Available online at http://scik.org
J. Math. Comput. Sci. 2023, 13:14
https://doi.org/10.28919/jmcs/8189
ISSN: 1927-5307

OUT-GRAPHIC TOPOLOGY ON DIRECTED GRAPHS

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Abstract. In this paper, we define a topology $\mathscr{T}_G^{out}$, for a digraph $G=(V,E)$ without isolated vertices called out-graphic topology, on the vertices' set. When the graph is locally finite, $\mathscr{T}_G^{out}$ will be an Alexandroff topology and we give some characterisations of the minimal basis. Then, we give some open sets and some closed ones. Functions between digraphs are studied and also those between graphic topological spaces and the relations between them. Finally, for a strongly connected digraph, we prove that the topological space $(V,\mathscr{T}_G^{out})$ can be disconnected but in other cases can be connected.

Keywords: directed graph; isolated vertex; graphic topology; connected components; homeomorphism; isomorphic digraphs.

2020 AMS Subject Classification: 05C38, 05C40, 05C60, 54A05, 54D05, 54D80.

1. Introduction

Graph theory is one of discrete mathematics structures. It is simple and can represent a lot of mathematical combinations. In 1736, L. Euler introduce graph theory for the first time for solving the problem of the Königsberg seven bridges [5]. Later, graph theory becomes a fundamental mathematical tool for a large of domain as chemistry, marketing and computers network.

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Received August 23, 2023

Making a topology on the graphs enriches its structures and gives a more applications as in economy domain, the traffick flow study [2, 8, 10], medical application and blood circulation [6, 9, 11, 12].

A topology is called an Alexandroff topology if any intersection of open sets is an open set [3, 13]. The graphic topology is an Alexandroff topology, and so the study of open sets, closed sets, homeomorphism, compactedness can be done through minimal basis.

In 2013, Jafarian Amiri et al. [7] introduced an Alexandroff topology on the set of vertices of a simple graph . This topology is known as graphic topology. In [14], the authors investigated the graphic topology and solved partially an open problem mentioned in [7] (Problem 2 page 658). In the paper [15] in 2022, Zoman et al. define the $\mathscr{X}$-graphic topology $\mathscr{X}_G$ which conserves the connectivity of the simple graph. Graphic topology was also be defined on fuzzy graph by Alzubaidi et al. in [4].

In the paper [1], Kacimi et al. defined two topologies on the edges set of directed graph called compatible and incompatible edges topologies. In this paper,we introduce the out-graphic topology on directed graph on the vertices set. Our motivation is the investigation of some properties of directed graphs by their corresponding topology. In Section 2, we give some preliminaries about directed graph theory and topology. Also, we introduce a subbasis for the out-graphic topology and give some examples. In section 3, we prove some elementary results. Section 4 is devoted to more properties of the out-graphic topology. In Section 5, functions between digraphs are investigated. Finally, Connectivity or disconnectivity of the out-graphic topology of some graphs are studied.

2. PRELIMINARIES

In this paper, we will define and study the out-graphic topology on directed graphs. Recall that a directed graph G (or digraph) is a given nonempty set V and a set of ordered pairs E, subset of $V \times V$. This means, a graph is called a directed graph if each edge $e \in E$ has a direction. When $e = (x,y) \in E$, x is called the tail of the edge e and y the head of the edge. We also say e is an edge from x to y and we write $x \to y$.

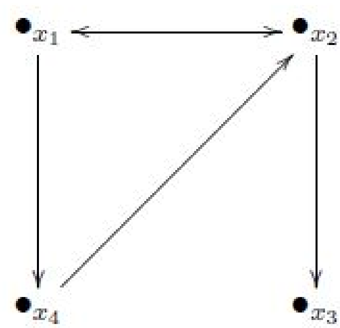
A directed graph $G = (V,E)$ is called simple if $(x,x) \notin E$ and there is no multiple edges from x to y, for any two vertices x,y of G.

Definition 2.1. *A digraph $G = (V,E)$ is called complete if it is simple and for any distinct $x,y \in V$, there exist a unique edge from x to y and a unique edge from y to x.*

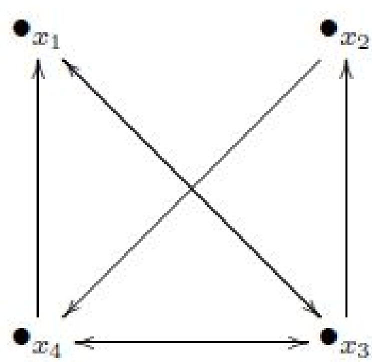
Definition 2.2. *A digraph $G = (V,E)$ is said an oriented graph if for all $x,y \in V$, at most one of (x,y) and (y,x) is in E. If G is an oriented graph such that for all $x,y \in V$, we have $(x,y) \in E$ iff $(y,x) \notin E$, then G is called a tournament.*

