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A MATHEMATICAL MODEL FOR CONSTRUCTING MAGIC SQUARES

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Abstract: In this paper, we develop a goal programming model that can be used to construct a magic square of any kind. A magic square is an m x m square array of numbers consisting of the first m^2 distinct positive integers arranged such that the sum of numbers in every row, every column, and every diagonal is the same number known as the magic total (or magic constant). The commonly used method for constructing magic squares are the Siamese method and Lozenge method for odd order square, the LUX method for singly even order square, and the cross diagonals method for doubly even order square. None of these methods uses a mathematical formula in the construction of the magic squares but uses the rule of thumb. The model developed in this paper is tested on the 3 x 3 magic square and is found to work perfectly well.

Key words: Magic square, magic total, goal programming, Siamese method, Lux method, Lozenge method

2000 AMS Subject Classification: 97M10

1. Introduction

A magic square is an m x m square array of numbers consisting of the first m^2 distinct positive integers arranged such that the sum of numbers in every row, every column, and every diagonal is the same number known as the magic total (or magic constant) Kraitchik (1942), Andrews (1960), Gardner (1961), Madachy (1979), Benson and Jacoby (1981), Ball and Coxeter (1987).

The magic total for a p^{th} order magic square is given by

$$MT(p) = \frac{p(p^2+1)}{2}$$

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and the magic total for a p^{th} order magic square starting with an integer A and with entries in an increasing arithmetic series with common difference D between terms is

$$MT(p;A,D) = \frac{p[2A + D(p^2 - 1)}{2}$$

(Hunter and Madachy 1975).

The magic square of any order (if it exist), except order 1 and 3, is not unique. The magic square of order 2 does not exist.There are 880 magic squares of order 4, 275305224 magic squares of order 5, Madachy (1979). Pinn and Wieczerkowski (1998) used Monte carlo simulation method and methods from statistical mechanics to estimate the number of 6^{th} order magic squares to lie between 1.7729×10^{19} and 1.7761×10^{19}. The 880 magic squares of order 4 were enumerated by Frenicle de Bessy in 1963. Methods for enumerating magic squares are discussed in Berlekamp et al (1982). The problem of determining the number of magic squares of any given order remains unsolved till date.

Magic squares are classified as even order magic square or odd order magic square. Even order magic squares are further classified as doubly even order (order 4 – multiple) magic squares or singly even order (non order 4 – multiple) magic squares.

Kraitchik (1942) came up with general techniques of constructing even and odd order p magic squares of different kind. He illustrated his technique with ‘3 x 3’, ‘4 x 4’, ‘5 x 5’, ‘6 x 6’ and ‘8 x 8’ magic squares. The problem with this technique is that it does not use any mathematical formula, but use the rule of thumb which does not account for constraints satisfaction in the process of assignment of numbers to cells. Construction of such squares is generally difficult, especially when squares are large.

In this paper, we introduce additional condition that a magic square must satisfy and then develop a goal programming model that would aid in the efficient construction of the magic squares. Our model is tested using a ‘3 x 3’ magic squares and is found to work perfectly well.

2. Preliminaries

2.1 Additional condition

We observed that when a magic square of even order p is divided into p/2 rectangles of dimension "p/2 x 2", the sum of the entries in each of the rectangles is also equal to the magic total.

Even though this condition was not taken into consideration while constructing the existing magic squares, the squares actually satisfy the condition.

The table below gives the summary of magic constants for magic squares of different orders.

Order(p)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
MT	**1**	**-**	**15**	**34**	**65**	**111**	**175**	**260**	**369**

2.2 Examples of magic square

(i) '3 x 3' magic square

8	**1**	**6**
3	**5**	**7**
4	**9**	**2**

(i) '4 x 4' magic square

16	**3**	**2**	**13**
5	**10**	**11**	**8**
9	**6**	**7**	**12**
4	**15**	**14**	**1**

(ii) '5 x 5' magic square

17	**24**	**1**	**8**	**15**
23	**5**	**7**	**14**	**16**
4	**6**	**13**	**20**	**22**
10	**12**	**19**	**21**	**3**
11	**18**	**25**	**2**	**9**

