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CONSTRUCTION OF AN ALEXANDROFF TOPOLOGY ON THE EDGE SET OF AN UNDIRECTED GRAPH

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Abstract. In this paper, we introduce a new topology $\mathscr{T}_G^E$ on the edge set of an undirected graph $G = (V,E)$ named E-graphic topology. We will study some properties of this topology in particular we prove that $\mathscr{T}_G^E$ is an Alexandroff topology. In addition, we investigate the compactness, connectedness and relation with functions between graphs. Finally, we give some characterization of homeomorphism between two E-graphic topologies.

Keywords: graph; edge; topology; connected components; homeomorphic topologies; isomorphic graphs.
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1. INTRODUCTION

In this paper, we construct a new topology on the edge set of an undirected graph. This topology is built by using subbases and under some assumptions, it is an Alexandroff topology. So, we have a minimal basis and therefore many properties are proved. Since any topology of a graph gives an arrangement of edges and vertices and influences its structure, it plays a crucial role in various applications and analyses. It can be used in connectivity analysis [9, 17], Fault Tolerance and Resilience [8], Network Design and Optimization [33] and Social Network Analysis and Security [35, 16, 32, 14] and human heart [24, 25, 26]. In computational geometry, a

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topology on the edge set indicates a spatial relationship between edges. It provides insights into the structure, connectivity, and behavior of that system or graph. A spatial analysis of this topology helps in understanding the geometric relationships between edges and can be crucial for Computational Geometry [11], and Geometry in Computer Science [15, 28]. In addition, topologies associated with the set of edges of graphs can be used to deal with some weighted graph problems. In Shokry [30], the author investigated a method to generate topology on graphs built on the choice of the distance between two vertices. His method can be applied in the graph of airline connections to give the least number of flights required to travel between two cities by determining the distance between two vertices (cities). These and others of applications of graphs with topologies give rise to many researches. For topologies on vertices sets we can refer to [1, 3, 6, 10, 13, 18, 19, 20, 29, 30, 37, 38]. For topologies on the edges set, we have fewer studies. For directed graphs, in 2018, Abdu and Kilicman [1] constructed two topologies on the set E of a directed graph $G(V,E)$. These two topologies are improvements to the graphic topology given in [19]. If the graphic topology respects the direction of two adjacent edges, they call it compatible edges topology and if not, it is called incompatible edges topology. For undirected graph, Othman and Alzubaidi [27] construct a topology on the edges set of undirected graph. This topology is built by path connected edges and it is the corresponding topology of Z-graphic topology introduced and given in [37]. In this paper, we define a new topology $\mathscr{T}_G^E$ on the edge set of undirected graph called edge graphic topology and which is larger than τ_E introduced in [27].

The outline of this paper is as follows. In section 2, we give some preliminaries about topological spaces and the main definitions and results that we will need later. Then, we write some basic definitions and properties in graph theory. In addition, we define the new topology $\mathscr{T}_G^E$ from a subbases. In section 3 some properties are presented after proving that the new topology is an Alexandroff topology. In section 4, we study the relation between isomorphic graphs, homeomorphic topological spaces and symmetric spaces. Finally,Compactness and connectedness are studied in section 5.

2. Basic Concepts

In this section, we give some basic definitions and properties for topological spaces and graph theory, for more details we cab refer to [7, 12, 23, 34]. Then, we introduce the new topolog on the edge set of an undirected graph.

Definition 2.1. *A topological space is a nonempty set X with a family $\mathscr{T}$ of subsets of it satisfying*

(i) $\emptyset$, $X \in \mathscr{T}$;

(ii) $\forall\, U_1,\, U_2 \in \mathscr{T}$, *we have* $U_1 \cap U_2 \in \mathscr{T}$;

(iii) $\forall\, \{U_i\}_{i\in I}$ *a family of elements in* $\mathscr{T}$, *the union* $\cup_{i\in I} U_i \in \mathscr{T}$.