Definition 2.3. *Let $G = (V,E)$ be a simple digraph. The complement of G is the digraph $\overline{G} = (V,\overline{E})$ defined by $(x,y) \in \overline{E}$ if, and only if $(x,y) \notin E$.*

Example 1. *The complement of the following graph*



is given by



For $x \in V$, we define the out-neighborhood set of x as

$$\mathscr{O}_x = \{y \in V, x \to y\} = \{y \in V, (x,y) \in E\}. \tag{1}$$

and the int-neighborhood set of x by

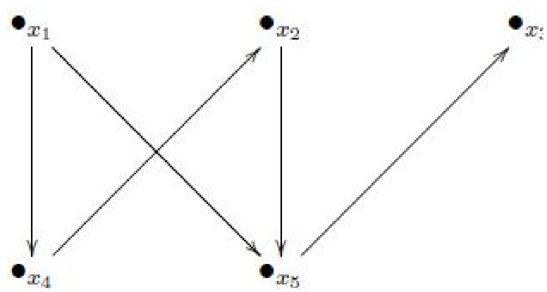
$$\mathscr{I}_x = \{y \in V, y \to x\} = \{y \in V, (y,x) \in E\}. \tag{2}$$

We remark that

$$y \in \mathscr{O}_x \text{ if and only if } x \in \mathscr{I}_y. \tag{3}$$

Definition 2.4. *A vertex x in G is called isolated if $\mathscr{I}_x = \emptyset$. That is, x does not dominates any $y \in V$.*

Example 2.



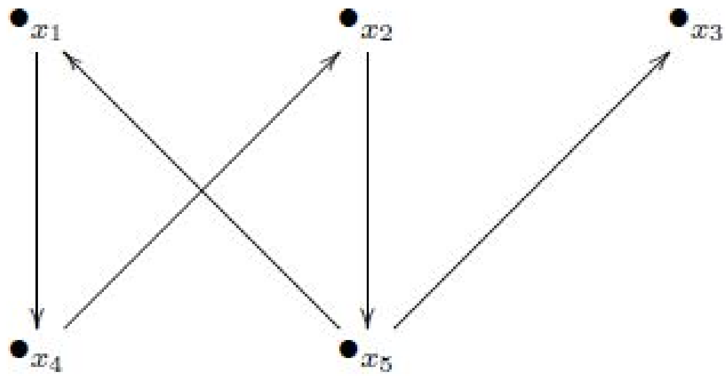
In this digraph, the vertex x_1 is an isolated vertex.

In order to define the out-graphic topology for V, we suppose that the digraph $G = (V,E)$ is simple and without isolated vertices. So, for all $x \in V$, $x \notin \mathscr{O}_x$.
Let us consider

$$\mathscr{S}_G^{out} = \{\mathscr{O}_x, x \in V\}. \tag{4}$$

Now, if $y \in V$ and since G without isolated vertices, we get $\mathscr{I}_y \neq \emptyset$. There exists $z \in \mathscr{I}_y$ and so, by (3) there exists $z \in V$ such that $y \in \mathscr{O}_z$. Therefore $V \subset \cup_{x \in V} \mathscr{O}_x$ and then $\cup_{x \in V} \mathscr{O}_x = V$.
Hence $\mathscr{S}_G^{out}$ is a subbasis for a topology of V called out-graphic topology and denoted by $\mathscr{T}_G^{out}$.

Example 3. *For the graph G given by*



$$\mathscr{O}_{x_1} = \{x_4\}, \mathscr{O}_{x_2} = \{x_5\}, \mathscr{O}_{x_3} = \emptyset, \mathscr{O}_{x_4} = \{x_2\} \textit{ and } \mathscr{O}_{x_5} = \{x_1, x_3\}.$$

Then, the basis of the topology $\mathscr{T}_G^{out}$ *is*

$$\mathscr{B} = \Big\{\{x_4\}, \{x_5\}, \{x_2\}, \{x_1, x_3\}\Big\}$$

such that

$$\mathscr{S}_G^{out} = \Big\{\emptyset, \{x_4\}, \{x_5\}, \{x_2\}, \{x_1, x_3\}\Big\}.$$

