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STRONGLY NONLINEAR PARABOLIC PROBLEMS WITH NATURAL GROWTH TERMS AND L^1 DATA IN MUSIELAK-ORLICZ-SOBOLEV SPACES

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Abstract. We prove in this paper the existence of solutions of strongly nonlinear parabolic problems with natural growth terms and L^1 data in Musielak-Orlicz-Sobolev spaces. An approximation and a compactness results in inhomogeneous Musielak-Orlicz-Sobolev spaces have also been provided.

Keywords: inhomogeneous Musielak-Orlicz-Sobolev spaces; parabolic problems; compactness.

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1. INTRODUCTION

Let Ω a bounded open subset of $\mathbb{R}^n$ and let Q be the cylinder $\Omega \times (0,T)$ with some given $T > 0$.

We consider the strongly nonlinear parabolic problem

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \text{ in } Q \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases}$$

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where $A = -\operatorname{div}(a(x,t,u,\nabla u))$ is an operator of Leray-Lions type, g is a nonlinearity with the sign condition but any restriction on its growth and $f \in L^1$.

This result generalizes analogous ones of Lions [26], Landes [22] when $g \equiv 0$ and of Brezis-Browder [11], Landes.Mustonen [23] for $g \equiv g(x,t,u)$. See also [9, 10] for related topics. In these results, the function a is supposed to satisfy a polynomial growth condition with respect to u and ∇u.

In the case where a satisfies a more general growth condition with respect to u and ∇u, it is shown in [14] that the adequate space in which (1) can be studied is the inhomogeneous Orlicz-Sobolev space $W^{1,x}L_M(Q)$ where the N-function M is related to the actual growth of a. The solvability of (1) in this setting is proved by Donaldson [14] for $g \equiv 0$ and by Robert [28] for $g \equiv g(x,t,u)$ when A is monotone, $t^2 \ll M(t)$ and $\overline{M}$ satisfies a Δ_2 condition and also by Elmahi [16] for $g = g(x,t,u,\nabla u)$ when M satisfies a Δ' condition and $M(t) \ll t^{N/(N-1)}$ as application of some L_M compactness results in $W^{1,x}L_M(Q)$, see [15].

The solvability of (1) in this setting is proved by Elmahi-Meskine [19] for $g \equiv 0$ and for $g \equiv g(x,t,u,\nabla u)$ in [18], without assuming any restriction on the N-function M.

In a recent work, the authors [3] have established an existence result for problems of the form (1), when $g \equiv 0$, without assuming any restriction on the Musielak function φ, and when $g \equiv g(x,t,u,\nabla u)$, in [2].

It is our purpose in this paper to prove, in the case where f belongs to $L^1(Q)$, the existence of solutions for problem (1) in the setting of Musielak-Orlicz spaces for general Musielak function φ with a nonlinearity $g(x,t,u,\nabla u)$ having natural growth with respect to the gradient. In section 3 some new approximation result in inhomogeneous Musielak-Orlicz-Sobolev spaces (see Theorem 1), and, on the other hand, to prove a trace result (see Lemma 3). In Section 4, we establish L^1-compactness results in the inhomogeneous Musielak-Orlicz-Sobolev spaces $W^{1,x}L_\varphi(Q)$. Section 5 contains the main result of this paper.

Our result generalizes that of the Elmahi-Meskine in [17] to the case of inhomogeneous Musielak- Orlicz-Sobolev spaces.

Let us point out that our result can be applied in the particular case when $\varphi(x,t)=t^{p}(x)$, in this case we use the notations $L^{p(x)}(\Omega)=L_{\varphi}(\Omega)$, and $W^{m,p(x)}(\Omega)=W^{m}L_{\varphi}(\Omega)$. These spaces are called Variable exponent Lebesgue and Sobolev spaces.

For some classical and recent results on elliptic and parabolic problems in Orlicz-sobolev spaces and a Musielak-Orlicz-Sobolev spaces, we refer to [1, 3, 4, 5, 8, 14, 16, 17, 18, 19, 20, 21, 29].

2. Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [27]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.

Musielak-Orlicz-Sobolev spaces: Let Ω be an open subset of $\mathbb{R}^n$.
A Musielak-Orlicz function φ is a real-valued function defined in $\Omega\times\mathbb{R}_+$ such that:

a): $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$ and

$$\lim_{t\longrightarrow 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0$$
$$\lim_{t\longrightarrow\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0.$$

b): $\varphi(.,t)$ is a Lebesgue measurable function

Now, let $\varphi_x(t)=\varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\phi_x^{-1})=t.$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering:

c): if there exists two positives constants c and T such that for almost everywhere $x\in\Omega$:

$$\varphi(x,t)\leq\gamma(x,ct) \text{ for } t\geq T$$

we write $\varphi\prec\gamma$ and we say that γ dominates φ globally if $T=0$ and near infinity if $T>0$.

d): if for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \text{ or } \lim_{t\to\infty}(\sup_{x\in\varphi}\frac{\varphi(x,ct)}{\gamma(x,t)}) = 0$$

we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near infinity respectively.

In the sequel the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability.

We define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \mapsto \mathbb{R}$ is a measurable function.

The set

$$K_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } /\rho_{\varphi,\Omega}(u) < +\infty\right\}.$$

is called the Musielak-Orlicz class (the generalized Orlicz class).

The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.

Equivelently:

$$L_\varphi(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ mesurable } /\rho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0\right\}$$

Let

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\},$$

ψ is the Musielak-Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ in the sense of Young with respect to the variable s.

On the space $L_\varphi(\Omega)$ we define the Luxemburg norm:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0/\int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx, \leq 1\}.$$

and the so-called Orlicz norm :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1}\int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [27].

The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [27].

The following conditions are equivalent:

e): $E_\varphi(\Omega) = K_\varphi(\Omega)$

f): $K_\varphi(\Omega) = L_\varphi(\Omega)$

g): φ has the Δ_2 property.

We recall that φ has the Δ_2 property if there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h , integrable in Ω such that $\varphi(x, 2t) \leq k\varphi(x,t) + h(x)$ for large values of t, or for all values of t, according to whether Ω has finite measure or not.

Let us define the modular convergence: we say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m \quad D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i; $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives.

The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Now, the functional

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}(D^\alpha u),$$

for $u \in W^m L_\varphi(\Omega)$ is a convex modular. and

$$||u||^m_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

is a norm on $W^m L_\varphi(\Omega)$.

The pair $\langle W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition:

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c,$$

as in [27].

The space $W^mL_\varphi(\Omega)$ will always be identified to a $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \prod L_\varphi$.
Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^mL_\varphi(\Omega)$.

Let $W^mE_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and let $W_0^mE_\varphi(\Omega)$ be the (norm) closure of $D(\Omega)$ in $W^mL_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m}L_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\}$$

$$W^{-m}E_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\}$$

As we did for $L_\varphi(\Omega)$, we say that a sequence of functions $u_n \in W^mL_\varphi(\Omega)$ is modular convergent to $u \in W^mL_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

From [27], for two complementary Musielak-Orlicz functions φ and ψ the following inequalities hold:

h) : the young inequality:

$$t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t,s \geq 0,\ x \in \Omega$$

i) : the Hölder inequality:

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leq ||u||_{\varphi,\Omega} |||v|||_{\psi,\Omega}.$$

for all $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$.

Inhomogeneous Musielak-Orlicz-Sobolev spaces:

Let Ω an bounded open subset of $\mathbb{R}^n$ and let $Q = \Omega\times]0,T[$ with some given T ¿ 0. Let φ be a Musielak function. For each $\alpha \in \mathbb{N}^n$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^n$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_\varphi(Q) = \{u \in L_\varphi(Q) : \forall|\alpha| \leq 1\ D_x^\alpha u \in L_\varphi(Q)\}$$

and

$$W^{1,x}E_{\varphi}(Q) = \{u \in E_{\varphi}(Q) : \forall |\alpha| \leq 1 \; D_x^{\alpha} u \in E_{\varphi}(Q)\}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \leq m} \|D_x^{\alpha} u\|_{\varphi,Q}.$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain [7]. These spaces are considered as subspaces of the product space $\Pi L_{\varphi}(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_{\varphi}, \Pi E_{\psi})$ and $\sigma(\Pi L_{\varphi}, \Pi L_{\psi})$. If $u \in W^{1,x}L_{\varphi}(Q)$ then the function : $t \longmapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1 L_{\varphi}(\Omega)$. If, further, $u \in W^{1,x}E_{\varphi}(Q)$ then this function is a $W^1 E_{\varphi}(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_{\varphi}(Q) \subset L^1(0,T; W^1 E_{\varphi}(\Omega))$. The space $W^{1,x}L_{\varphi}(Q)$ is not in general separable, if $u \in W^{1,x}L_{\varphi}(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_{\varphi}(Q)$ is defined as the (norm) closure in $W^{1,x}E_{\varphi}(Q)$ of $\mathscr{D}(Q)$. We can easily show as in [7] that when Ω a Lipschitz domain then each element u of the closure of $\mathscr{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_{\varphi}, \Pi E_{\psi})$ is limit, in $W^{1,x}L_{\varphi}(Q)$, of some subsequence $(u_i) \subset \mathscr{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, (\frac{D_x^{\alpha} u_i - D_x^{\alpha} u}{\lambda}))\, dx\, dt \to 0 \text{ as } i \to \infty,$$

this implies that (u_i) converges to u in $W^{1,x}L_{\varphi}(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\psi})$. Consequently

$$\overline{\mathscr{D}(Q)}^{\sigma(\Pi L_{\varphi}, \Pi E_{\psi})} = \overline{\mathscr{D}(Q)}^{\sigma(\Pi L_{\varphi}, \Pi L_{\psi})},$$

this space will be denoted by $W_0^{1,x}L_{\psi}(Q)$. Furthermore, $W_0^{1,x}E_{\varphi}(Q) = W_0^{1,x}L_{\varphi}(Q) \cap \Pi E_{\varphi}$.

We have the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_{\varphi}(Q) & F \\ W_0^{1,x}E_{\varphi}(Q) & F_0 \end{pmatrix},$$

F being the dual space of $W_0^{1,x}E_{\varphi}(Q)$. It is also, except for an isomorphism, the quotient of ΠL_{ψ} by the polar set $W_0^{1,x}E_{\varphi}(Q)^{\perp}$, and will be denoted by $F = W^{-1,x}L_{\psi}(Q)$ and it is shown that

$$W^{-1,x}L_{\psi}(Q)=\Big\{f=\sum_{|\alpha|\leq 1}D_x^{\alpha}f_{\alpha}:f_{\alpha}\in L_{\psi}(Q)\Big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\|=\inf\sum_{|\alpha|\leq 1}\|f_{\alpha}\|_{\psi,Q}$$

where the inf is taken on all possible decompositions

$$f=\sum_{|\alpha|\leq 1}D_x^{\alpha}f_{\alpha},\quad f_{\alpha}\in L_{\psi}(Q).$$

The space F_0 is then given by

$$F_0=\Big\{f=\sum_{|\alpha|\leq 1}D_x^{\alpha}f_{\alpha}:f_{\alpha}\in E_{\psi}(Q)\Big\}$$

and is denoted by $F_0=W^{-1,x}E_{\psi}(Q)$.