$\mathscr{T}$ *is called a topology on X and an element of $\mathscr{T}$ is called an open set of X.*

A topology on a set X can be defined from a basis or a subbases. First, let us consider the following definitions.

Definition 2.2. *Let X a nonempty set.*
If (X,T) is a topological space and $\mathscr{B} \subset \mathscr{P}(X)$. Then, we say that $\mathscr{B}$ is a bases of T if for all $A \in T$, A can be written as union of elements of $\mathscr{B}$ and the unions of sets in $\mathscr{B}$ are also in T.

Conversely, we have the following result.

Proposition 2.1. *Let X be a nonempty set and let $\mathscr{B} \subset \mathscr{P}(X)$ a family of subsets of X. If for all $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$ and for all $B_1, B_2 \in \mathscr{B}$ and for all $x \in B_1 \cap B_2$ there exists $B \in \mathscr{B}$ such that $x \in B \subset B_1 \cap B_2$, then the set T of all possible unions of $\mathscr{B}$ is a topology on X.*
The topology T has $\mathscr{B}$ as basis.

Definition 2.3. *Let X a nonempty set.*
Suppose that (X,T) is a topological space with bases $\mathscr{B}$ and $\beta \subset \mathscr{P}(X)$. We say that β is a subbases of T (or for $\mathscr{B}$) if the sets in $\mathscr{B}$ are precisely the intersections of sets in β.

Proposition 2.2. *Let X be a nonempty set and let $\beta \subset \mathscr{P}(X)$ a family of subsets of X. If the union of all elements of β is X, then the set of all possible intersections of sets in β is a basis for a topology T on X.*

Definition 2.4. *A graph G is a pair of sets $G=(V,E)$ such that $E\subset V\times V$. An element of V is called vertex and of E is said edge.*
When we identify (x,y) with (y,x) in $V\times V$, for all x and y in V, we say that the graph G is undirected graph.
If e is an edge joining two vertices x and y, we write $e=(x,y)$ or $e=xy$.

In this paper, a graph means an undirected graph.

Definition 2.5. *Let $G=(V,E)$ be a graph. Two edges e_1 and e_2 of G are called adjacent if they have a common end. We write $e_1\sim e_1$ and $e_1\cap e_1\neq\emptyset$) (see* [27]*).*

Definition 2.6. *Let $G=(V,E)$ be a graph. An edge e of G is called isolated if it does not have an adjacent edge.*

Let $G=(V,E)$ be a graph and let $e\in E$ be an edge of G. We define

$$\mathscr{A}_e=\{e'\in E, e'\sim e\} \tag{1}$$

the set of the edges adjacent to e in G.
It is clear that

$$e'\in\mathscr{A}_e \text{ if and only if } e\in\mathscr{A}_{e'}.$$

Theorem 2.1. *Let $G=(V,E)$ be a graph without isolated edges. The set*

$$\mathscr{E}_G=\{\mathscr{A}_e, e\in E\}$$

is a subbases for a topology of E.

Proof. From the Proposition 2.2, we have to prove that

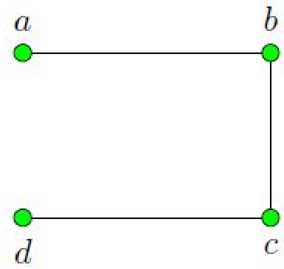
$$E\subset\bigcup_{e\in E}\mathscr{A}_e.$$

Let $f\in E$. Since f is not isolated, there exists $e'\in E$ such that $f\sim e'$ and so $f\in\mathscr{A}_{e'}$. Therefore, $f\in\cup_{e\in E}\mathscr{A}_e$ and the result follows.

□

The topology induced by the subbases $\mathscr{E}_G$ is called edge-graphic topology and denoted $\mathscr{T}_G^E$.

Example 2.1. *For the following undirected graph*



we have $\mathscr{E}_G = \Big\{\{bc\}, \{ab, cd\}\Big\}$ *the sabbases of the edge-graphic topology* $\mathscr{T}_G^E$ *and then* $\mathscr{T}_G^E = \Big\{\emptyset, \{bc\}, \{ab, cd\}, \{ab, bc, cd\}\Big\}$.