We get

$$\begin{aligned}\mathscr{T}_G^{out} = &\Big\{\emptyset, \{x_4\}, \{x_5\}, \{x_2\}, \{x_1, x_3\}, \{x_4, x_5\}, \{x_4, x_2\}, \{x_4, x_1, x_3\}, \{x_5, x_2\}, \\ &\{x_5, x_1, x_3\}, \{x_2, x_1, x_3\}, \{x_2, x_4, x_5\}, \{x_4, x_5, x_1, x_3\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_3, x_4\}, V\Big\}.\end{aligned}$$

For a vertex x of G, the *out-degree* of x is defined as

$$d^+(x) = card(\mathscr{O}_x),$$

the cardinal of the out-neighborhood $\mathscr{O}_x$ and the *int-degree* of x is

$$d^-(x) = card(\mathscr{I}_x),$$

the cardinal of the int-neighborhood $\mathscr{I}_x$.

The minimum out-degree and the minimum int-degree of a digraph $G = (V, E)$ are given by

$$\delta^+(G) = \min\{d^+(x),\ x \in V\},$$

and

$$\delta^-(G) = \min\{d^-(x),\ x \in V\}.$$

However, the maximum out-degree and maximum int-degree of G are respectively

$$\Delta^+(G) = \max\{d^+(x),\ x \in V\},$$

and

$$\Delta^-(G) = \max\{d^-(x),\ x \in V\}.$$

Definition 2.5. *A digraph G is called locally finite if for all vertex x of G, the int-degree $d^-(x)$ is finite. That is, $\mathscr{I}_x$ is a finite set for all $x \in V$.*

We remark that any finite digraph is locally finite.

Definition 2.6. *Let $(V, \mathscr{T})$ be a topological space and $X \subset V$.*

(i) The set $X^c = \{x \in V;\ x \notin X\}$ is called the complement of X in V.
(ii) X is called a closed set of V if X^c is an open set of V.
(iii) The closure of X in V is the smallest closed set of V containing X. It will be denoted by $\overline{X}$.

Definition 2.7. *Let $G = (V, E)$ a directed graph.*

(i) A directed path P from x_0 to x_n in G is a sequence of the form $x_0, e_0, x_1, e_1, \cdots, x_{n-1}, e_{n-1}, x_n$, where $x_i \in V$ and e_i an edge from x_i to x_{i+1}, $i = 0, \cdots, n-1$.
(ii) Two vertices x and y are said connected if there is a directed path from x to y and a directed path from y to x.
(iii) The directed graph G is called strongly connected if any two distinct vertices are connected.

In this paper, a digraph means a simple locally finite digraph without isolated vertex.

3. PRELIMINARY RESULTS

Theorem 3.1. *Let G be a digraph. Then $(V, \mathscr{T}_G^{out})$ is an Alexandroff space.*

Proof. In order to prove that $\mathscr{T}_G^{out}$ is an Alexandroff topology, it is sufficient to prove that any intersection of elements in the subbasis $\mathscr{S}_G^{out}$ is an open set. Let U a subset of V and consider $\cap_{x\in U}\mathscr{O}_x$.

(i) If $\cap_{x\in U}\mathscr{O}_x = \emptyset$, then $\cap_{x\in U}\mathscr{O}_x$ is an open set.

(ii) If $\cap_{x\in U}\mathscr{O}_x \neq \emptyset$, then let $z \in \cap_{x\in U}\mathscr{O}_x$. We have $z \in \mathscr{O}_x$, for all $x \in U$. So, $\forall x \in U, x \in \mathscr{I}_z$, this means $U \subset \mathscr{I}_z$. Since G is locally finite, we deduce that U is a finite set and so $\cap_{x\in U}\mathscr{O}_x$ is an open set.

Then, $(V, \mathscr{T}_G^{out})$ is an Alexandroff space.

□

Let G be a digraph, then the out-graphic topology $\mathscr{T}_G^{out}$ of V has a minimal basis

$$\mathscr{U}_G = \{V_x;\ x \in V\}, \tag{5}$$

where V_x is the intersection of all open sets containing x, i.e. the smallest open set containing the vertex x.

The first characterisation of the minimal basis is the following.

Theorem 3.2. *Let G be a digraph and let $x \in V$. Then, $V_x = \cap_{y\in\mathscr{I}_x}\mathscr{O}_y$ and V_x is a finite set.*

Proof. Since G is without isolated vertices, $\mathscr{I}_x \neq \emptyset$.

Let $y \in \mathscr{I}_x$, then from (3), we have $x \in \mathscr{O}_y$ and so $\cap_{y\in\mathscr{I}_x}\mathscr{O}_y$ is an open set containing x. Since V_x is the smallest open set containing x, we get

$$V_x \subset \bigcap_{y\in\mathscr{I}_x}\mathscr{O}_y.$$

Conversely, since $\mathscr{S}_G^{out}$ is a subbasis for the topology and V_x is the smallest open set containing x, there exists $U \subset V$ such that $V_x = \cap_{y\in U}\mathscr{O}_y$.