3. Main Results

4. Approximation Theorem and Trace Result

In this section, Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ with the segment property and I is a subinterval of $\mathbb{R}$ (both possibly unbounded) and $Q=\Omega\times I$. It is easy to see that Q also satisfies Lipschitz domain.

Definition 1. We say that $u_n\to u$ in $W^{-1,x}L_{\psi}(Q)+L^2(Q)$ for the modular convergence if we can write

$$u_n=\sum_{|\alpha|\leq 1}D_x^{\alpha}u_n^{\alpha}+u_n^0 \text{ and } u=\sum_{|\alpha|\leq 1}D_x^{\alpha}u^{\alpha}+u^0$$

with $u_n^{\alpha}\to u^{\alpha}$ in $L_{\psi}(Q)$ for modular convergence for all $|\alpha|\leq 1$
and $u_n^{\alpha}\to u^{\alpha}$ strongly in $L^2(Q)$.

We shall prove the following approximation theorem, which plays a fundamental role when the existence of solutions for parabolic problems is proved.

Theorem 1. If $u\in W^{1,x}L_{\varphi}(Q)\cap L^2(Q)$ (respectively $W_0^{1,x}L_{\varphi}(Q)\cap L^2(Q)$)
and $\frac{\partial u}{\partial t}\in W^{-1,x}L_{\psi}(Q)+L^2(Q)$, then there exists a sequence (v_j) in $\mathscr{D}(\overline{Q})$ (respectively $\mathscr{D}((\bar{I}),\mathscr{D}(\Omega))$) such that $v_j\to u$ in $W^{1,x}L_{\varphi}(Q)\cap L^2(Q)$ and

$\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q)+L^2(Q)$ for the modular convergence.

Proof. Let $u \in W^{1,x}L_\varphi(Q)\cap L^2(Q)$ such that $\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q)+L^2(Q)$

and let $\varepsilon>0$ be given. Writing $\frac{\partial u}{\partial t}=\sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_\psi(Q)$

for all $|\alpha|\leq 1$ and $u^0 \in L^2(Q)$, we will show that there exists $\lambda>0$(depending only on u and N)

and there exists $v\in \mathscr{D}(\overline{Q})$ for which we can write $\frac{\partial v}{\partial t}=\sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathscr{D}(\overline{Q})$ such that

$$\int_Q \varphi(x,\frac{D_x^\alpha v - D_x^\alpha u}{\lambda})dxdt \leq \varepsilon, \forall |\alpha|\leq 1, \tag{2}$$

$$||v-u||_{L^2(Q)} \leq \varepsilon, \tag{3}$$

$$||v^0-u^0||_{L^2(Q)} \leq \varepsilon, \tag{4}$$

$$\int_Q \psi(x,\frac{v^\alpha - u^\alpha}{\lambda})dxdt \leq \varepsilon, \forall |\alpha|\leq 1, \tag{5}$$

The equation (3) flows from a slight adaptation of the arguments of [7],

(4) and (5) flow also from classical approximation results.

Regrading the equation (6) it is enough to prove that $\mathscr{D}(\overline{Q})$ is dense in $L_\psi(Q)$ for this end.

We use the fact that the log-HÖlder continuity(commutes with the complementarity) i.e: if φ is log-HÖlder the its complementary ψ also it is, and proceed as in [7] (with φ and ψ interchanged) and using of course $\mathbb{R}^{N+1}$ instead of $\mathbb{R}^N$ and $Q=\Omega\times(0,T)$ instead of Ω.

These facts lead us to prove that

$$||K_\varepsilon f||_{\psi,Q} \leq C||f||_{\psi,Q}, \forall f \in L_\psi(Q)$$

(with $K_\varepsilon f(x,t)=k_\varepsilon^{-1}\int_Q K_\varepsilon(x-y)f(k_\varepsilon y,t)dy$,$K_\varepsilon(x)=\frac{1}{\varepsilon^N}K(\frac{x}{\varepsilon})$ and $K(x)$ is a measurable function with support in the ball $B_R=B(0,R)$ see [7]).

And then we deduce that $\mathscr{D}(\overline{Q})$ is dense in $L_\psi(Q)$ for the modular convergence which gives the desired conclusion.

The case of $W_0^{1,x}L_\varphi(Q)\cap L^2(Q)$ is similar to the above arguments as in [7].

Remark 1. If, in the statement of Theorem 1, one consider $\Omega\times\mathbb{R}$ instead of Q,

we have $\mathscr{D}(\Omega\times\mathbb{R})$ is dense in $u\in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R}) : \frac{\partial u}{\partial t}\in W_0^{1,x}L_\psi(\Omega\times\mathbb{R})+$

$L^2(\Omega\times\mathbb{R})$ for the modular convergence. This follows trivially from the fact that $\mathscr{D}(\mathbb{R},\mathscr{D}(\Omega))\equiv\mathscr{D}(\Omega\times\mathbb{R})$.

A first application of Theorem 1 is the following trace result generalizing a classical result which states that if u belong to $L^2(a,b;H_0^1(\Omega))$ and $\frac{\partial u}{\partial t}$ belongs to $L^2(a,b;H^{-1}(\Omega))$, then u is in $C([a,b],L^2(\Omega))$.

Lemma 1. Let $a<b\in\mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then $\{u\in W_0^{1,x}L_\varphi(\Omega\times(a,b))\cap L^2(\Omega\times(a,b)) : \frac{\partial u}{\partial t}\in W^{-1,x}L_\psi(\Omega\times(a,b))+L^2(\Omega\times(a,b))\}$ is a subset of $C([a,b],L^2(\Omega))$.

Proof. Let $u\in W_0^{1,x}L_\varphi(\Omega\times(a,b))\cap L^2(\Omega\times(a,b))$ such that $W^{-1,x}L_\psi(\Omega\times(a,b))+L^2(\Omega\times(a,b))$. After two consecutive reflection first with respect to $t=b$ and then with respect to $t=b$,

$\hat{u}(x,t)=u(x,t)\chi_{(a,b)}+u(x,2b-t)\chi_{(b,2b-a)}$ on $\Omega\times(a,2b-a)$

$\tilde{u}(x,t)=\hat{u}(x,t)\chi_{(a,2b-a)}+\hat{u}(x,2a-t)\chi_{(3a-2b,a)}$ on $\Omega\times(3a-2b,2b-a)$,

we get a function $\tilde{u}\in W_0^{1,x}L_\varphi(\Omega\times(3a-2b,2b-a))\cap L^2(\Omega\times(3a-2b,2b-a))$

such that $\frac{\partial\tilde{u}}{\partial t}\in W^{-1,x}L_\psi(\Omega\times(3a-2b,2b-a))+L^2(\Omega\times(3a-2b,2b-a))$. Now, by letting a function

$\eta\in\mathscr{D}(\mathbb{R})$ with $\eta=1$ on $[a,b]$ and supp$\eta\subset(3a-2b,2b-a)$, setting $\overline{u}=\eta\tilde{u}$,

and using standard arguments (see [[11], Lemme IV, Remarque 10, p. 158]), we have $\overline{u}=u$ on $\Omega\times(a,b)$ $\tilde{u}\in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R})$ $\frac{\partial\tilde{u}}{\partial t}\in W^{-1,x}L_\psi(\Omega\times\mathbb{R})+L^2(\Omega\times\mathbb{R})$.

Now let $v_j\in\mathscr{D}(\Omega\times\mathbb{R})$ be the sequence given by Theorem 1 corresponding to $\overline{u}$,

that is,

$$v_j\to\overline{u}\in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^2(\Omega\times\mathbb{R}) \text{ and } \frac{\partial v_j}{\partial t}\to\frac{\partial\overline{u}}{\partial t}\in W^{-1,x}L_\psi(\Omega\times\mathbb{R})+L^2(\Omega\times\mathbb{R})$$

for the modular convergence.

We have

$$\int_\Omega(v_i(\tau)-v_j(\tau))^2dx=2\int_\Omega\int_{-\infty}^{\tau}(v_i-v_j)(\frac{\partial v_i}{\partial t}-\frac{\partial v_j}{\partial t})dxdt\to 0,\text{ as } i,j\to\infty$$

from which one deduces that v_j is a Cauchy sequence in $C(\mathbb{R},L^2(\Omega))$, and since the limit of v_j in $L^2(\Omega\times\mathbb{R})$ is $\overline{u}$, we have $v_j\to\overline{u}$ in$C(\mathbb{R},L^2(\Omega))$. Consequently, $u\in C([a,b],L^2(\Omega))$.

In order to deal with the time derivative, we introduce a time mollification of a function $u\in L_\varphi(Q)$.

Thus we define, for all $\mu > 0$ and all $(x,t) \in Q$

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))ds, \tag{6}$$

where $\tilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s)$ is the zero extension of u.

Throughout the paper the index μ always indicates this mollification.

Proposition 1. If $u \in L_\varphi(Q)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u-u_\mu)$ and if $u \in \mathscr{L}_\varphi(Q)$ then

$$\int_Q \varphi(x,u_\mu)dxdt \le \int_Q \varphi(x,u)dxdt.$$

Proof. Since $(x,t,s) \mapsto u(x,s)exp(\mu(s-t))$ is measurable in $\Omega \times [0,T] \times [0,T]$, we deduce that u_μ is measurable by Fubini's theorem. By Jensen's integral inequality we have, since $\int_{-\infty}^{0} exp(\mu s)ds = 1$,

$$\begin{aligned}\varphi(x,\int_{-\infty}^{t} \mu\tilde{u}(x,s)exp(\mu(s-t))ds) &= \varphi(x,\int_{-\infty}^{0} \mu exp(\mu s)\tilde{u}(x,s+t)ds)\\ &\le \int_{-\infty}^{0} \mu exp(\mu s)\varphi(x,\tilde{u}(x,s+t))ds\end{aligned}$$

which implies

$$\begin{aligned}\int_Q \varphi(x,u_\mu(x,t))dxdt &\le \int_{\Omega\times\mathbb{R}} (\int_{-\infty}^{0} \mu exp(\mu s)\varphi(x,\tilde{u}(x,s+t)ds))dxdt\\ &\le \int_{-\infty}^{0} \mu exp(\mu s)(\int_{\Omega\times\mathbb{R}} \varphi(x,\tilde{u}(x,s+t))dxdt)ds\\ &\le \int_{-\infty}^{0} \mu exp(\mu s)(\int_Q \varphi(x,u(x,t))dxdt)ds\\ &= \int_Q \varphi(x,u)dxdt.\end{aligned}$$

Furthermore
$\frac{\partial u_\mu}{\partial t} = \lim_{\delta\to 0} \frac{1}{\delta}(exp(-\mu\delta)-1)u_\mu(x,t) + \lim_{\delta\to 0} \frac{1}{\delta}\int_t^{t+\delta} u(x,s)exp(\mu(s-(t+\delta))ds = -\mu u_\mu + \mu u.$

Proposition 2. (1) If $u \in L_\varphi(Q)$ then $u_\mu \to u$ as $\mu \to \infty$ in $L_\varphi(Q)$ for the modular convergence.
(2) If $u \in W^{1,x}L_\varphi(Q)$ then $u_\mu \to u$ as $\mu \to \infty$ in $W^{1,x}L_\varphi(Q)$ for the modular convergence.