Definition 2.7. *Let* $G = (V, E)$ *be simple graph, that is without multiple edges nor loops. Let* $e \in E$ *be an edge of* G*. The number of the edges adjacent to* e *is called the degree of* e *in* G*.*

$$deg(e) = |\mathscr{A}_e|$$

The graph $G = (V, E)$ is called edge-locally finite graph if for all $e \in E$, the edge e has a finite number of adjacent edges. In what follows, we consider undirected,edge-locally finite and simple graphs without isolated edges.

3. Alexandroff Space and Minimal Basis

When we deal with an Alexandroff topology, many properties can be proved and studied by using minimal bases [4, 5, 21]. So, our first result is the following.

Theorem 3.1. *Suppose that* $G = (V, E)$ *is a graph. Then,* $(E, \mathscr{T}_G^E)$ *is an Alexandroff topological space.*

Proof. From Theorem 2.1, the topology $\mathscr{T}_G^E$ has $\mathscr{E}_G$ as subbases. So, in order to prove that any intersection of open sets is an a open set it is sufficient to prove this for open sets in the subbasis. Let $A \subset E$ and consider the family $\{\mathscr{A}_e;\ e \in A\}$.
If $f \in \cap_{e \in A} \mathscr{A}_e$, we have $f \in \mathscr{A}_e$, for all $e \in A$. Then, for all $e \in A$, $e \in \mathscr{A}_f$. Therefore, $A \subseteq \mathscr{A}_f$ and so A is also finite. Hence, $\cap_{e \in A} \mathscr{A}_e$ is an open set.

□

For all $e \in E$, the intersection of all open sets containing e, U_e, is an open set of the topological space E. Let

$$\mathscr{U} = \{U_e;\ e \in E\}$$

Then, $\mathscr{U}$ is a basis of $\mathscr{T}_G^E$ and it is minimal, that is, if $\mathscr{B}$ is a basis of $\mathscr{T}_G^E$ then $\mathscr{U} \subseteq \mathscr{B}$. For more details and properties we can refer to [4, 5, 21, 31, 38].
In the topological space $(E, \mathscr{T}_G^E)$, we have a characterisation of U_e using the subbases and so for a graph, we use the neighbors.

Theorem 3.2. *Let $G = (V,E)$ be a graph and $e \in E$. Then, the minimal open set containing e is finite and satisfies*

$$U_e = \cap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'}.$$

Proof. e is not an isolated edge, so $\mathscr{A}_e \neq \emptyset$. For all $e' \in \mathscr{A}_e$, we have $e \in \mathscr{A}_{e'}$ and hence $e \in \cap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'}$.
Since we deal in Alexandroff space, $\cap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'}$ is an open set. We get

$$U_e \subseteq \bigcap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'}.$$

Conversely, U_e is the minimal open set containing e and so U_e is in the basis following from the original subbases $\mathscr{E}_G$. Therefore there exists a subset A of E such that

$$U_e = \bigcap_{e' \in A} \mathscr{A}_{e'}.$$

Now, if $e' \in A$ then $e \in \mathscr{A}_{e'}$ which is equivalent to $e' \in \mathscr{A}_e$. It follows that $A \subseteq \mathscr{A}_e$ and we get

$$U_e = \bigcap_{e' \in A} \mathscr{A}_{e'} \subseteq \bigcap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'}$$

and the theorem is proved.

□

Next, we give some methods to find the minimal open set U_e of e in $\mathscr{T}_G^E$ due to Theorem 3.2.

Corollary 3.1. *Let $G = (V,E)$ be a graph and $e \in E$.*

(a) If $deg(e) = 1$ and $f \sim e$, then $U_e = \mathscr{A}_f$ the set of neighbors of f in the edges set E.
(b) If $f \in \mathscr{A}_e$, then $U_e \subset \mathscr{A}_f$.
(c) If $U_f \subset \mathscr{A}_e$, then $U_e \subset \mathscr{A}_f$.