(iii) ‘6 x 6’ magic square

35	**1**	**6**	**26**	**19**	**24**
3	**32**	**7**	**21**	**23**	**25**
31	**9**	**2**	**22**	**27**	**20**
8	**28**	**33**	**17**	**10**	**15**
30	**5**	**34**	**12**	**14**	**16**
4	**36**	**29**	**13**	**18**	**11**

(ii) ‘8 x 8’ magic square

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22

55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	59	48	33	32	17

3. Model development

3.1 Model for odd order magic square

Constraints formulations

Let x_{ij}, i, j = 1, 2, . . . p, be the entry in row i and column j where p is odd

We define and formulate the following constraints,

(i) Every row entries add up to the magic total (MT)

$$\sum_{j=1}^{p} x_{ij} = MT, i = 1, 2, . . . , p \qquad (1)$$

(ii) Every column entries add up to the magic total

$$\sum_{i=1}^{p} x_{ij} = MT, j = 1, 2, . . . , p \qquad (2)$$

(iii) The entries on the principal diagonal add up to the magic total

$$\sum_{i=1}^{p} x_{ii} = MT \qquad (3)$$

(iv) The entries on the secondary diagonal add up to the magic total

$$\sum_{i=1}^{p} x_{i(p+1-i)} = MT \qquad (4)$$

(v) Every entry is distinct

i.e. $x_{ij} \neq x_{rs}$, for all $(i, j) \neq (r, s)$ (5)

These conditions imply that, either

$$X_{ij} - x_{rs} \geq 1, \text{ for all } (i, j) \neq (r, s) \tag{6}$$

Or

$$X_{ij} - x_{rs} \leq -1, \text{ for all } (i, j) \neq (r, s) \tag{7}$$

Since it is either (6) or (7) that must be satisfied for every pair of cells (i, j) and (r, s), we introduce binary variables x_{ijrs} and y_{ijrs} in (6) and (7), respectively, with the following interpretations.

$$x_{ijrs} = \begin{cases} 0, & \text{if (6) is satisfied} \\ 1, & \text{otherwise} \end{cases}$$

$$y_{ijrs} = \begin{cases} 0, & \text{if (7) is satisfied} \\ 1, & \text{otherwise} \end{cases}$$

For a very large number M, we therefore write constraints (6) and (7) as follows

$$x_{ij} - x_{rs} - Mx_{ijrs} \geq 1 \tag{8}$$

$$x_{ij} - x_{rs} - My_{ijrs} \leq -1 \tag{9}$$

Where

$$x_{ijrs} + y_{ijrs} = 1 \tag{10}$$

The introduction of the large number M is to ensure that one of the constraints is satisfied if the other is violated.

In the above formulations, we defined (8) and (9) to be our goal objectives. So, introducing the deviational variables a_{ijrs} and b_{ijrs} in constraints (8) and c_{ijrs} and d_{ijrs} in constraints (9), we obtain the following goal constraints.

$$x_{ij} - x_{rs} - Mx_{ijrs} + a_{ijrs} - b_{ijrs} = 1 \qquad (11)$$

$$x_{ij} - x_{rs} - My_{ijrs} + c_{ijrs} - d_{ijrs} = -1 \qquad (12)$$

Where

a_{ijrs} : = Positive deviation from 1

b_{ijrs} : = Negative deviation from 1

c_{ijrs} : = Positive deviation from -1

d_{ijrs} : = Negative deviation from -1

(vi) Every entry in a cell lies between 1 and p, inclusive, and is integer.

i.e. for every cell (i, j)

$$1 \le x_{ij} \le p \qquad (13)$$

and

$$x_{ij} \text{ is integer,} \qquad (14)$$

Formulation of the objective function

We seek to minimize the sum of all negative deviational variables in equation (11) and the sum of all positive deviational variables in equation (12). That is, our objective function is defined as

$$D = \sum_{(i,\, j) \neq (r,\, s)} (b_{ijrs} + c_{ijrs}) \qquad (15)$$

The model

The complete odd order p - magic square model is therefore

$$\operatorname{Min} D = \sum_{(i,\, j) \neq (r,\, s)} (b_{ijrs} + c_{ijrs})$$

Subject to:

$$\sum_{j=1}^{p} x_{ij} = MT, \; i = 1, 2, \ldots, p$$

$$\sum_{i=1}^{p} x_{ij} = MT, \; j = 1, 2, \ldots, p$$

$$\sum_{i=1}^{p} x_{ii} = MT$$

$$\sum_{i=1}^{p} x_{i(p+1-i)} = MT$$

$$x_{ij} - x_{rs} - Mx_{ijrs} + a_{ijrs} - b_{ijrs} = 1$$

$$x_{ij} - x_{rs} - My_{ijrs} + c_{ijrs} - d_{ijrs} = -1$$

$$x_{ijrs} + y_{ijrs} = 1$$

$1 \leq x_{ij} \leq p$

x_{ij} is integer,

x_{ijrs} and y_{ijrs} are binary variables.

3.2 Model for even order magic square

In the even order p magic square, in addition to the constraints in the odd order p magic square we introduce p sub partition constraints.

i.e. $[\text{Sum of sub-partition}\, i \text{ variables}] = MT$, I = 1, 2, . . . , P/2

The complete model for the even order p magic square is presented below

$$\operatorname{Min} D = \sum_{(i,\, j) \neq (r,\, s)} (b_{ijrs} + c_{ijrs})$$

Subject to:

$$\sum_{j=1}^{p} x_{ij} = MT,\ i = 1, 2, \ldots, p$$

$$\sum_{i=1}^{p} x_{ij} = MT,\ j = 1, 2, \ldots, p$$

$$\sum_{i=1}^{p} x_{ii} = MT$$

$$\sum_{i=1}^{p} x_{i(p+1-i)} = MT$$

$$[\text{Sum of sub-partition}\, i \text{ variables}] = MT, i = 1, 2, \ldots, p/2$$

$$x_{ij} - x_{rs} - Mx_{ijrs} + a^{+}_{ijrs} - a^{-}_{ijrs} = 1$$

$$x_{ij} - x_{rs} - My_{ijrs} + b^{+}_{ijrs} - b^{-}_{ijrs} = -1$$

$$x_{ijrs} + y_{ijrs} = 1$$

$1 \leq x_{ij} \leq p$

x_{ij} is integer,

x_{ijrs} and y_{ijrs} are binary variables.

3.3 The Model for ‘3 x 3’ Magic square matrix

In the ‘3 x 3’ magic square matrix the magic total is 15.

$$\operatorname{Min} D = \sum_{(i,\, j) \neq (r,\, s)} (b_{ijrs} + c_{ijrs})$$

Subject to:

$X_{11} + x_{12} + x_{13} = 15$

$X_{21} + x_{22} + x_{23} = 15$

$X_{31} + x_{32} + x_{33} = 15$

$X_{11} + x_{21} + x_{31} = 15$

$X_{12} + x_{22} + x_{32} = 15$

$X_{13} + x_{23} + x_{33} = 15$

$X_{11} + x_{22} + x_{33} = 15$

$X_{13} + x_{22} + x_{31} = 15$

$X_{11} - x_{12} - Mx_{1112} + a_{1112} - b_{1112} = 1$

$X_{11} - x_{13} - M x_{1113} + a_{1113} - b_{1113} = 1$

$X_{11} - x_{21} - Mx_{1121} + a_{1121} - b_{1121} = 1$

$X_{11} - x_{22} - Mx_{1122} + a_{1122} - b_{1122} = 1$

$X_{11} - x_{23} - Mx_{1123} + a_{1123} - b_{1123} = 1$

$X_{11} - x_{31} - Mx_{1131} + a_{1131} - b_{1131} = 1$

$X_{11} - x_{32} - Mx_{1132} + a_{1132} - b_{1132} = 1$

$X_{11} - x_{33} - Mx_{1133} + a_{1133} - b_{1133} = 1$

$X_{12} - x_{13} - Mx_{1213} + a_{1213} - b_{1213} = 1$

$X_{12} - x_{21} - M x_{1221} + a_{1221} - b_{1221} = 1$

$X_{12} - x_{22} - Mx_{1222} + a_{1222} - b_{1222} = 1$

$X_{12} - x_{23} - Mx_{1223} + a_{1223} - b_{1223} = 1$

$X_{12} - x_{31} - Mx_{1231} + a_{1231} - b_{1231} = 1$

$X_{12} - x_{32} - Mx_{1232} + a_{1232} - b_{1232} = 1$

$X_{12} - x_{33} - Mx_{1233} + a_{1233} - b_{1233} = 1$

$X_{13} - x_{21} - Mx_{1321} + a_{1321} - b_{1321} = 1$

$X_{13} - x_{21} - M\,x_{1321} + a_{1321} - b_{1321} = 1$

$X_{13} - x_{22} - Mx_{1322} + a_{1322} - b_{1322} = 1$

$X_{13} - x_{23} - Mx_{1323} + a_{1323} - b_{1323} = 1$

$X_{13} - x_{31} - Mx_{1331} + a_{1331} - b_{1331} = 1$

$X_{13} - x_{32} - Mx_{1332} + a_{1332} - b_{1332} = 1$

$X_{13} - x_{33} - Mx_{1333} + a_{1333} - b_{1333} = 1$

$X_{21} - x_{23} - Mx_{2123} + a_{2123} - b_{2123} = 1$

$X_{21} - x_{31} - Mx_{2131} + a_{2131} - b_{2131} = 1$

$X_{21} - x_{32} - Mx_{2132} + a_{2132} - b_{2132} = 1$

$X_{21} - x_{33} - Mx_{2133} + a_{2133} - b_{2133} = 1$

$X_{22} - x_{23} - Mx_{2223} + a_{2223} - b_{2223} = 1$

$X_{22} - x_{31} - Mx_{2231} + a_{2231} - b_{2231} = 1$

$X_{22} - x_{32} - Mx_{2232} + a_{2232} - b_{2232} = 1$

$X_{22} - x_{33} - Mx_{2233} + a_{2233} - b_{2233} = 1$

$X_{23} - x_{31} - Mx_{2331} + a_{2331} - b_{2331} = 1$

$X_{23} - x_{32} - Mx_{2332} + a_{2332} - b_{2332} = 1$

$X_{23} - x_{33} - Mx_{2333} + a_{2333} - b_{2333} = 1$

$X_{31} – x_{32} – Mx_{3132} + a_{3132} – b_{3132} = 1$

$X_{31} – x_{33} – Mx_{3133} + a_{3133} – b_{3133} = 1$

$X_{32} – x_{33} – Mx_{3233} + a_{3233} – b_{3233} = 1$

$X_{11} – x_{12} – My_{1112} + c_{1112} – d_{1112} = -1$

$X_{11} – x_{13} – M\ y_{1113} + c_{1113} – d_{1113} = -1$

$X_{11} – x_{21} – My_{1121} + c_{1121} – d_{1121} = -1$

$X_{11} – x_{22} – My_{1122} + c_{1122} – d_{1122} = -1$

$X_{11} – x_{23} – My_{1123} + c_{1123} – d_{1123} = -1$

$X_{11} – x_{31} – My_{1131} + c_{1131} – d_{1131} = -1$

$X_{11} – x_{32} – My_{1132} + c_{1132} – d_{1132} = -1$

$X_{11} – x_{33} – My_{1133} + c_{1133} – d_{1133} = -1$

$X_{12} – x_{13} – My_{1213} + c_{1213} – d_{1213} = -1$

$X_{12} – x_{21} – M\ y_{1221} + c_{1221} – d_{1221} = -1$

$X_{12} – x_{22} – My_{1222} + c_{1222} – d_{1222} = -1$

$X_{12} – x_{23} – My_{1223} + c_{1223} – d_{1223} = -1$

$X_{12} – x_{31} – My_{1231} + c_{1231} – d_{1231} = -1$

$X_{12} – x_{32} – My_{1232} + c_{1232} – d_{1232} = -1$

$X_{12} – x_{33} – My_{1233} + c_{1233} – d_{1233} = -1$