Proof. (1) Let $(\phi_k) \subset \mathscr{D}(Q)$ such that $\phi_k \to u$ in $L_\varphi(Q)$ for the modular convergence.

Let $\lambda > 0$ large enough such that

$$\frac{u}{\lambda} \in \mathscr{L}_\varphi(Q) \text{ and } \int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt \to 0 \text{ as } k \to \infty.$$

For a.e. $(x,t) \in Q$ we have

$$|(\phi_k)_\mu(x,t) - (\phi_k)(x,t)| = \frac{1}{\mu}|\frac{\partial \phi_k}{\partial t}(x,t)| \leq \frac{1}{\mu}||\frac{\partial \phi_k}{\partial t}||_\infty.$$

On the other hand

$$\begin{aligned}\int_Q \varphi(x, \frac{u_\mu - u}{3\lambda})dxdt &\leq \frac{1}{3}\int_Q \varphi(x, \frac{u_\mu - (\phi_k)_\mu}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{(\phi_k)_\mu - \phi_k}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt \\ &\leq \frac{1}{3}\int_Q \varphi(x, \frac{(\phi_k - u)_\mu}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{(\phi_k)_\mu - \phi_k}{\lambda})dxdt \\ &+\frac{1}{3}\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt.\end{aligned}$$

This implies that

$$\int_Q \varphi(x, \frac{u_\mu - u}{3\lambda})dxdt \leq \frac{2}{3}\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt + \frac{1}{3}\varphi(x, \frac{1}{\mu\lambda}||\frac{\partial \phi_k}{\partial t}||_\infty)meas(Q).$$

Let $\varepsilon > 0$. There exists k such that

$$\int_Q \varphi(x, \frac{\phi_k - u}{\lambda})dxdt \leq \varepsilon,$$

and there exists μ_0 such that

$$\varphi(x, \frac{1}{\mu\lambda}||\frac{\partial \phi_k}{\partial t}||_\infty)meas(Q) \leq \varepsilon \text{ for all } \mu \geq \mu_0.$$

Hence

$$\int_Q \varphi(x, \frac{u_\mu - u}{3\lambda})dxdt \leq \varepsilon \text{ for all } \mu \geq \mu_0.$$

(2) Since $\forall \alpha, |\alpha| \leq 1$, we have $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$, consequently, the first part above applied on each $D_x^\alpha u$, gives the result.

Remark 2. If $u \in E_\varphi(Q)$, we can choose λ arbitrary small since $\mathscr{D}(Q)$ is (norm) dense in $E_\varphi(Q)$.

Thus,for all $\lambda > 0$

$$\int_Q \varphi(x, \frac{u_\mu - u}{\lambda})dxdt \to 0 \text{ as } \mu \to \infty$$

and $u_\mu \to u$ strongly in $E_\varphi(Q)$.Idem for $W^{1,x}E_\varphi(Q)$.

Proposition 3. If $u_n \to u$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence) then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence).

Proof. For all $\lambda > 0$ (resp., for some $\lambda > 0$),

$$\int_Q \varphi(x, \frac{D_x^\alpha((u_n)\mu) - D_x^\alpha(u)_\mu}{\lambda})dxdt \leq \int_Q \varphi(x, \frac{D_x^\alpha(u_n) - D_x^\alpha u}{\lambda})dxdt \to 0 \text{ as } n \to \infty,$$

then $(u_n)_\mu \to u_\mu$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence).

5. Compactness Results

In this section, we shall prove some compactness theorems in inhomogeneous Musielak-Orlicz- Sobolev spaces which will be applied to get existence theorem for parabolic problems.

For each $h > 0$, define the usual translated $\tau_h f$ of the function f by$\tau_h f(t) = f(t+h)$. If f is defined on $[0,T]$ then $\tau_h f$ is defined on $[-h, T-h]$.

First of all, recall the following compactness result proved by Simon [30].

Lemma 2. Let φ be a Musielak function. Let Y be a Banach space such that the following continuous imbedding holds $L^1(\Omega) \subset Y$. Then for all $\varepsilon > 0$ and all $\lambda > 0$, there is $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x}L_\varphi(Q)$, with $\frac{|\nabla u|}{\lambda} \in \mathscr{L}_\varphi(Q)$,

$$||u||_{L^1(Q)} \leq \varepsilon\lambda(\int_Q \varphi(x, \frac{|\nabla u|}{\lambda})dxdt + T) + C_\varepsilon ||u||_{L^1(0,T;Y)}.$$

Proof. Since $W_0^1 L_\varphi(\Omega) \subset L^1(\Omega)$ with compact imbedding, then for all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all $v \in W_0^1 L_\varphi(\Omega)$:

$$||v||_{L^1(\Omega)} \leq \varepsilon ||\nabla v||_{L_\varphi(\Omega)} + C_\varepsilon ||v||_Y. \tag{7}$$

Indeed, if the above assertion holds false, there is $\varepsilon_0 > 0$ and $v_n \in W_0^1 L_\varphi(\Omega)$ such that

$$||v_n||_{L^1(\Omega)} \geq \varepsilon_0 ||\nabla v_n||_{L_\varphi(\Omega)} + n||v_n||_Y.$$

This gives, by setting $w_n = \frac{v_n}{||\nabla v_n||_{L_\varphi(\Omega)}}$:

$$||w_n||_{L^1(\Omega)} \geq \varepsilon_0 + n||w_n||_Y, ||\nabla w_n||_{L_\varphi(\Omega)} = 1.$$

Since (w_n) is bounded in $W_0^1 L_\varphi(\Omega)$ then for a subsequence

$$w_n \rightharpoonup w \text{ in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \text{ and strongly in } L^1(\Omega).$$

Thus $||w_n||_{L^1(\Omega)}$ is bounded and $||w_n||_Y \to 0$ as $n \to \infty$.We deduce $w_n \to 0$ in Y and that $w = 0$ implying that $\varepsilon_0 \leq ||w_n||_{L^1(\Omega)} \to 0$, a contradiction.

Using $v = u(t)$ in (7) for all $u \in W_0^{1,x} L_\varphi(Q)$ with $\frac{|\nabla u|}{\lambda} \in \mathscr{L}_\varphi(Q)$ and a.e. t in $(0,T)$, we have

$$||u(t)||_{L^1(\Omega)} \leq \varepsilon||\nabla u(t)||_{L_\varphi(\Omega)} + C_\varepsilon||u(t)||_Y.$$

Since $\int_Q \varphi(x, \frac{|\nabla u(x,t)|}{\lambda})dxdt < \infty$ we have thanks to Fubini's theorem $\int_\Omega \varphi(x, \frac{|\nabla u(x,t)|}{\lambda})dx < \infty$ for a.e t in $(0,T)$, and then

$$||\nabla u(t)||_{L_\varphi(\Omega)} \leq \lambda(\int_\Omega \varphi(x, \frac{|\nabla u(x,t)|}{\lambda})dx + 1),$$

which implies that

$$||u(t)||_{L^1(\Omega)} \leq \varepsilon\lambda(\int_\Omega \varphi(x, \frac{|\nabla u(x,t)|}{\lambda})dx + 1) + C_\varepsilon||u(t)||_{Y)}.$$

Integrating this over $(0,T)$ yields

$$||u||_{L^1(Q)} \leq \varepsilon\lambda(\int_Q \varphi(x, \frac{|\nabla u(x,t)|}{\lambda})dxdt + T) + C_\varepsilon \int_0^T ||u(t)||_{Y)}dt$$

and finally

$$||u||_{L^1(Q)} \leq \varepsilon\lambda(\int_Q \varphi(x, \frac{|\nabla u|}{\lambda})dxdt + T) + C_\varepsilon||u||_{L^1(,0,T;Y)}.$$

We also prove the following lemma which allows us to enlarge the space Y whenever necessary.

Lemma 3. Let Y be a Banach space such that $L^1(\Omega) \subset Y$ with continuous imbedding.

If F is bounded in $W_0^{1,x} L_\varphi(Q)$ and is relatively compact in $L^1(0,T;Y)$ then F is relatively compact in $L^1(Q)$ (and also in $E_\gamma(Q)$ for all Musielak function $\gamma \ll \varphi$).

Proof. Let $\varepsilon > 0$ be given. Let $C > 0$ be such that $\int_Q \varphi(x, \frac{|\nabla f|}{C})dxdt \leq 1$ for all $f \in F$.

By the previous lemma, there exists $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x} L_\varphi(Q)$ with $\frac{|\nabla u|}{C} \in \mathscr{L}_\varphi(Q)$,

$||u(t)||_{L^1(Q)} \leq \frac{2\varepsilon C}{4C(1+T)}(\int_Q \varphi(x, \frac{|\nabla u|}{2C})dxdt + T) + C_\varepsilon ||u||_{L^1(0,T;Y)}$.

Moreover, there exists a finite sequence (fi) in F satisfying

$$\forall f \in F, \exists f_i \text{ such that } ||f - f_i||_{L^1(0,T;Y)} \leq \frac{\varepsilon}{2C_\varepsilon}$$

so that

$$||f - f_i||_{L^1(Q)} \leq \frac{\varepsilon}{2(1+T)}(\int_Q \varphi(x, \frac{|\nabla f - \nabla f_i|}{2C})dxdt + T) + C_\varepsilon ||f - f_i||_{L^1(0,T;Y)} \leq \varepsilon$$

and hence F is relatively compact in $L^1(Q)$.

Since $\gamma \ll \varphi$ then by using Vitali's theorem, it is easy to see that F is relatively compact in $E_\gamma(Q)$.

Remark 3(see [16]). If $F \subset L^1(0,T;B)$ is such that $\{\frac{\partial f}{\partial t} : f \in F\}$ is bounded in $F \subset L^1(0,T;B)$ then

$||\tau_h f - f||_{L^1(0,T;B)} \to 0$ as $h \to 0$ uniformly with respect to $f \in F$.

Theorem 2. Let φ be a Musielak function. If F is bounded in $W^{1,x}L_\varphi(Q)$ and $\{\frac{\partial f}{\partial t} : f \in F\}$ is bounded in $W^{-1,x}L_\psi(Q)$, then F is relatively compact in $L^1(Q)$.

Proof. Let γ and θ be Musielak functions such that $\gamma \ll \varphi$ and $\theta \ll \psi$ near infinity.