For all $y \in U$, $x \in \mathscr{O}_y$. Therefore, for all $y \in U$, $y \in \mathscr{I}_x$. Then, $U \subset \mathscr{I}_x$ and so,

$$\bigcap_{y\in\mathscr{I}_x}\mathscr{O}_y \subset \bigcap_{y\in U}\mathscr{O}_y = V_x.$$

The result follows.

□

Corollary 3.1. *Let G be a digraph and let $x, y \in V$ two distinct vertices.*

(i) If $\mathscr{I}_x = \{y\}$, then $V_x = \mathscr{O}_y$.
(ii) If $y \in \mathscr{I}_x$, then $V_x \subset \mathscr{O}_y$.
(iii) If $V_y \subset \mathscr{I}_x$, then $V_x \subset \mathscr{O}_y$.

Proof.

(i) If $\mathscr{I}_x = \{y\}$, then $V_x = \cap_{z \in \mathscr{I}_x} \mathscr{O}_z = \mathscr{O}_y$.
(ii) $V_x = \cap_{z \in \mathscr{I}_x} \mathscr{O}_z$, so $V_x \subset \mathscr{O}_z$ for all $z \in \mathscr{I}_x$. Therefore, if $y \in \mathscr{I}_x$, we get $V_x \subset \mathscr{O}_y$.
(iii) If $V_y \subset \mathscr{I}_x$, then $y \in V_y \subset \mathscr{I}_x$. From (ii), we obtain that $V_x \subset \mathscr{O}_y$.

□

Proposition 3.1. *Let G be a digraph and let $x, y \in V$. Then, $y \in V_x$ if, and only if, $\mathscr{I}_x \subset \mathscr{I}_y$. That is,*

$$V_x = \{y \in V;\ \mathscr{I}_x \subset \mathscr{I}_y\}.$$

Proof. Since $V_x = \cap_{z \in \mathscr{I}_x} \mathscr{O}_z$, we have

$$\begin{aligned} y \in V_x &\Leftrightarrow y \in \mathscr{O}_z, \forall z \in \mathscr{I}_x \\ &\Leftrightarrow \forall z \in \mathscr{I}_x, z \in \mathscr{I}_y \\ &\Leftrightarrow \mathscr{I}_x \subset \mathscr{I}_y. \end{aligned}$$

□

Proposition 3.2. *Let G be a digraph and $x \in V$. Then, $V_x \cap \mathscr{I}_x = \emptyset$. Also, if $V_y \subset \mathscr{I}_x$, we have $V_x \cap V_y = \emptyset$.*

Proof.

(i) Suppose that there exists $z \in V_x \cap \mathscr{I}_x$. Since $z \in V_x$, we get from Proposition 3.1 $\mathscr{I}_x \subset \mathscr{I}_z$. But $z \in \mathscr{I}_x$ and so $z \in \mathscr{I}_z$: this is impossible since the graph is simple. We deduce that $V_x \cap \mathscr{I}_x = \emptyset$.

(ii) If $V_y \subset \mathscr{I}_x$, then $V_y \cap V_x \subset \mathscr{I}_x \cap V_x$. From (i), we get $V_x \cap V_y = \emptyset$.

$\square$

Using the above result, we remark that $\overline{\{x\}} \subset \overline{V_x} \subset \overline{\mathscr{I}_x^c}$, $\overline{\mathscr{I}_x} \subset V_x^c$ and if G is a tournament, $\mathscr{I}_x^c = \mathscr{O}_x^c \cup \{x\}$.

Finally, we have the following result.

Proposition 3.3. *Let G be a directed graph and $x \in V$. We have $y \in \overline{\{x\}}$ if and only if $\mathscr{I}_y \subset \mathscr{I}_x$. This means, $\overline{\{x\}} = \{y \in V;\ \mathscr{I}_y \subset \mathscr{I}_x\}$.*

Proof. $y \in \overline{\{x\}}$ if and only if, for all open set O containing y, $O \cap \{x\} \neq \emptyset$. But, this is equivalent to $V_y \cap \{x\} \neq \emptyset$. So, $y \in \overline{\{x\}}$ if and only if $x \in V_y$ and the result follows by Proposition 3.1.

$\square$

4. Some Properties of Graphic Topology

A topological space V is called compact if for any family $\{A_i\}_{i\in I}$ of open sets satisfying

$$V \subset \bigcup_{i\in I} A_i$$

there exists a finite set $J \subset I$ such that

$$V \subset \bigcup_{i\in J} A_i.$$

For diagraphs, we have the following result.