Proof.

(a) From Theorem 3.2, we have $U_e = \bigcap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'} = \mathscr{A}_f$.

(b) We have U_e is a subset $\mathscr{A}_{e'}$, for all $e' \in \mathscr{A}_e$. Then, $\mathscr{A}_e \subset \mathscr{A}_f$.

(c) Since $U_f \subset \mathscr{A}_e$, we get $f \in U_f \subset \mathscr{A}_e$ and so $U_e \subset \mathscr{A}_f$, using (b).

□

We have the next characterisation of the minimal open sets for the topology $\mathscr{T}_G^E$.

Proposition 3.1. *Suppose that G is a graph and e an edge of G. Then*

$$U_e = \{f \in E;\ \mathscr{A}_e \subset \mathscr{A}_f\}.$$

Proof. We have $U_e = \bigcap_{e' \in \mathscr{A}_e} \mathscr{A}_{e'}$. Then, $f \in U_e$ if and only if $f \in \mathscr{A}_{e'}, \forall e' \in \mathscr{A}_e$. This means, $\forall e' \in \mathscr{A}_e$, we have $e' \in \mathscr{A}_f$. This is equivalent to $\mathscr{A}_e \subset \mathscr{A}_f$.

□

Corollary 3.2. *Suppose that G is a graph and $e \in E$ an edge. If $f \in U_e$, then $deg(e) \leq deg(f)$.*

Proof. Follows from the fact that $f \in U_e \Leftrightarrow \mathscr{A}_e \subset \mathscr{A}_f$ and the Definition 2.7.

□

Proposition 3.2. *Suppose that $G = (V,E)$ a graph and let $\Delta = \max\{deg(e),\ e \in E\}$. The set*

$$U = \{e \in E,\ deg(e) = \Delta\}$$

is an open set for the space $(E, \mathscr{T}_G^E)$.

Proof. Suppose that $e \in U$ and consider $e' \in U_e$. Using Corollary 3.2, we get $deg(e) \leq deg(e')$ and so $deg(e') = deg(e) = \Delta$.
Therefore $e' \in U$ and then $U_e \subset U$. The result follows.

□

4. Topology and Functions

Definition 4.1. *Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. G and G' are said isomorphic ($G\approx_i G'$) if there exists $f=(h,g)$ a pair of bijective maps, $h:V\to V'$ and $g:E\to E'$, satisfying*

$$g(xy)=h(x)h(y),\ \textit{for all}\ x,y\in V.$$

We call f an isomorphism between the two graphs G and G'.

Lemma 4.1. *Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. If $G\approx_i G'$ and $f=(h,g)$ is an isomorphism between them, then $g(e)\sim g(e')\Leftrightarrow e\sim e'$, for all $e,e' inE$.*

Proof. Let $e,e'\in E$ two edges in the graph G. Then, $e=xy$ and $e'=x'y'$ for some vertices $x,y,x',y'\in V$.
First, suppose that $g(e)\sim g(e')$ and so $f(x)f(y)\sim f(x')f(y')$. Therefore, the two edges have a common end. Without loss of generality, we can suppose $f(x)=f(y')$ and so $x=y'$. We get $e\sim e'$.
Conversely, if $e\sim e'$, then e and e' have a common vertex. Let us take $x=x'$ as example and so $g(e')=g(xy')=f(x)f(y')$. Hence, $g(e)\sim g(e')$.

Definition 4.2. *Let $(X,\mathscr{T})$ and $(X',\mathscr{T}')$ be two topological spaces. A function $g:X\to X'$ is called continuous if for all $A\in\mathscr{T}'$, $g^{-1}(A)\in\mathscr{T}$.*
We say that the two topologies $\mathscr{T}$ and $\mathscr{T}'$ are homeomorphic ($\mathscr{T}\approx_h\mathscr{T}'$) if there exists a continuous bijective function $g:X\to X'$ and g^{-1} is also continuous.