For all $0 < t_1 < t_2 < T$ and all $f \in F$, we have

$$\begin{aligned} ||\int_{t_1}^{t_2} f(t)dt||_{W_0^1 E_\gamma(\Omega)} &\leq \int_0^T ||f(t)||_{W_0^1 E_\gamma(\Omega)} dt \\ &\leq C_1 ||f||_{W_0^{1,x} E_\gamma(Q)} \leq C_2 ||f||_{W_0^{1,x} E_\varphi(Q)} \leq C, \end{aligned}$$

where we have used the following continuous imbedding:

$$W_0^{1,x} L_\varphi(Q) \subset W_0^{1,x} E_\gamma(Q) \subset L^1(0,T; W_0^1 E_\gamma(\Omega)).$$

Since the imbedding $W_0^1 L_\gamma(\Omega) \subset L^1(\Omega)$ is compact we deduce that $(\int_{t_1}^{t_2} f(t)dt)_{f \in F}$ is relatively compact in $L^1(\Omega)$ and in $W^{-1,1}(\Omega)$ as well.

On the other hand $\{\frac{\partial f}{\partial t} : f \in F\}$ is bounded in $W^{-1,x}L_\psi(Q)$ and $L^1(0,T;W^{-1,1}(\Omega)$ as well, since

$$W^{-1,x}L_\psi(Q) \subset W^{-1,x}E_\theta(Q) \subset L^1(0,T;W^{-1}E_\theta(\Omega)) \subset L^1(0,T;W^{-1,1}(\Omega))$$

with continuous imbedding.

By Remark 3 of [16], we deduce that $||\tau_h f - f||_{L^1(0,T;W^{-1,1}(\Omega))} \to 0$ uniformly in $f \in F$ when

$h \to 0$ and by using Theorem 2 of [16],F is relatively compact in $L^1(0,T;W^{-1,1}(\Omega))$.
Since $L^1(\Omega) \subset W^{-1,1}(\Omega))$ with continuous imbedding we can apply Lemma 3 to conclude that F is relatively compact in $L^1(Q)$.

Corollary 1. Let φ be a Musielak function.
Let (u_n) be a sequence of $W^{1,x}L_\varphi(Q)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\psi)$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathscr{D}'(Q)$$

with h_n bounded in $W^{-1,x}L_\psi(Q)$ and (k_n) bounded in the space $\mathscr{M}(Q)$ of measures on Q.
then $u_n \to u$ strongly in $L^1_{loc}(Q)$.
If further $u_n \in W_0^{1,x}L_\varphi(Q)$ then $u_n \to u$ strongly in $L^1(Q)$.
Proof. It is easily adapted from that given in [10] by using Theorem 2 and Remark 3 instead of Lemma 8 of [30].

6. Existence Result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$, $T > 0$ and set $Q = \Omega \times (0,T)$.
Throughout this section, we denote $Q_\tau = \Omega \times (0,\tau)$ for every $\tau \in [0,T]$.
Let φ and γ two Musielak-Orlicz functions such that $\gamma \ll \varphi$.
Consider a second-order operator $A : D(A) \subset W^{1,x}L_\varphi(Q) \to W^{-1,x}L\psi(Q)$ of the form

$$A(u) = -diva(x,t,u,\nabla u),$$

where $a : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, for almost every$(x,t) \in \Omega \times [0,T]$ and all $s \in \mathbb{R}, \xi \neq \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leq \beta(c_1(x,t) + \psi_x^{-1}\gamma(x,\vartheta|s|) + \psi_x^{-1}\varphi(x,\vartheta|\xi|)) \tag{8}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0 \tag{9}$$

$$a(x,t,s,\xi)\xi \geq \alpha_1\varphi(x, \frac{|s|}{\lambda}) \tag{10}$$

$$a(x,t,s,\xi)\xi \geq \alpha_2\varphi(x, \frac{|\xi|}{\lambda}) - d(x,t) \tag{11}$$

with $c_1(x,t) \in E_\psi(Q), c_1 \geq 0, d(x,t) \in L^1(Q), \alpha_1, \alpha_2, \beta, \vartheta > 0$.
Assume that $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, for almost every$(x,t) \in \Omega \times [0,T]$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \leq b(|s|)(c_2(x,t) + \varphi(x,|\xi|)) \tag{12}$$

$$g(x,t,s,\xi)s \geq 0 \tag{13}$$

with $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function. Furtheremore let

$$f \in L^1(Q). \tag{14}$$

Consider then the following parabolic initial-boundary value problem.

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f \text{ in } Q \\ u(x,t) = 0 \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases} \tag{15}$$

where u_0 is a given function in $L^1(\Omega)$.

Definition 2. A measurable function $u : \Omega \times (0,T) \to \mathbb{R}$ is called entropy solution of (15) if u belongs to $L^\infty(0,T;L^1(\Omega)), T_k(u)$ belongs to $D(A) \cap W_0^{1,x}L_\varphi(Q)$ for every $k > 0$,$S_k(u(.,t))$ belongs to $L^1(\Omega)$ for every $t \in [0,T]$ and every $k > 0$, $g(x,t,u,\nabla u)$ is in $L^1(Q)$ and u satisfies:

$$\begin{aligned} &\int_\Omega S_k(u-v)(\tau)dx + \langle \frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-v)dxdt \\ &+ \int_{Q_\tau} g(x,t,u,\nabla u)T_k(u-v)dxdt \leq \int_{Q_\tau} fT_k(u-v)dxdt + \int_\Omega S_k(u_0 - v(0))dx \end{aligned} \tag{16}$$

for every $\tau \in [0,T], k > 0$, and for all $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ such that $\frac{\partial v}{\partial t}$ belongs to $W^{-1,x}L_\psi(Q) + L^1(Q)$(recall that T_k is the usual truncation at height k defined on $\mathbb{R}$ by $T_k(s) = \min(k, \max(s,-k))$ and that $S_k(s) = \int_0^s T_k(t)dt$ is its primitive vanishing on 0).
Note that, all the terms in (16) make sense since $T_k(u-v)$ belongs to $W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$. Moreover Lemma 1 implies that $v \in \mathscr{C}([0,T], L^1(\Omega))$ and then the first and last terms are well defined.

We shall prove the following existence theorem:

Theorem 3. Assume that (8)-(14) hold true. Then the problem (15) admits at least one entropy solution solution $u \in \mathscr{C}(([0,T], L^1(\Omega))$ satisfying $u(x,0) = u_0(x)$ for almost every $x \in \Omega$.

Proof of Theorem 3. We divide the proof in four steps.

Step 1. A priori estimates.

Let (f_n) be a sequence of smooth functions such that $f_n \to f$ in $L^1(Q)$ and let (u_{0n}) be a sequence in $L^2(\Omega)$ such that $u_{0n} \to u_0$ in $L^1(\Omega)$

Consider the sequence of approximate problems:

$$(17) \qquad \begin{cases} u_n \in D(A) \cap W_0^{1,x} L_\varphi(Q) \cap \mathscr{C}(([0,T], L^2(\Omega)), u_n(x,0) = u_0(x) \\ \frac{\partial u_n}{\partial t} - \text{ div } (a(x,t,T_n(u_n), \nabla u_n)) + g_n(x,t,u_n,\nabla u_n) v dxdt = f_n \end{cases}$$

where

$$g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi))$$

. Note that $g_n(x,t,s,\xi)s \geq 0, |g_n(x,t,s,\xi)| \leq |g(x,t,s,\xi)|$ and $|g_n(x,t,s,\xi)| \leq n$.

Since g_n is bounded for any fixed $n > 0$, then, by Theorem 3 of [2], there exists at last one solution u_n of (17).

Note also that $\langle u_n^{'}, v\rangle$ is defined in the sense of distributions(where $u_n^{'} = \frac{\partial u_n}{\partial t}$ means for the time derivative of u_n).Since $u_n^{'} = f - A(u_n) - g_n$ is in $W^{-1,x} L_\psi(Q)$ we can extend $\langle u_n^{'}, v\rangle$ to all $v \in W_0^{1,x} L_\varphi(Q)$.

Using in (17) the test function $T_k(u_n)\chi_{(0,\tau)}$, we get, for every $\tau \in (0,T)$

$$(18) \qquad \int_\Omega S_k(u_n(\tau))dx + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla u_n)\nabla T_k(u_n) dxdt \leq C_1 k$$

where here and below C_1 denote positive constants not depending on n and k.

Consider now for $\theta, \varepsilon > 0$ a function $\rho_\theta^\varepsilon \in \mathscr{C}^1(\mathbb{R})$ such that

$$\rho_\theta^\varepsilon(s) = \begin{cases} 0 \text{ if } |s| \leq \theta, \\ \text{sign}(s) \text{ if } |s| \geq \theta + \varepsilon, \end{cases}$$

$$(\rho_\theta^\varepsilon(s))' \geq 0 \forall s \in \mathbb{R}$$

then, by using $\rho_\theta^\varepsilon(u_n)$ as a test function in (17) and following [25], we can see that

$$\int_{\{|u_n|>\theta\}} |g_n(x,t,u_n,\nabla u_n)|dxdt \le \int_{\{|u_n|>\theta\}} |f_n|dxdt + \int_{\{|u_0n|>\theta\}} |u_0n|dxdt \tag{19}$$

and so by letting $\theta \to 0$ and using Fatou's lemma, we deduce that $g_n(x,t,u_n,\nabla u_n)$ is a bounded sequence in $L^1(Q)$.

Moreover, we have from (10) and (18) that $(T_k(u_n))_n$ is bounded in $W_0^{1,x}L_\varphi(Q)$ for every $k > 0$. Take a $\mathscr{C}^2(\mathbb{R})$, and nondecreasing function ζ_k such that $\zeta_k(s) = s$ for $|s| \le \frac{k}{2}$ and $\zeta_k(s) = k$ sign(s) for $|s| \ge k$. Multiplying the approximating equation by $\zeta_k'(u_n)$, we get

$$\frac{\partial}{\partial t}(\zeta_k(u_n)) - \text{ div } (a(x,t,u_n,\nabla u_n)\zeta_k'(u_n)) + a(x,t,u_n,\nabla u_n)\zeta_k''(u_n)$$
$$+ g_n(x,t,u_n,\nabla u_n)\zeta_k'(u_n)) = f_n\zeta_k'(u_n))$$

in the sense of distributions. This implies, thanks to (18) and the fact that ζ_k' has compact support, that $\zeta_k(u_n)$ is bounded in $W_0^{1,x}L_\varphi(Q)$ while its time derivative $\frac{\partial}{\partial t}(\zeta_k(u_n))$ is bounded in $W^{-1,x}L_\psi(Q) + L^1(Q)$, hence Corollary 1 allows us to conclude that $\zeta_k(u_n)$ is compact in $L^1(Q)$.