Proposition 4.1. *Let $G = (V,E)$ be a digraph. Then, $(V, \mathscr{T}_G^{out})$ is a compact topological space if and only if V is finite.*

Proof. First, suppose that V is a compact topological space. Consider the minimal basis $\mathscr{U}_G$ given by (5). $\mathscr{U}_G$ is an open cover of V, so there exists a finite subcover $\mathscr{M}$ of $\mathscr{U}_G$. Since it is minimal as basis, $\mathscr{M} = \mathscr{U}_G$. Therefore, V is finite from (5).

Conversely, if V is finite, from any open cover we have a finite subcover. Hence, the result follows.

$\square$

Proposition 4.2. *Let $G=(V,E)$ be a digraph. Then, $U^-=\{x\in V,\ d^-(x)=\Delta^-(G)\}$ is an open set for the graphic topology on G.*

Proof. The idea is to prove that for all $x\in U^-$, we have $x\in V_x\subset U^-$.
Let $x\in U^-$ and $y\in V_x$, the smallest open set containing x. From Proposition 3.1, $\mathscr{I}_x\subset\mathscr{I}_y$. We get $d^-(x)\leq d^-(y)$. Since $d^-(x)=\Delta^-(G)$, we have $d^-(y)=\Delta^-(G)$ and so $y\in U^-$.

□

Proposition 4.3. *Let $G=(V,E)$ be a digraph. Then $F^-=\{x\in V,\ d^-(x)=\delta^-(G)\}$ is a closed set for the graphic topology on G.*

Proof. We will prove that $\overline{F^-}=F^-$. Let $x\in\overline{F^-}$, we have $V_x\cap F^-\neq\emptyset$.
Consider an element $z\in V_x\cap F^-$. We have $\mathscr{I}_x\subset\mathscr{I}_z$ and $d^-(z)=\delta^-(G)$.
So, $d^-(x)\leq d^-(z)=\delta^-(G)$. Then $d^-(x)=\delta^-(G)$ and so $x\in F^-$ and the result follows.

□

Proposition 4.4. *Let $G=(V,E)$ be a finite digraph. Then, the following results hold.*

(i) $\mathscr{T}_G^c=\{A;\ A^c\in\mathscr{T}_G^{out}\}$ is a topology for V.

(ii) If G is an oriented graph and $\mathscr{T}_G^{out}=\mathscr{T}_G^c$, then $\mathscr{T}_G^{out}$ is the discrete topology.

Proof. (i) • We have $\emptyset=V^c$ and $V=\emptyset^c$ and $V,\emptyset\in\mathscr{T}_G^{out}$.
• If A and B two elements of $\mathscr{T}_G^c$, then $(A\cap B)^c=A^c\cup B^c\in\mathscr{T}_G^{out}$ and so, $A\cap B\in\mathscr{T}_G^c$.
• For any family $\{A_i\}$ in $\mathscr{T}_G^c$, we have $(\cup_iA_i)^c=\cap_i(A_i^c)$. Since $\mathscr{T}_G^{out}$ is an Alexandroff topology, $\cup_iA_i\in\mathscr{T}_G^c$.
(ii) Let $x\in V$. When G is an oriented graph, we have $\Big(\mathscr{I}_x\cup\{x\}\Big)^c=\mathscr{O}_x$. That is, $\Big(\mathscr{I}_x\cup\{x\}\Big)^c\in\mathscr{T}_G^{out}$. Since $\mathscr{T}_G^{out}=\mathscr{T}_G^c$, we have $\mathscr{I}_x\cup\{x\}\in\mathscr{T}_G^{out}$. Therefore, $V_x\subset\mathscr{I}_x\cup\{x\}$. From Proposition 3.2, we have get $V_x\subset\{x\}$ and so $V_x=\{x\}$ and the result follows.

□

5. On Functions Between Digraphs

Definition 5.1. *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two digraphs. We say that G_1 and G_2 are isomorphic if there exists a bijection $\phi:V_1\to V_2$ satisfying*

$$(x,y)\in E_1 \text{ if, and only if } (\phi(x),\phi(y))\in E_2. \tag{6}$$

Definition 5.2. *Let (V_1,T_1) and (V_2,T_2) be two topological spaces. A function $\phi : V_1 \to V_2$ is called continuous if for all $U \in T_2$, we have $\phi^{-1}(U) \in T_1$.*
Further more, the two spaces V_1 and V_1 are said homeomorphic if there exists a continuous bijective $\phi : V_1 \to V_2$ such that ϕ^{-1} is also continuous.