Theorem 4.1. *Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. If $G\approx_i G'$, Then $\mathscr{T}_G^E\approx_h\mathscr{T}_{G'}^E$.*

Proof. We set $f=(h,g)$ an isomorphism from G to G'. We are going to prove that the function g is an homeomorphism. Since we deal with a topology having subbases, let $A\in\mathscr{E}_{G'}$ an open set in the subbases of the topology of E'. Then, there exists $e'\in E'$ such that $A=\mathscr{A}_{e'}$. By Lemma 4.1, we get

$$\begin{aligned} e\in g^{-1}(A) &\Leftrightarrow g(e)\in A \\ &\Leftrightarrow g(e)\in\mathscr{A}_{e'} \end{aligned}$$

$$\Leftrightarrow g(e) \sim e'$$

$$\Leftrightarrow e \sim g^{-1}(e')$$

$$\Leftrightarrow e \in \mathscr{A}_{g^{-1}(e')}$$

So, $g^{-1}(A)$ is an open set of E and so, the function g is continuous. We have also

$$g^{-1}(\mathscr{A}_{e'}) = \mathscr{A}_{g^{-1}(e')}, \forall e' \in E'. \tag{2}$$

In the equation (2) we replace g by g^{-1}, we get

$$g(\mathscr{A}_{e}) = \mathscr{A}_{g(e)}, \forall e \in E \tag{3}$$

and then the function g^{-1} is continuous.

□

The converse of the above result is not true. If we consider the two following graphs

Definition 4.3. [18] *Let $(X, \mathscr{T})$ and $(X', \mathscr{T}')$ be two topological spaces. We say that the two topologies $\mathscr{T}$ and $\mathscr{T}'$ are symmetric ($\mathscr{T} \approx_s \mathscr{T}'$) if*

(i) $|\mathscr{T}| = |\mathscr{T}'|$

and

(ii) For all $A_1 \in \mathscr{T}$, there exists $A_2 \in \mathscr{T}'$ with $|A_1| = |A_2|$, and conversely.

Theorem 4.2. *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $G \approx_i G'$, Then $\mathscr{T}_G^E \approx_s \mathscr{T}_G^{E'}$.*

Proof. Let $f = (h, g)$ an isomorphism from G to G'. From the equation (2) and (3), we have a bijection between the two subbases of the topologies $\mathscr{T}_G^E$ and $\mathscr{T}_G^{E'}$. Then they will be symmetric.

□

5. ALEXANDROFF SPACE AND MINIMAL BASIS

Recall that a topological space is called compact if each open cover of the space has a finite subcover. We have the following result for the edge graphic topology.

Theorem 5.1. *Suppose that $G = (V,E)$ is a graph. The topological space $(E, \mathscr{T}_G^E)$ is compact if and only if E is finite.*

Proof. If E is finite, then from any open cover we have a finite subcover since we can choose an open set for any element in E.
Conversely, suppose that $(E, \mathscr{T}_G^E)$ is a compact topological space and consider the open cover given by the minimal bases. This open cover has a finite subcover. So, E is finite.

□

Definition 5.1. *We say that a topological space $(X, \mathscr{T})$ is connected if there does not exist nonempty disjoint open subsets A and B of X such that $x = A \cup B$.*

Example 5.1. *For $X = \{a,b,c\}$, $\mathscr{T}_1 = \{\emptyset, \{b\}, \{b,a\}, \{b,c\}, X\}$ the space is connected but for $\mathscr{T}_2 = \{\emptyset, \{b\}, \{a,c\}, X\}$, the space $(X, \mathscr{T}_2)$ is disconnected since $X = \{b\} \cup \{a,c\}$.*

Definition 5.2. *A graph $G = (V,E)$ is called connected if for all $x, y \in V$ there exists a path joining x and y.*

Example 5.2. *The following first graph is connected but the second is disconnected one.*

Definition 5.3. *Let $G = (V,E)$ be a graph and let $H_i = (V_i, E_i)$, $i = 1, \ldots, k$, a family of connected subgraphs of G such that*

(i) $V = \cup_{i=1}^{k} V_i$,
(ii) $E = \cup_{i=1}^{k} E_i$,
(iii) $V_i \cap V_j = \emptyset$ *and* $E_i \cap E_j = \emptyset$, *if* $i \neq j$.