By (10) and (18), we have

$$||T_k(u_n)||_{W_0^{1,x}L_\varphi(Q)} \le C_2.$$

We show that $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have

$$k\text{meas}\{|u_n| > k\} = \int_{\{|u_n|>k\}} |T_k(u_n)|dxdt \le \int_Q |T_k(u_n)|dxdt \le C_3||T_k(u_n)||_{W_0^{1,x}L_\varphi(Q)},$$

therefore,

$$\text{meas}\{|u_n| > k\} \le C_4, \tag{20}$$

where C_4 is a constant that does not depend on k. Since for all $\delta > 0$,

$$\text{meas}\{|u_n - u_m| > \delta\} \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\},$$

using (20), we get that for all $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$\text{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \text{ and } \text{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3} \forall k \ge k_0(\varepsilon). \tag{21}$$

Since the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,x}L_\varphi(Q)$, then there exists a subsequence still denoted $(T_k(u_n))_n$ such that

$$T_k(u_n) \rightharpoonup v_k \text{ in } W_0^{1,x}L_\varphi(Q) \text{ as } n \to \infty$$

and by the compact embedding (by a slight adaptation of the context of Theorem 6. of [8]), we obtain

$$T_k(u_n) \to v_k \text{ in } L_\varphi(Q) \text{ and a.e. in} Q.$$

Therefore, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q, then for all $k > 0$ and $\delta, \varepsilon > 0$ there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \forall n, m \geq n_0. \tag{22}$$

Combining (21) and (22), we obtain that for all $k > 0$ and $\delta, \varepsilon > 0$ there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \frac{\varepsilon}{3} \forall n, m \geq n_0,$$

it follows that $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n \to u \text{ a.e. in } Q.$$

We obtain

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,x}L_\varphi(Q), \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \to T_k(u) \text{ strongly in } L_\varphi(Q) \text{ and a.e. in } Q. \end{cases} \tag{23}$$

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,x}L_\varphi(Q), \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \to T_k(u) \text{ strongly in } L^1(Q) \text{ and a.e. in } Q. \end{cases} \tag{24}$$

To prove that $a(x,t,T_k(u_n), \nabla T_k(u_n))$ is a bounded sequence in $(L_\psi(Q))^N$. Let $\phi \in (E_\varphi(Q))^N$ with $||\phi||_{\varphi,Q} = 1$.

In view of (9), we have

$$\int_Q [a(x,t,T_k(u_n), \nabla T_k(u_n)) - a(x,t,T_k(u_n), \phi)][\nabla T_k(u_n) - \phi] dxdt \geq 0,$$

which gives

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\phi\,dxdt \leq$$

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt - \int_Q a(x,t,T_k(u_n),\phi)[\nabla T_k(u_n)-\phi]dxdt.$$

On the one hand, by (18), we have

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq C,$$

where here and below C denote positive constants not depending on n.

On the other hand, using (8), we see that

$$\psi(x,\frac{|a(x,t,T_k(u_n),\phi)|}{2\beta(k)}) \leq \psi(x,c_1(x,t)) + \varphi(x,\vartheta|\phi|)$$

and hence $a(x,t,T_k(u_n),\phi)$ is bounded in $(L_\psi(Q))^N$, implying that, since $T_k(u_n)$ is bounded in $W_0^{1,x}L_\varphi(Q)$

$$|\int_Q a(x,t,T_k(u_n),\phi)[\nabla T_k(u_n)-\phi]dxdt| \leq C,$$

and so, by using the dual norm, $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is a bounded sequence in $(L_\psi(Q))^N$. Thus, up to subsequence

(25) $$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi),$$

for some $h_k \in (L_\psi(Q))^N$.

Step 2. Almost everywhere convergence of gradients.

Fix $k>0$ and let $\phi(s) = s\exp(\delta s^2), \delta>0$. It is well known that when $\delta \geq (\frac{b(k)}{2\alpha})^2$ one has

(26) $$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \geq \frac{1}{2} \text{ for all } s \in \mathbb{R}$$

Let $v_j \in \mathscr{D}(Q)$ be a sequence such that

(27) $$v_j \to u \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for the modular convergence}$$

and let $w_i \in \mathscr{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^2(\Omega)$.

Set $\omega^i_{\mu,j} = T_k(v_j)_\mu + \exp(-\mu t)T_k(w_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$, see (6).

Note that $\omega^i_{\mu,j}$ is a smooth function having the following properties:

$$\begin{cases} \frac{\partial}{\partial t}(\omega^i_{\mu,j}) = \mu(T_k(v_j) - \omega^i_{\mu,j}), \omega^i_{\mu,j}(0) = T_k(v_j), |\omega^i_{\mu,j}| \leq k, \\ \omega^i_{\mu,j} \to T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for the modular convergence as } j \to \infty, \\ T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \to T_k(u) \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for the modular convergence as } \mu \to \infty. \end{cases}$$

Let now the function ρ_m defined on $\mathbb{R}$ by

$$\rho_m(s) = \begin{cases} 1 \text{ if } |s| \leq m, \\ m+1-|s| \text{ if } m \leq |s| \leq m+1, \\ 0 \text{ if } |s| \geq m+1, \end{cases}$$

where $m > k$. Let $\theta^{\mu,i}_{n,j} - \omega^i_{\mu,j}$ and $Z^{\mu,i}_{n,j,m} = \phi(\theta^{\mu,i}_{n,j})\rho_m(u_n)$.

Using in (17) the test function $Z^{\mu,i}_{n,j,m}$, we get(u'_n denotes by the distributional time derivative of u_n),

$$\begin{aligned} \langle u'_n, Z^{\mu,i}_{n,j,m}\rangle + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla\omega^i_{\mu,j}]\phi'(\theta^{\mu,i}_{n,j})\rho_m(u_n)dxdt \\ + \int_Q a(x,t,u_n,\nabla u_n)\phi(\theta^{\mu,i}_{n,j})\rho'_m(u_n)dxdt \\ + \int_Q g_n(x,t,u_n,\nabla u_n)Z^{\mu,i}_{n,j,m}dxdt = \int_Q f_n Z^{\mu,i}_{n,j,m}, \end{aligned}$$

which implies since $g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n) - \omega^i_{\mu,j})\rho_m(u_n) \geq 0$ on $|u_n| > k$:

$$\begin{aligned} \langle u'_n, Z^{\mu,i}_{n,j,m}\rangle + \int_Q a(x,t,u_n,\nabla u_n)[\nabla T_k(u_n) - \nabla\omega^i_{\mu,j}]\phi'(\theta^{\mu,i}_{n,j})\rho_m(u_n)dxdt \\ + \int_Q a(x,t,u_n,\nabla u_n)\phi(\theta^{\mu,i}_{n,j})\rho'_m(u_n)dxdt \\ + \int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n) - \omega^i_{\mu,j})\rho_m(u_n)dxdt \leq \int_Q f_n Z^{\mu,i}_{n,j,m}dxdt \end{aligned} \tag{28}$$

In the sequel and throughout the paper, we will omit for simplicity the dependence on x and t in the function $a(x,t,s,\xi)$ and denote $\varepsilon(n,j,\mu,i,s,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{s\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon(n,j,\mu,i,s,m) = 0$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then j,μ,i,s and finally m. Similarly, we will write only $\varepsilon(n)$, or $\varepsilon(n,j)$,... to mean that the limits are made only on the specified parameters.

We will deal with each term of (23). First of all, observe that

$$\int_Q f_n\phi(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt=\varepsilon(n,j,\mu) \tag{29}$$

since $\phi(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)\rightharpoonup\phi(T_k(u)-\omega^i_{\mu,j})\rho_m(u)$ weakly in $L^\infty(Q)$ as $n\to\infty$, and $\phi(T_k(u)-\omega^i_{\mu,j})\rho_m(u)\to\phi(T_k(u)-T_k(u)_\mu+\exp(-\mu t)T_k(w_i))\rho_m(u)$ weakly in $L^\infty(Q)$ as $j\to\infty$, and finally $\phi(T_k(u)-T_k(u)_\mu+\exp(-\mu t)T_k(w_i))\rho_m(u)\rightharpoonup 0$ weakly in $L^\infty(Q)$ as $\mu\to\infty$.

On the one hand, from (17) one deduces that $u_n\in W^{1,x}_0L_\varphi(Q)$ and $\frac{\partial u_n}{\partial t}\in W^{-1,x}L_\psi(Q)+L^1(Q)$ and then, by theorem 1, there exists a smooth function $u_{n\sigma}$ such that,

as $\sigma\to 0^+$,$u_{n\sigma}\to u_n$ in $W^{1,x}_0L_\varphi(Q)$ and $\frac{\partial u_{n\sigma}}{\partial t}\to\frac{\partial u_n}{\partial t}$ in $W^{-1,x}L_\psi(Q)+L^1(Q)$ for the modular convergence,$\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\rho_m(u_{n\sigma})\to Z^{\mu,i}_{n,j,m}$ in $W^{1,x}_0L_\varphi(Q)$ for the modular convergence and weakly in $L^\infty(Q)$. This implies

$$\begin{aligned}\langle u'_n,Z^{\mu,i}_{n,j,m}\rangle&=\lim_{\sigma\to0^+}\int_Q u'_{n\sigma}\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\rho_m(u_{n\sigma})dxdt\\&=\lim_{\sigma\to0^+}\int_Q[(R_m(u_{n\sigma}))']\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dxdt,\end{aligned}$$

where $R_m(s)=\int_0^s\rho_m(\eta)d\eta$. Hence

$$\begin{aligned}\langle u'_n,Z^{\mu,i}_{n,j,m}\rangle&=\lim_{\sigma\to0^+}[\int_Q(R_m(u_{n\sigma})-T_k(u_{n\sigma}))'\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dxdt\\&\qquad+\int_Q(T_k(u_{n\sigma}))'\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dxdt]\\&=\lim_{\sigma\to0^+}([\int_Q(R_m(u_{n\sigma})-T_k(u_{n\sigma}))\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dx]_0^T\\&\quad-\int_Q(R_m(u_{n\sigma})-T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma})-\omega^i_{\mu,j})(T_k(u_{n\sigma})-\omega^i_{\mu,j})'dxdt\\&\qquad+\int_Q(T_k(u_{n\sigma}))'\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dxdt)\\&=\lim_{\sigma\to0^+}\{I_1(\sigma)+I_2(\sigma)+I_3(\sigma)\}.\end{aligned}$$

Observe that for $|s|\le k$ we have $R_m(s)=T_k(s)=s$ and for $|s|>k$ we have $|R_m(s)|\ge|T_k(s)|$ and, since both $R_m(s)$ and $T_k(s)$ have the same sign of s, we deduce that sign $(s)(R_m(s)-T_k(s))\ge0$.