Theorem 5.1. *Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two digraphs. If G_1 and G_2 are isomorphic, then the topological spaces $(V_1, \mathscr{T}^{out}_{G_1})$ and $(V_2, \mathscr{T}^{out}_{G_2})$ are homeomorphic.*

Proof. Let $\phi : V_1 \to V_2$ the bijection map satisfying (6) and let U an open set in V_2. Without loss of generality we can suppose that U is in the subbasis $\mathscr{S}_{G_2}$ and so $U = \mathscr{O}_y$, for some $y \in V_2$. Let $x = \phi^{-1}(y)$, we have

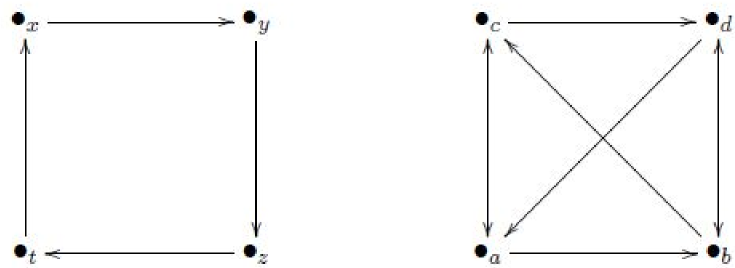
$$\begin{aligned}
\phi^{-1}(U) &= \{z \in V_1,\ \phi(z) \in \mathscr{O}_y\} \\
&= \{z \in V_1,\ (y,\phi(z)) \in E_2\} \\
&= \{z \in V_1,\ (\phi(x),\phi(z)) \in E_2\} \\
&= \{z \in V_1,\ (x,z) \in E_1\} \\
&= \{z \in V_1,\ x \to z\} \\
&= \mathscr{O}_x.
\end{aligned}$$

So, the function ϕ is continuous. Now, let $U = \mathscr{O}_x$, $x \in V_1$. Denote $y = \phi(x)$, we have

$$\begin{aligned}
\left(\phi^{-1}\right)^{-1}(U) &= \{z \in V_2,\ \phi^{-1}(z) \in \mathscr{O}_x\} \\
&= \{z \in V_2,\ (x,\phi^{-1}(z)) \in E_1\} \\
&= \{z \in V_2,\ (\phi^{-1}(y),\phi^{-1}(z)) \in E_1\} \\
&= \{z \in V_2,\ (y,z) \in E_2\} \\
&= \{z \in V_2,\ y \to z\} \\
&= \mathscr{O}_y.
\end{aligned}$$

□

In general, the converse is not true. Let us consider the following two graphs:



They have the same out-graphic topology (the discrete topology) but they are not isomorphic.

Proposition 5.1. *Assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two digraphs and $\phi : V_1 \to V_2$ a function. Then, ϕ is a continuous function from $(V_1, \mathscr{T}_{G_1}^{out})$ to $(V_2, \mathscr{T}_{G_2}^{out})$ if, and only if $\mathscr{I}_x \subset \mathscr{I}_y \Longrightarrow \mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(y)}$, for all $x, y \in V_1$.*

Proof. (i) Suppose that ϕ is continuous. If $\mathscr{I}_x \subset \mathscr{I}_y$, then $y \in V_x$ (Proposition 3.1).
Consider the open set $V_{\phi(x)}$, we have $x \in \phi^{-1}\left(V_{\phi(x)}\right)$ and so $V_x \subset \phi^{-1}\left(V_{\phi(x)}\right)$.
We get $y \in \phi^{-1}\left(V_{\phi(x)}\right)$, that is, $\phi(y) \in V_{\phi(x)}$. From the Proposition 3.1, we obtain $\mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(y)}$.
(ii) Suppose that for all $x, y \in V_1$, if $\mathscr{I}_x \subset \mathscr{I}_y$ then $\mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(y)}$. Consider $A \in \mathscr{T}_{G_2}^{out}$ and let $x \in \phi^{-1}(A)$, we claim that $V_x \in \phi^{-1}(A)$. Indeed, let $z \in V_x$. We have $\mathscr{I}_x \subset \mathscr{I}_z$ and so $\mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(z)}$. Then, $\phi(z) \in V_{\phi(x)} \subset A$. Therefore, $z \in \phi^{-1}(A)$ and the result follows.

□

We have also the following characterisation of homeomorphic graphic topology spaces.