Each subgraph H_i is called a connected component of the graph G.

Remark 5.1. *A graph G is connected if and only if it has one connected component.*

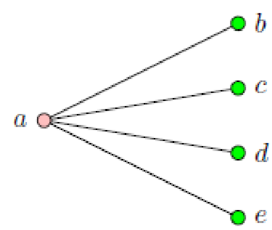
Theorem 5.2. *Let $G = (V,E)$ be a graph. If G is disconnected, then $(E, \mathscr{T}_G^E)$ is a disconnected topological space.*

Proof. Suppose that G is disconnected. So, as in Definition 5.3 we have $E = \bigcup_{i\in I} E_i$, where the cardinal of I satisfies $|I| \geq 2$. In addition, $E_i = \bigcup_{e\in E_i} \mathscr{A}_e$, then E_i is an open set. So, we have at least two nonempty disjoint open sets where their union is E.

□

Remark 5.2. *The converse of the Theorem 5.2 is not true.*

Example 5.3. *For the following connected graph, the edge graphic topology is the discrete topology.*



Proposition 5.1. *Let $G = (V,E)$ be a cycle of order $n \geq 4$ and $n \neq 4$, then $\mathscr{T}_G^E$ is the discrete topology and therefore disconnected.*

Proof. Let G be a cycle $a_1, \cdots, a_n$. We have
$E = \{a_1a_2, a_2a_3, \cdots, a_na_1\}$ and so $\mathscr{A}_{a_ia_{i+1}} = \{a_{i-1}a_i, a_{i+1}a_{i+2}\}$, $i = 1,\ldots,n$, with the convention $a_{n+1} = a_1, a_{n+2} = a_2$ and $a_0 = a_n$. Then, $\mathscr{T}_G$ is the discrete topology.

□

Conclusion

In this work, we introduced a new topology on edge set of an undirected graph. We studied its properties and mainly we proved that it is an Alexandroff space. In addition, we investigated this topology and related it with isomorphic and symmetric graphs. As perspectives, we can study the density in this topology.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

REFERENCES

[1] K.A. Abdu, A. Kilicman, Bitopological spaces on undirected graphs, J. Math. Computer Sci. 18 (2018), 232-241.

[2] G. Agnarsson, R. Greeniaw, Graph theory: Modeling application, and algorithms, Person Education Inc., (2007).

[3] A. Ali, Z. Raza, A.A. Bhatti, On the augmented Zagreb index, Kuwait J. Sci. 43 (2016), 123-138.

[4] P. Alexandroff, Diskrete räume, Mat. Sb. (N.S.) 2 (1937), 501-518.

[5] F.G. Arenas, Alexandroff spaces, Acta Math. Univ. Comenian. 68 (1999), 17-25.

[6] A.M. Alzubaidi, M. Dammak, Graphic topology on fuzzy graphs, Adv. Math.: Sci. J. 11 (2022), 853-868.

[7] J.A. Bondy, U.S.R. Murty, Graph theory with applications, Elsevier, New York, 1976.

[8] G. Coulouris, J. Dollimore, T. Kindberg, Distributed systems: concepts and design, Addison-Wesley, United States 2012.

[9] T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein, Introduction to algorithms, fourth edition, MIT Press, United States, 2022.

[10] R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs, Appl. Math. Comput. 399 (2021), 126018.

[11] M. de Berg, O. Cheong, M. van Kreveld, M. Overmars, Computational geometry: algorithms and applications, third edition, Heidelberg, Springer, 2008.