Consequently

$$I_1(\sigma) = [\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})dx]_0^T$$
$$\geq -\int_{\{|u_{n\sigma}(0)|>k\}} (R_m(u_{n\sigma}(0)) - T_k(u_{n\sigma}(0)))\phi(T_k(u_{n\sigma}(0)) - \omega^i_{\mu,j}(0))dx$$

and so, by letting $\sigma \to 0^+$ in the last integral, we get

$$\limsup_{\sigma\to 0^+} I_1(\sigma) \geq -\int_{\{|u_{0n}|>k\}} (R_m(u_{0n}) - T_k(u_{0n}))\phi(T_k(u_{0n}) - T_k(w_i))dx.$$

Letting $n \to \infty$, the right hand side of the above inequality clearly tends to

$$-\int_{\{|u_0|>k\}} (R_m(u_0) - T_k(u_0))\phi(T_k(u_0) - T_k(w_i))dx$$

which obviously goes to 0 as $i \to \infty$. We deduce the that

$$\limsup_{\sigma\to 0^+} I_1(\sigma) \geq \varepsilon(n,i).$$

About $I_2(\sigma)$, we have, since $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma})' = 0$

$$I_2(\sigma) = \int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(\omega^i_{\mu,j})'dxdt$$
$$= \mu \int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(T_k(v_j) - \omega^i_{\mu,j})dxdt$$
$$\geq \mu \int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega^i_{\mu,j})(T_k(v_j) - T_k(u_{n\sigma}))dxdt$$

by using the fact $\phi' \geq 0$ and that $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega^i_{\mu,j})\chi_{\{|u_{n\sigma}|>k\}} \geq 0$ and so,by letting $\sigma \to 0^+$ in the last integral

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \geq \mu \int_{\{|u_n|\geq k\}} (R_m(u_n) - T_k(u_n))\phi'(T_k(u_n) - \omega^i_{\mu,j})(T_k(v_j) - T_k(u_n))dxdt$$

and since, as it can be easily seen, the last integral is of the form $\varepsilon(n, j)$ we deduce that

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \geq \varepsilon(n, j).$$

For what concerns $I_3(\sigma)$, one

$$I_3(\sigma) = \int_Q (R_m(u_{n\sigma}) - \omega^i_{\mu,j})'\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})dxdt$$
$$+ \int_Q (\omega^i_{\mu,j})'\phi(T_k(u_{n\sigma}) - \omega^i_{\mu,j})dxdt$$

and then, by setting $\Phi(s)=\int_0^s\phi(\eta)d\eta$ and integrating by parts

$$I_3(\sigma)=[\int_\Omega\Phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})(t)dx]_0^T+\mu\int_Q(T_k(v_j)-\omega^i_{\mu,j})\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dxdt,$$

which implies, since $\Phi\geq 0$ and $(T_k(v_j)-\omega^i_{\mu,j})\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})\geq 0$

$$I_3(\sigma)\geq -\int_\Omega\Phi(T_k(u_{n\sigma}(0))-T_k(w_i))dx$$
$$+\mu\int_Q(T_k(v_j)-T_k(u_{n\sigma})\phi(T_k(u_{n\sigma})-\omega^i_{\mu,j})dxdt$$

so that

$$\limsup_{\sigma\to 0^+}I_3(\sigma)\geq -\int_\Omega\Phi(T_k(u_{0n})-T_k(w_i))dx$$
$$+\mu\int_Q(T_k(v_j)-T_k(u_n)\phi(T_k(u_n)-\omega^i_{\mu,j})dxdt,$$

and hence, by letting $n\to\infty$ in the last side, we obtain

$$\limsup_{\sigma\to 0^+}I_3(\sigma)\geq -\int_\Omega\Phi(T_k(u_0)-T_k(w_i))dx$$
$$+\mu\int_Q(T_k(v_j)-T_k(u)\phi(T_k(u)-\omega^i_{\mu,j})dxdt+\varepsilon(n).$$

since the first integral of the last side is of the from $\varepsilon(i)$ while the second one is of the form $\varepsilon(j)$ we deduce that

$$\limsup_{\sigma\to 0^+}I_3(\sigma)\geq\varepsilon(n,j,i).$$

where we have used the fact that (recall that $|\omega^i_{\mu,j}|\leq k$)

$$\int_Q G_k(u)\phi'(T_k(u)-\omega^i_{\mu,j})(T_k(u)-\omega^i_{\mu,j})dxdt$$
$$=\int_{\{u>k\}}(u-k)\phi'(k-\omega^i_{\mu,j})(k-\omega^i_{\mu,j})dxdt$$
$$+\int_{\{u<-k\}}(u+k)\phi'(-k-\omega^i_{\mu,j})(-k-\omega^i_{\mu,j})dxdt\geq 0.$$

Combining these estimates, we conclude that

$$\langle u_n',\phi(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)\rangle\geq\varepsilon(n,j,i). \tag{30}$$

On the other hand, the second term of the left hand side of (28) read as

$$\begin{aligned}
&\int_Q a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\
&=\int_{\{|u_n|\le k\}} a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\
&+\int_{\{|u_n|>k\}} a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\
&=\int_Q a(T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt\\
&+\int_{\{|u_n|>k\}} a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt
\end{aligned}$$

where we have used the fact that, since $m>k,\rho_m(u_n)=1$ on $\{|u_n|\le k\}$. Setting for $s>0$, set $Q^s=\{(x,t)\in Q:|\nabla T_k(u)|\le s\}$ and $Q^s_j=\{(x,t)\in Q:|\nabla T_k(v_j)|\le s\}$ and denote by χ^s and χ^s_j the characteristic functions of Q^s and Q^s_j respectively, we deduce that

$$\begin{aligned}
&\int_Q a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\
&=\int_Q[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\times\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt\\
&+\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt\\
&+\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^s_j\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt\\
&-\int_Q a(u_n,\nabla u_n)\nabla\omega^i_{\mu,j}\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\
&:=J_1+J_2+J_3+J_4.
\end{aligned}$$

We shall go to the limit as n,j,μ and $s\to\infty$ in the last three integrals of the last side.

Starting with J_2, we have by letting $n\to\infty$

$$J_2=\int_Q a(T_k(u),\nabla T_k(v_j)\chi^s_j)[\nabla T_k(u)-\nabla T_k(v_j)\chi^s_j]\phi'(T_k(u)-\omega^i_{\mu,j})\rho_m(u)dxdt+\varepsilon(n),$$

since $a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)\to a(T_k(u),\nabla T_k(v_j)\chi^s_j)$ strongly in $(E_\psi(Q))^N$ by using (8), (27) and Lebesgue theorem while $\nabla T_k(u_n)\chi^s_j\rightharpoonup\nabla T_k(u)\chi^s$ strongly in $(L_\varphi(Q))^N$.

$$J_2=\varepsilon(n,j).$$

About $J_3(n,j,\mu,s)$,we have by letting $n\to\infty$ and using (25)

$$J_3=\int_Q h_k\nabla T_k(v_j)\chi_j^s\phi'(T_k(u)-\omega^i_{\mu,j})\rho_m(u)dxdt+\varepsilon(n)$$

which gives by letting $j\to\infty$, thanks to (27) (recall that $\rho_m(u)=1$ on $\{|u|\le k\}$),

$$J_3=\int_Q h_k\nabla T_k(u)\chi^s\phi'(T_k(u)-T_k(u)_\mu-\exp(-\mu t)T_k(w_i)dxdt+\varepsilon(n,j),$$

implying that, by letting $\mu\to\infty$, $J_3=\int_Q h_k\nabla T_k(u)\chi^s dxdt+\varepsilon(n,j,\mu)$, and thus

$$J_3=\int_Q h_k\nabla T_k(u)dxdt+\varepsilon(n,j,\mu,s).$$

For what concerns J_4 we can write, since $\rho_m(u)=1$ on $\{|u|>m+1\}$

$$\begin{aligned}J_4=&-\int_Q a(T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)\\=&-\int_{\{|u_n|\le k\}}a(T_k(u_n),\nabla T_k(u_n))\nabla\omega^i_{\mu,j}\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\&-\int_{\{k<|u_n|\le m+1\}}a(T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\end{aligned}$$

and, as above, by letting $n\to\infty$

$$\begin{aligned}J_4=&-\int_{\{|u|\le k\}}h_k\nabla\omega^i_{\mu,j}\phi'(T_k(u)-\omega^i_{\mu,j})dxdt\\&-\int_{\{k<|u|\le m+1\}}h_{m+1}\nabla\omega^i_{\mu,j}\phi'(T_k(u)-\omega^i_{\mu,j})\rho_m(u)dxdt+\varepsilon(n)\end{aligned}$$

which implies that, by letting $j\to\infty$

$$\begin{aligned}J_4=&-\int_{\{|u|\le k\}}h_k[\nabla T_k(u)_\mu-\exp(-\mu t)\nabla T_k(w_i)]\phi'(T_k(u)-T_k(u)_\mu-\exp(-\mu t)\nabla T_k(w_i))dxdt+\varepsilon(n,j)\\&-\int_{\{k<|u|\le m+1\}}h_{m+1}[\nabla T_k(u)_\mu-\exp(-\mu t)\nabla T_k(w_i)]\phi'(T_k(u)-T_k(u)_\mu-\exp(-\mu t)\nabla T_k(w_i))\rho_m(u)dxdt\end{aligned}$$

so that, by letting $\mu\to\infty$

$$J_4=-\int_Q h_k\nabla T_k(u)dxdt+\varepsilon(n,j).$$

We conclude then that

$$\begin{aligned}&\int_Q a(u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla\omega^i_{\mu,j}]\phi'(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\\&=\int_Q[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\\&\qquad\times\phi'(T_k(u_n)-\omega^i_{\mu,j})dxdt+\varepsilon(n,j,\mu,s).\end{aligned}\tag{31}$$

To deal with the third term of the left-hand side of (28), observe that

$$\begin{aligned} &| \int_Q a(x,t,u_n,\nabla u_n)\phi(\theta_{n,j}^{\mu,i})\rho_m'(u_n)dxdt \,| \\ &\leq \phi(2k)\int_{\{m\leq |u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt. \end{aligned}$$

On the other hand, using $\theta_m(u_n)$ as a test function in (17) where $\theta_m(s) = T_1(s-T_m(s))$, we get

$$\begin{aligned} \langle u_n', \theta_m(u_n)\rangle + \int_Q a(u_n,\nabla u_n)\nabla u_n \theta_m'(u_n)dxdt + \int_Q g_n(u_n,\nabla u_n)\theta_m(u_n)dxdt \\ = \int_Q f_n \theta_m(u_n)dxdt \end{aligned}$$

which gives, by setting $\Theta_m(s) = \int_0^s \theta_m(\eta)d\eta$ (observe that $\theta_m(s)s \geq 0$)

$$[\int_\Omega \Theta_m(u_n(t))dx]_0^T + \int_{\{m\leq |u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt \leq \int_{\{m\leq |u_n|\leq m+1\}} |f_n|dxdt$$

and since $\Theta_m \geq 0$, we deduce that

$$\int_{\{m\leq |u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt \leq \int_\Omega \Theta_m(u_{0n})dx + \int_{\{m\leq |u_n|\leq m+1\}} |f_n|dxdt.$$

Since, as it can be easily seen, each integral of the right hand side is of the form $\varepsilon(n,m)$ we obtain

$$| \int_Q a(x,t,u_n,\nabla u_n)\phi(\theta_{n,j}^{\mu,i})\rho_m'(u_n)dxdt| \leq \varepsilon(n,m). \tag{32}$$