Theorem 5.2. *Assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two digraphs and $\phi : V_1 \to V_2$ a function. Then the following properties are equivalent.*

(i) *ϕ is an homeomorphism between the topological spaces $(V_1, \mathscr{T}_{G_1}^{out})$ and $(V_2, \mathscr{T}_{G_2}^{out})$*
(ii) $\mathscr{I}_x \subset \mathscr{I}_y \Longleftrightarrow \mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(y)}$, *for all $x, y \in V_1$.*

Proof. First, suppose that ϕ is an homeomorphism and let $x, y \in V_1$. If $\mathscr{I}_x \subset \mathscr{I}_y$, from the Proposition 5.1 we get $\mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(y)}$.
Now, we suppose that $\mathscr{I}_{\phi(x)} \subset \mathscr{I}_{\phi(y)}$. By using the Proposition 5.1 for the continuous function ϕ^{-1}, we get $\mathscr{I}_x \subset \mathscr{I}_y$.

Conversely, suppose that the result (ii) is true.

It is clear that the function ϕ is continuous by using Proposition 5.1.

We want to prove that ϕ^{-1} is also continuous. Let $x', y' \in V_2$ such that $\mathscr{I}_{x'} \subset \mathscr{I}_{y'}$.

We have

$$\mathscr{I}_{\phi(\phi^{-1}(x'))} \subset \mathscr{I}_{\phi(\phi^{-1}(y'))},$$

and so

$$\mathscr{I}_{\phi^{-1}(x')} \subset \mathscr{I}_{\phi^{-1}(y')}$$

Again, from Proposition 5.1, the function ϕ^{-1} is continuous.

□

6. Graphic Topology and Connectedness

Definition 6.1. *Let $(X, \mathscr{T})$ a topological space. We say that X is connected if whenever $X = U \cup V$ and $U \cap V = \emptyset$, we have $U = \emptyset$ or $V = \emptyset$. That is, X can not be the union of two disjoint proper open sets.*

Definition 6.2. *A digraph $G = (V, E)$ is called strongly connected if for all $x, y \in V$ there exist a path from x to y and a path from y to x.*

For a digraph $G = (V, E)$, we define the connected components as follows.

Definition 6.3. *Let $G = (V, E)$ be a digraph. Let $V_1, V_2, \cdots$ be subsets of V such that*

(i) $V = \cup_i V_i$;

(ii) $V_i \cap V_j = \emptyset$, *for all* $i \neq j$;

(iii) For $i = 1, 2, \cdots$, *for all* $x, y \in V_i$, *there exist a path from x to y and a path from y to x.*

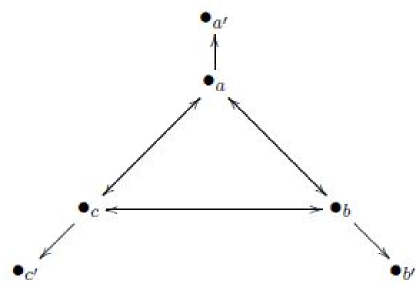
(iv) For all $x \in V_i$, $y \in V_j$ and $i \neq j$, there is no pair of paths: one from x to y and one from y to x.

Then, each subset V_i is called connected component of the digraph G.

As a particular cases, a strongly connected digraph has one connected component. Also, a finite digraph has a finite connected components.

When the graph $G = (V, E)$ is undirected, and it is not connected, the connected components are open sets for the graphic topology $\mathscr{T}_G$ and so, $(V, \mathscr{T}_G)$ is a disconnected topological space. But if $G = (V, E)$ is not strongly connected digraph, the topology $\mathscr{T}_G^{out}$ can be connected as in the following example.

Example 4. *Consider the following non strongly connected graph G:*



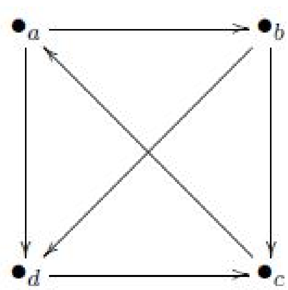
We have

x	O_x	I_x	V_x
a	$\{a', b, c\}$	$\{b, c\}$	$\{a\}$
a'	$\emptyset$	$\{a\}$	$\{a', b, c\}$
b	$\{a, b', c\}$	$\{a, c\}$	$\{b\}$
b'	$\emptyset$	$\{b\}$	$\{a, b', c\}$
c	$\{a, b, c'\}$	$\{a, b\}$	$\{c\}$
c'	$\emptyset$	$\{c\}$	$\{a, b, c'\}$

and so, $\mathscr{T}_G^{out}$ is connected.

In the following example, the graph is strongly connected but $\mathscr{T}_G^{out}$ is not connected.

Example 5.