[12] J. Dugundji, Topology, Allyn and Bacon Inc., Boston, 1966.

[13] A. Gamorez, C.G. Nianga, S. Canoy, Topologies induced by neighborhoods of a graph under some binary operations, Eur. J. Pure Appl. Math. 12 (2019), 749-755.

[14] C.E. Ginestet, T.E. Nichols, E.T. Bullmore, et al. Brain network analysis: separating cost from topology using cost-integration, PLoS One. 6 (2011), e21570.

[15] J.E. Goodman, J. O'Rourke, Handbook of discrete and computational geometry, Second Edition, CRC Press LLC, Boca Raton, FL, 2004.

[16] I. Grigorik, High-performance browser networking, O'Reilly Media, Inc., California, 2013.

[17] J.L. Gross, J. Yellen, M. Anderson, Graph theory and its applications, 3rd Edition, Chapman and Hall/CRC, New York 2018.

[18] A.M. Hamza, S.N. Al-khafaji, Construction a topology on graphs, J. Al-Qadisiyah Computer Sci. Math. 5 (2013), 39-46.

[19] S.M. Jafarian Amiri, A. Jafarzadeh, H. Khatibzadeh, An Alexandroff topology on graphs, Bull. Iran. Math. Soc. 39 (2013), 647-662.

[20] A. Kilicman, K. Abdulkalek, Topological spaces associated with simple graphs, J. Math. Anal. 9 (2018), 44-52.

[21] Y. Li, J. Li, J. Feng, et al. Minimal bases and minimal sub-bases for topological spaces, Filomat. 33 (2019), 1957-1965.

[22] J.M. Møller, General topology, Matematisk Institut, Universitetsparken 5, 2007.

[23] J.R. Munkres, Topology, 2nd ed., PHI Learning Private Limited, (2000).

[24] A.S. Nawar, A.E.F.A. El Atik, A model of a human heart via graph nano topological spaces, Int. J. Biomath. 12 (2019), 1950006.

[25] D. Nogly, M. Schladt, Digital topology on graphs, Comput. Vis. Image. Und. 63 (1996), 394-396.

[26] H.A. Othman, M.M. Al-Shamiri, A. Saif, et al. Pathless directed topology in connection to the circulation of blood in the heart of human body, AIMS Math. 7 (2022), 18158-18172.

[27] H.A. Othman, A.M. Alzubaidi, Construction topologies on the edges set of undirected graphs, Palestine J. Math. 11 (2022), 1-6.

[28] F.P. Preparata, M.I. Shamos, Computational geometry: an introduction, Springer-Verlag, Berlin, Heidelberg, 1985.

[29] D. Sasikala, A. Divya, An Alexandroff bitopological space on undirected graphs, Int. J. Sci. Res. Rev. 8 (2019), 3720-3728.

[30] M. Shokry, Generating topology on graphs by operations on graphs, Appl. Math. Sci. 9 (2015), 2843–2857.

[31] T. Speer, A short study of Alexandroff spaces, http://arxiv.org/abs/0708.2136, 2007.

[32] W. Stallings, Network security essentials: applications and standards, 6th edition, Pearson, London, 2021.

[33] A.S. Tanenbaum, D.J. Wetherall, Computer networks, fifth Edition, Pearson, London, 2010.

[34] C. Vasudev, Graph theory with applications, New Age International Publishers, New Delhi, 2006.

[35] S. Wasserman, K. Faust, Social network analysis: methods and applications, Cambridge University Press, England, 1994.

[36] H.O. Zomam, Out-graphic topology on directed graphs, J. Math. Comput. Sci. 13 (2023), 14.

[37] H.O. Zomam, M. Dammak, Z-graphic topology on undirected graph, Kuwait J. Sci. 50 (2023), 1-8.

[38] H.O. Zomam, H.A. Othman, M. Dammak, Alexandroff spaces and graphic topology, Adv. Math. Sci. J. 10 (2021), 2653-2662.