We now turn to the fourth term of the left hand side of (28). We can write

$$\begin{aligned} &| \int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n)-\omega_{\mu,j}^i)\rho_m(u_n)|dxdt \\ &\leq b(k)\int_Q c_2(x,t)|\phi(T_k(u_n)-\omega_{\mu,j}^i)|dxdt \\ &+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n)|\phi(T_k(u_n)-\omega_{\mu,j}^i)|dxdt. \end{aligned} \tag{33}$$

Since $c_2(x,t)$ belongs to $L^1(Q)$ it is easy to see that

$$b(k)\int_Q c_2(x,t)|\phi(T_k(u_n)-\omega_{\mu,j}^i)|dxdt = \varepsilon(n,j,\mu).$$

On the other hand, the second term of the right hand side of (33) reads as

$$\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt$$

$$=\frac{b(k)}{\alpha}\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt$$

$$+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt$$

$$+\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^s_j)|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt$$

and, as above, by letting successively first n, then j,μ and finally s go to infinity, we can easily see that each one of last two integrals of the right-hand side of the last equality is of the form $\varepsilon(n,j,\mu)$. This implies that

$$\begin{aligned}&|\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)dxdt\ |\\ &\le\frac{b(k)}{\alpha}\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]|\phi(T_k(u_n)-\omega^i_{\mu,j})|dxdt+\varepsilon(n,j,\mu).\end{aligned}\tag{34}$$

Combining (28),(29),(30),(31),(32) and (34), we get

$$\begin{aligned}&\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]\\ &\times[\phi'(T_k(u_n)-\omega^i_{\mu,j})-\frac{b(k)}{\alpha}|\phi(T_k(u_n)-\omega^i_{\mu,j})|]dxdt\le\varepsilon(n,j,\mu,i,s,m).\end{aligned}$$

and so, thanks to (26),

$$\begin{aligned}&\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]dxdt\le 2\varepsilon(n,j,\mu,i,s,m).\end{aligned}\tag{35}$$

On the other hand, we have

$$\begin{aligned}&\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt\\ &-\int_Q [a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi^s_j)][\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j]dxdt\\ &=\int_Q a(T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_j)\chi^s_j-\nabla T_k(u)\chi^s]dxdt\\ &-\int_Q a(T_k(u_n),\nabla T_k(u)\chi^s)[\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt\end{aligned}$$

$$+\int_Q a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]dxdt$$

and, as it can be easily seen, each integral of the right-hand side is of the form $\varepsilon(n,j,s)$, implying that

$$\begin{aligned}\int_Q&[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt\\&=\int_Q[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)]\\&\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]dxdt+\varepsilon(n,j,s).\end{aligned}\tag{36}$$

For $r\leq s$, we have

$$\begin{aligned}0&\leq\int_{Q^r}[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\\&\leq\int_{Q^s}[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\\&=\int_{Q^s}[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt\\&\leq\int_{Q}[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt\\&=\int_Q[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(v_j)\chi_j^s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_j^s]dxdt+\varepsilon(n,j,s)\\&\leq\varepsilon(n,j,\mu,i,s,m),\end{aligned}$$

hence, by passing to the limit sup over n, we get

$$\begin{aligned}0\leq\limsup_{n\to\infty}\int_{Q^r}&[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\\&\leq\limsup_{n\to\infty}\varepsilon(n,j,\mu,i,s,m),\end{aligned}$$

in which we let successively $j\to\infty,\mu\to,i\to\infty,s\to\infty$, and $m\to\infty$, to obtain

$$\int_{Q^r}[a(T_k(u_n),\nabla T_k(u_n))-a(T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]dxdt\to 0\text{ as }n\to\infty$$

and thus, as in the elliptic case(see [4]), there exists a subsequence also denote by u_n such that

$$\nabla u_n\to\nabla u\text{ a.e. in}Q.\tag{37}$$

We deduce then that, for all $k > 0$

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))$$
$$\text{weakly in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi) \tag{38}$$

Step 3. Modular convergence of the truncations and equi-integrability of the nonlinearities.

Thanks to (33) and (36), we can write

$$\begin{aligned}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \\ \leq \int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi^s dxdt \\ +\int_Q a(T_k(u_n),\nabla T_k(u)\chi^s)[\nabla T_k(u_n)-\nabla T_k(u)\chi^s]dxdt \\ +\varepsilon(n,j,\mu,i,s,m),\end{aligned}$$

and then

$$\begin{aligned}\limsup_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \\ \leq \int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)\chi^s dxdt \\ +\int_Q a(T_k(u_n),\nabla T_k(u)\chi^s)[1-\chi^s]dxdt \\ +\lim_{n\to\infty}\varepsilon(n,j,\mu,i,s,m),\end{aligned}$$

in which we can pass to the limit as $j,\mu,i,s,m\to\infty$ to obtain

$$\limsup_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq \int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt.$$

On the other hand, Fatou's lemma implies

$$\int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt \leq \liminf_{n\to\infty}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt,$$

and thus, as $n\to\infty$,

$$\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \to \int_Q a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt.$$

Since $a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \geq d(x,t)\in L^1(Q)$ we deduce that

$$a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \to a(T_k(u),\nabla T_k(u))\nabla T_k(u)dxdt \text{ in } L^1(Q), \tag{39}$$

as $n \to \infty$; implying by using (11) and Vitali's theorem that

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ in } (L_\varphi(Q))^N \text{ for the modular convergence .}$$

We shall now prove that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u_n,\nabla u_n)$ strongly in $L^1(Q)$ by using Vitli's theorem. Since $g_n(x,u_n,\nabla u_n) \to g(x,u_n,\nabla u_n)$ a.e. in Q, thanks to (24)and (30), it suffices to prove that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q.

Let $E \subset Q$ be a measurable subset of Q. We have for any $m > 0$:

$$\begin{aligned}\int_E |g_n(x,t,u_n,\nabla u_n)|dxdt &= \int_{E\cap\{|u_n|\le m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt + \int_{E\cap\{|u_n|>m\}} |g_n(x,t,u_n,\nabla u_n)|dxdt.\\ &\le \frac{b(m)}{\alpha}\int_E a(T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt + b(m)\int_E [c_2(x,t)+\frac{1}{\alpha}d(x,t)]dxdt\\ &\quad + \int_{\{|u_n|>m\}} |f_n|dxdt + \int_{\{|u_{0n}|>m\}} |u_{0n}|dxdt,\end{aligned}$$

where we have used (12) and (19). Therefore, it is easy to see that there exists ν such that

$$|E| < \nu \Rightarrow \int_E |g_n(x,t,u_n,\nabla u_n)|dxdt \le \varepsilon \forall n,$$

which shows that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q as required.

Step 4. Passage to the limit and regularity of the solution.

Let $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$. There exists a prolongation $\bar{v}$ of v such that (see proof of Lemma 1)

$$\bar{v} = v \text{ on } Q, \bar{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^2(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}),$$

and

$$\frac{\partial \bar{v}}{\partial t} = v \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^2(\Omega\times\mathbb{R}). \tag{40}$$

By Theorem1(see also Remark 1), there exists a sequence $(w_j \subset \mathscr{D}(\Omega\times\mathbb{R}))$ such that

$$w_j \to \bar{v} \text{ in } W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^2(\Omega\times\mathbb{R}),$$

and

$$\frac{\partial w_j}{\partial t} \to \frac{\partial \bar{v}}{\partial t} \text{ in } W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^2(\Omega\times\mathbb{R}), \tag{41}$$

for the modular convergence and $||w_j||_{\infty,\Omega\times\mathbb{R}} \leq (N+2)||\bar{v}||_{\infty,\Omega\times\mathbb{R}}$.

Go back to approximate equations (17) and use $T_k(u_n-w_j)\chi_{(0,\tau)}$, for every $\tau\in[0,T]$(which belongs to $W_0^{1,x}L_\varphi(Q)$) as a test function one has

$$\langle u_n', T_k(u_n-w_j)\rangle_{Q_\tau} + \int_{Q_\tau} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\nabla T_k(u_n-w_j)dxdt$$
$$+\int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n-w_j)dxdt = \int_{Q_\tau} f_n T_k(u_n-w_j)dxdt, \tag{42}$$

where $\overline{k}=k+C||v||_{\infty,Q}$.

The second term of the left hand side of (42) reads as

$$\begin{aligned}&\int_{Q_\tau} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\nabla T_k(u_n-w_j)dxdt\\ &=\int_{Q_\tau\cap\{|u_n-w_j|\leq k\}} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\nabla u_n dxdt\\ &-\int_{Q_\tau\cap\{|u_n-w_j|\leq k\}} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\nabla w_j dxdt\end{aligned}$$

and by using Fatou's lemma in the first integral of the last side and (38) in the second one, we deduce that

$$\begin{aligned}&\int_{Q_\tau} a(T_{\overline{k}}(u),\nabla T_{\overline{k}}(u))\nabla T_k(u-w_j)dxdt\\ &\leq \liminf_{0\to\infty}\int_{Q_\tau} a(T_{\overline{k}}(u_n),\nabla T_{\overline{k}}(u_n))\nabla T_k(u_n-w_j)dxdt.\end{aligned}$$

Since $\nabla T_k(u_n-w_j)\to\nabla T_k(u-w_j)$ weakly in $L^\infty(Q)$ as $n\to\infty$, we have (as $n\to\infty$)

$$\int_{Q_\tau} g_n(u_n,\nabla u_n)T_k(u_n-w_j)dxdt \to \int_{Q_\tau} g(u,\nabla u)T_k(u-w_j)dxdt \text{ and}$$
$$\int_{Q_\tau} f_n T_k(u_n-w_j)dxdt \to \int_{Q_\tau} fT_k(u-w_j)dxdt.$$

For what concerns the first term of (42), we have, by setting $S_k(s)=\int_0^s T_k(\eta)d\eta$

$$\begin{aligned}&\langle u_n', T_k(u_n-w_j)\rangle_{Q_\tau} = \langle u_n'-w_j', T_k(u_n-w_j)\rangle_{Q_\tau} + \langle w_j', T_k(u_n-w_j)\rangle_{Q_\tau}\\ &=\int_\Omega S_k(u_n-w_j)(\tau)dx - \int_\Omega S_k(u_{0n}-w_j(0))dx + \int_{Q_\tau}\frac{\partial w_j}{\partial t}T_k(u_n-w_j)dxdt,\end{aligned} \tag{43}$$

and, in order to pass to the limit (as $n\to\infty$) in (43), we will first prove that $u_n\to u$ in $\mathscr{C}([0,T],L^1(\Omega))$ (implying, in particular, that $u\in\mathscr{C}([0,T],L^1(\Omega))$).