$X_{13} – x_{21} – My_{1321} + c_{1321} – d_{1321} = -1$

$X_{13} – x_{21} – M\ y_{1321} + c_{1321} – d_{1321} = -1$

$X_{13} – x_{22} – My_{1322} + c_{1322} – d_{1322} = -1$

$X_{13} - x_{23} - My_{1323} + c_{1323} - d_{1323} = -1$

$X_{13} - x_{31} - My_{1331} + c_{1331} - d_{1331} = -1$

$X_{13} - x_{32} - My_{1332} + c_{1332} - d_{1332} = -1$

$X_{13} - x_{33} - My_{1333} + c_{1333} - d_{1333} = -1$

$X_{21} - x_{23} - My_{2123} + c_{2123} - d_{2123} = -1$

$X_{21} - x_{31} - My_{2131} + c_{2131} - d_{2131} = -1$

$X_{21} - x_{32} - My_{2132} + c_{2132} - d_{2132} = -1$

$X_{21} - x_{33} - My_{2133} + c_{2133} - d_{2133} = -1$

$X_{22} - x_{23} - My_{2223} + c_{2223} - d_{2223} = -1$

$X_{22} - x_{31} - My_{2231} + c_{2231} - d_{2231} = -1$

$X_{22} - x_{32} - My_{2232} + c_{2232} - d_{2232} = -1$

$X_{22} - x_{33} - My_{2233} + c_{2233} - d_{2233} = -1$

$X_{23} - x_{31} - My_{2331} + c_{2331} - d_{2331} = -1$

$X_{23} - x_{32} - My_{2332} + c_{2332} - d_{2332} = -1$

$X_{23} - x_{33} - My_{2333} + c_{2333} - d_{2333} = -1$

$X_{31} - x_{32} - My_{3132} + c_{3132} - d_{3132} = -1$

$X_{31} - x_{33} - My_{3133} + c_{3133} - d_{3133} = -1$

$X_{32} - x_{33} - My_{3233} + c_{3233} - d_{3233} = -1$

$X_{1112} + y_{1112} = 1$

$X_{1113} + y_{1113} = 1$

$X_{1121} + y_{1121} = 1$

$X_{1122} + y_{1122} = 1$

$X_{1123} + y_{1123} = 1$

$X_{1131} + y_{1131} = 1$

$X_{1132} + y_{1132} = 1$

$X_{1133} + y_{1133} = 1$

$X_{1213} + y_{1213} = 1$

$X_{1221} + y_{1221} = 1$

$X_{1222} + y_{1222} = 1$

$X_{1223} + y_{1223} = 1$

$X_{1231} + y_{1231} = 1$

$X_{1232} + y_{1232} = 1$

$X_{1233} + y_{1233} = 1$

$X_{1321} + y_{1321} = 1$

$X_{1322} + y_{1322} = 1$

$X_{1333} + y_{1333} = 1$

$X_{2122} + y_{2122} = 1$

$X_{2123} + y_{2123} = 1$

$X_{2223} + y_{2223} = 1$

$X_{2331} + y_{2331} = 1$

$X_{2332} + y_{2332} = 1$

$X_{2333} + y_{2333} = 1$

$X_{3132} + y_{3132} = 1$

$X_{3133} + y_{3133} = 1$

$X_{3233} + y_{3233} = 1$

$1 \leq x_{ij} \leq 9$

x_{ij} is integer,

x_{ijrs} and y_{ijrs} are binary variables.

$a_{ijrs}, b_{ijrs}, c_{ijrs}, d_{ijrs} \geq 0$,

for all $(i, j) \neq (r, s)$; $i, j = 1, 2, 3$

3.4 Solving the model

We solved our model using a LINGO computer software version 13.0 x64 on Pentium IV 600 CPU with 320 MB RAM Windows XP computers. The summary of the solution output of our model is given below.

8	**1**	**6**
3	**5**	**7**
4	**9**	**2**

3.5 Conclusion

The model developed in this paper can be used to construct a magic square of any order. It can even be used to construct magic rectangles which have not yet been dealt with. The enumeration of all possible magic squares of any given order is possible and made simple with the help of our model by using the multiple optimal solution facility in the LINDO computer software. The major disadvantage of our model is that it contains too many constraints and variables even for a magic square of small order. Our limitation in this work is the inability to obtain license to use the full version of LINGO software which has capacity to solve mathematical model of several variables and constraints.

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