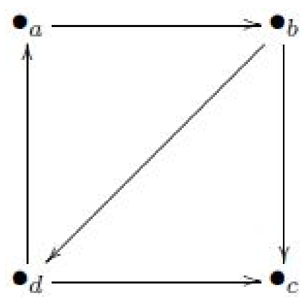


We get in this example

x	O_x	I_x	V_x
a	$\{b,c\}$	$\{d\}$	$\{a\}$
b	$\{c,d\}$	$\{a\}$	$\{b,c\}$
c	$\emptyset$	$\{a,b\}$	$\{c\}$
d	$\{a\}$	$\{b\}$	$\{c,d\}$

and so, $\mathscr{T}_G^{out}$ is not connected.

Example 6. *This is an example of not strongly connected graph with disconnected out-graphic topology.*

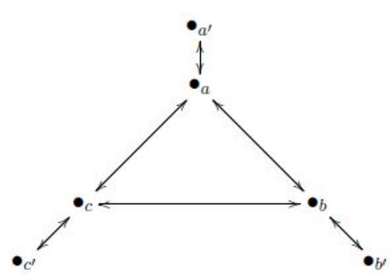


We have

x	O_x	I_x	V_x
a	$\{b\}$	$\{d\}$	$\{a,c\}$
b	$\{c,d\}$	$\{a\}$	$\{b\}$
c	$\emptyset$	$\{b,d\}$	$\{c\}$
d	$\{a,c\}$	$\{b\}$	$\{c,d\}$

and so, $\mathscr{T}_G^{out}$ is not connected.

Example 7. *In this example, we have a strongly connected graph with connected out-graphic topology.*



Since we have

x	O_x	I_x	V_x
a	$\{a',b,c\}$	$\{a',b,c\}$	$\{a\}$
a'	$\{a\}$	$\{a\}$	$\{a',b,c\}$
b	$\{a,b',c\}$	$\{a,b',c\}$	$\{b\}$
b'	$\{b\}$	$\{b\}$	$\{a,b',c\}$
c	$\{a,b,c'\}$	$\{a,b,c'\}$	$\{c\}$
c'	$\{c\}$	$\{c\}$	$\{a,b,c'\}$

then, the topological space $(V,\mathscr{T}_G^{out})$ *is connected.*

Theorem 6.1. *Let* $G=(V,E)$ *be a bipartite digraph, then* $\mathscr{T}_G^{out}$ *is disconnected.*

Proof. Suppose that $V=U_1\cup U_2$, with $U_1\cap U_2=\emptyset$ and

$$xy \in E \text{ and } (x,y)\notin A\times B \Rightarrow (x,y)\in B\times A.$$

Set $V_1=\bigcup_{x\in A} O_x \subset U_2$ and $V_2=\bigcup_{x\in B} O_x\subset U_1$.
We have $V_1\neq\emptyset$, $V_2\neq\emptyset$ and $V=V_1\bigcup V_2$ since $\mathscr{S}_G^{out}$ is a subbasis for the topology $\mathscr{T}_G^{out}$.
The result follows from the fact that $V_1\cap V_2\subset U_1\cap U_2=\emptyset$.

□

In fact, we have the following result.

Proposition 6.1. *For a strongly connected digraph* $G=(V,E)$*, the space* $(V,\mathscr{T}_G^{out})$ *is disconnected.*

Proposition 6.2. *If* $G=(x_1,\cdots,x_n)$ *is a cycle of order* $n\geq 3$ *(strongly connected), then* $\mathscr{T}_G^{out}$ *is disconnected.*

Proof. Since G is a cycle $(x_1,\cdots,x_n)$. We have $V_{x_i}=\{x_i\}$, for all $i=1,\ldots,n$ and so $\mathscr{T}_G^{out}$ is the discrete topology.

□

CONCLUSIONS

In this work, we introduced the out-graphic topology for the vertex's set of a directed graph $G=(V,E)$. We consider $\mathscr{S}_G^{out}$ the set of all out-neighborhoods and we suppose that there not exist a vertex without int-neighbor, that is, dominated at least one vertex. We prove that $\mathscr{S}_G^{out}$ is a subbasis for a topology denoted by $\mathscr{T}_G^{out}$. When the graph is locally finite, this topology is an Alexandroff topology. Working with minimal basis helps to discover this minimal basis, characterise it and study a lot of topological properties. In particular, we investigate a necessary and sufficient condition for two graphs to be homeomorphic and the connectivity of the topological space $(V,\mathscr{T}_G^{out})$. As future work, we estimate solve the problem: are there some necessary and sufficient conditions for the connectivity of $(V,\mathscr{T}_G^{out})$?

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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