | 1.80210116 |

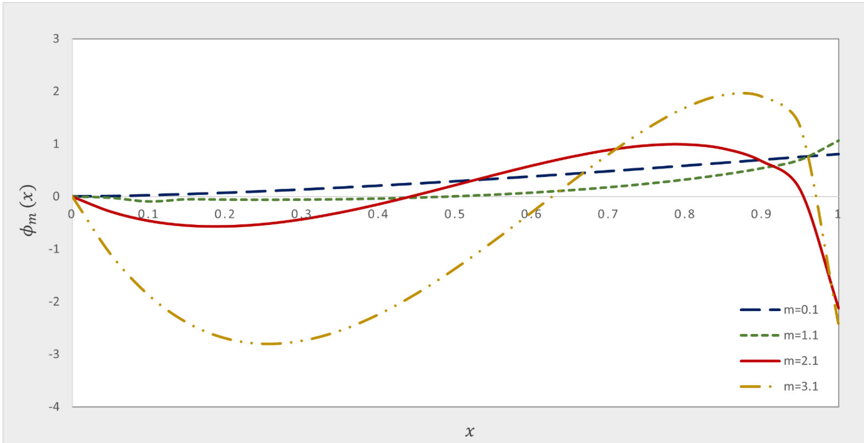


**Figure 2**

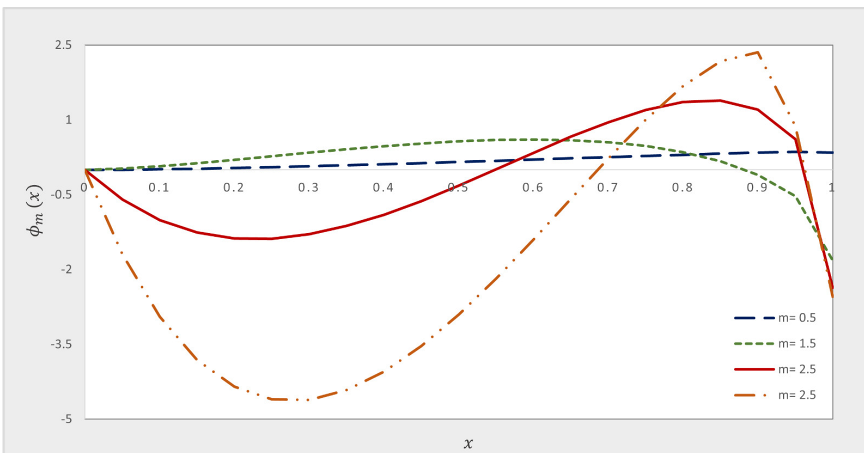


**Figure 3**

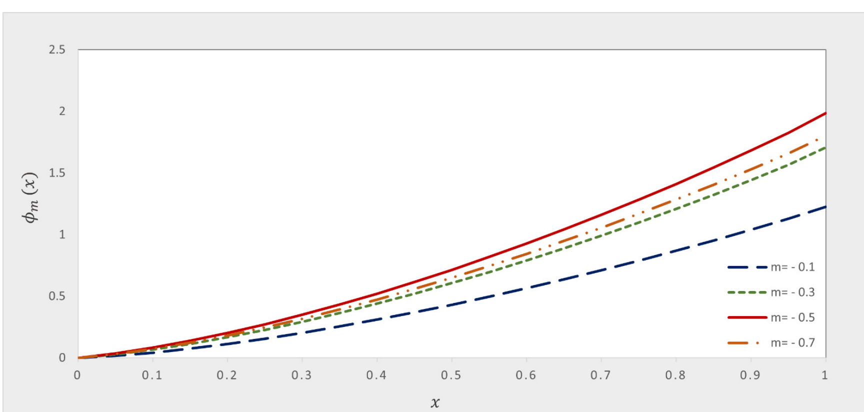


**Figure 4**

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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