Let now, for every $l>0$ $\omega_{j,\mu}^{i,l}=T_l(v_j)_\mu+\exp(-\mu t)T_l(w_i)$ and $\omega_\mu^{i,l}=T_l(u)_\mu+\exp(-\mu t)T_l(w_i)$, where $v_j^l \in \mathscr{D}(Q)$ is a sequence such that: $v_j^l \to T_l(u)$ in $W_0^{1,x}L_\varphi(Q)$ for the modular convergence as $j \to \infty$.

We have for every $\tau \in (0,T]$

$$
\begin{aligned}
\langle (\omega_{j,\mu}^{i,l})', T_k(u_n - \omega_{j,\mu}^{i,l})\rangle_{Q_\tau} &= \mu \int_{Q_\tau} (T_l(v_j) - \omega_{j,\mu}^{i,l}) T_k(u_n - \omega_{j,\mu}^{i,l}) dxdt \\
&\to \mu \int_{Q_\tau} (T_l(v_j) - \omega_{j,\mu}^{i,l}) T_k(u_n - \omega_{j,\mu}^{i,l}) dxdt \\
&\to \mu \int_{Q_\tau} (T_l(u) - \omega_{j,\mu}^{i,l}) T_k(u_n - \omega_{j,\mu}^{i,l}) dxdt \geq 0,
\end{aligned} \tag{44}
$$

as first $n \to \infty$ and then $j \to \infty$ and where we have used the fact that $|\omega_\mu^{i,l}| \leq l$ to get the positiveness of last integral.

On the other hand, by using (17)

$$
\begin{aligned}
\langle u_n', T_k(u_n - \omega_{j,\mu}^{i,l})\rangle_{Q_\tau} = &\int_Q a(x,t,u_n,\nabla u_n)[\nabla \omega_{j,\mu}^{i,l} - \nabla u_n]\chi_{\{|u_n - \omega_{j,\mu}^{i,l}| \leq k\}} dxdt \\
&+ \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n) T_k(\omega_{j,\mu}^{i,l} - u_n) dxdt \\
&+ \int_{Q_\tau} f_n T_k(u_n - \omega_{j,\mu}^{i,l}) dxdt,
\end{aligned}
$$

in which we can use Fatou's lemma and Lebesgue theorem to pass to the limit sup first over n and then over j,μ,l, to get, for every $k>0$,

$$
\langle u_n', T_k(u_n - \omega_{j,\mu}^{i,l})\rangle_{Q_\tau} \leq \varepsilon(n,j,\mu,l) \text{ not depending on } \tau. \tag{45}
$$

Therefore, by writing

$$
\begin{aligned}
\int_\Omega S_k(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau))dx &= \langle u_n' - (\omega_{j,\mu}^{i,l})', T_k(u_n - \omega_{j,\mu}^{i,l})\rangle_{Q_\tau} + \int_\Omega S_k(u_0 - T_l(w_i))dx \\
&= \langle u_n', T_k(u_n - \omega_{j,\mu}^{i,l})\rangle_{Q_\tau} - \langle (\omega_{j,\mu}^{i,l})', T_k(u_n - \omega_{j,\mu}^{i,l})\rangle_{Q_\tau} + \int_\Omega S_k(u_0 - T_l(w_i))dx
\end{aligned}
$$

and using (44) and (45), we see that, for every fixed $k>0$,

$\int_\Omega S_k(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau))dx \leq \varepsilon(n,j,\mu,l,i)$ not depending on τ

which implies, by writing (recall that S_k is a convex function)

$$
\int_\Omega S_k[\frac{1}{2}(u_n(\tau) - u_m(\tau))]dx \leq \int_\Omega S_k(u_n(\tau) - \omega_{j,\mu}^{i,l}(\tau))dx + \int_\Omega S_k(u_m(\tau) - \omega_{j,\mu}^{i,l}(\tau))dx,
$$

that

$$\int_{\Omega} S_k[\frac{1}{2}(u_n(\tau)-u_m(\tau))]dx \leq \varepsilon_1(n,m),$$

where $\varepsilon_i(n,m)(i=1,2)$ is a term not depending on τ and which tends to 0 as n and m go to infinity.

We deduce then that (see for instance, the proof of Theorem 1.1 of [25]),

$\int_{\Omega}|u_n(\tau)-u_m(\tau))|dx \leq \varepsilon_2(n,m)$ not depending on τ

and thus,u_n is a Cauchy sequence in $C([0,T],L^1(\Omega))$(the space of continuous functions from $[0,T]$ into $L^1(\Omega)$ equipped with topology of uniform convergence).

Since the limit of u_n in $L^1(Q)$ is u, we have

$$u_n \to u \text{ in } C([0,T],L^1(\Omega)).$$

Moreover, since $S_k(u_n-w_j)(\tau) \leq k|u_n(\tau)|+k|w_j(\tau)|$, we have by using Lebesgue theorem

$$\int_{\Omega} S_k(u_n-w_j)(\tau)dx \to \int_{\Omega} S_k(u-w_j)(\tau)dx \text{ as } n\to\infty$$

therefore we can pass to the limit in n in each term of the right hand side of (43) to get

$$\lim_{n\to\infty}\langle u_n', T_k(u_n-w_j)\rangle_{Q_\tau}$$
$$= \int_{\Omega} S_k(u-w_j)(\tau)dx - \int_{\Omega} S_k(u_0-w_j(0))dx + \int_{Q_\tau}\frac{\partial w_j}{\partial t}T_k(u-w_j)dxdt$$

and thus, by passing to the limit inf over n in (42), we have

$$\int_{\Omega} S_k(u-w_j)(\tau)dx + \int_{Q_\tau}\frac{\partial w_j}{\partial t}T_k(u-w_j)dxdt$$
$$+\int_{Q_\tau} a(u,\nabla u)\nabla T_k(u-w_j)dxdt + \int_{Q_\tau} g(u,\nabla u)T_k(u-w_j)dxdt$$
$$\leq \int_{Q_\tau} fT_k(u-w_j)dxdt + \int_{\Omega} S_k(u_0-w_j(0))dx. \tag{46}$$

To go to the limit in (46) as $j\to\infty$, observe that, thanks to (41), we have

$$\int_{Q_\tau}\frac{\partial w_j}{\partial t}T_k(u-w_j)dxdt \to \langle\frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_\tau}.$$

Moreover, for every $\tau\in[0,T]$

$$\int_{\Omega} S_1(w_i-w_j)(\tau)dx = \int_{\Omega}\int_{-\infty}^{0} T_1(w_i-w_j)(\frac{\partial w_i}{\partial t}-\frac{\partial w_j}{\partial t})dxdt \to 0 \text{ as } i,j\to\infty,$$

implying, as above, that $||w_i(\tau)-w_j(\tau)||_{L^1(\Omega)} \to 0$ as $i, j \to \infty$ and so $||w_j(\tau)-v(\tau)||_{L^1(\Omega)} \to 0$ as $j \to \infty$.

Therefore, we can go to the limit, as $j \to \infty$, in each integral of (46), to get

$$\begin{aligned}
&\int_\Omega S_k(u-v)(\tau)dx + \langle \frac{\partial v}{\partial t}, T_k(u-v)\rangle_{Q_\tau} \\
&+ \int_{Q_\tau} a(u,\nabla u)\nabla T_k(u-v)dxdt + \int_{Q_\tau} g(u,\nabla u)T_k(u-v)dxdt \\
&\leq \int_{Q_\tau} fT_k(u-v)dxdt + \int_\Omega S_k(u_0-v(0))dx,
\end{aligned}$$

where for the first and last integrals, we have used the fact that $S_k(u-w_j)(\tau) \leq S_k(u(\tau)) + k|w_j(\tau)|$, and thus, u is an entropy solution of (15).This completes the proof of theorem 3.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A.O. Ahmed Oubeid, M. Sidi El Vally, Nonlinear parabolic equations involving measure data in Musielak-Orlicz-Sobolev spaces, Appl. Math. Sci. 18 (2024), 205–221.

[2] M.L. Ahmed Oubeid, A. Benkirane, M.S. El Vally, Strongly nonlinear parabolic problems in Musielak-Orlicz-Sobolev spaces, Bol. Soc. Paran. Mat. 33 (2014), 191–223.

[3] M.L. Ahmed Oubeid, A. Benkirane, M.Sidi El Vally, Parabolic equations in Musielak-Orlicz-Sobolev spaces, Int. J. Anal. Appl. 4 (2014), 174–191.

[4] M. L. Ahmed Oubeid, A. Benkirane, M. Sidi El Vally, Nonlinear elliptic equations involving measure data in Musielak-Orlicz-Sobolev spaces, J. Abstr. Differ. Equ. Appl. 4 (2013), 43–57.

[5] A. Benkirane, M. Ould Mohamedhen Val, An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 57–75.

[6] A. Benkirane, J. Douieb, M. Ould Mohamedhen Val, An approximation theorem in Musielak-Orlicz-Sobolev spaces, Comment. Math. (Prace Matem.) 51 (2011), 109–120.

[7] A. Benkirane, M. Ould Mohamedhen Val, Some approximation properties in Musielak-Orlicz-Sobolev spaces, Thai. J. Math. 10 (2012), 371–381.

[8] A. Benkirane, M. Ould Mohamedhen Val, Variational inequalities in Musielak-Orlicz-Sobolev spaces, Bull. Belg. Math. Soc. Simon Stevin 21 (2014), 787–811.

[9] L. Boccardo, F. Murat, Strongly nonlinear Cauchy problems with gradient dependent lower order nonlinearity, Pitman Res. Notes Math. 208 (1989), 247–254.

[10] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients, Nonlinear Anal. 19 (1992), 581–597.

[11] H.Brézis, Analyse fonctionnelle, théorie et applications, 3rd end, Masson, Paris, 1992.

[12] H. Brézis, F. E. Browder, Strongly nonlinear parabolic initial boundary value problems, Proc. Nat. Acad. Sci. U.S.A. 76 (1979), 38–40.

[13] A. Dall'aglio, L. Orsina, Nonlinear parabolic equations with natural growth conditions and L^1 data, Nonlinear Anal. TMA 27 (1996), 59–73.

[14] T. Donaldson, Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial boundary value problems, J. Differ. Equ. 16 (1974), 201–256.

[15] A. Elmahi, Compactness results in inhomogeneous Orlicz-Sobolev spaces, Lecture Notes in Pure and Applied Mathematics, vol. 229, Marcel Dekker, New York, pp. 207-221, 2002.

[16] A. Elmahi, Strongly nonlinear parabolic initial-boundary value problems in Orlicz spaces, Electron. J. Differ. Equ. 09 (2002), 203-220.

[17] A. Elmahi, D. Meskine, Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces, Portugaliae Math. 62 (2005), 143–182.

[18] A. Elmahi, D. Meskine, Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces, Nonlinear Anal. 60 (2005), 1-35.

[19] A. Elmahi, D. Meskine, Parabolic equations in Orlicz spaces, J. London Math. Soc. 72 (2005), 410-428.

[20] S. Heidari, A. Razani, Infinitely many solutions for nonlocal elliptic systems in Orlicz–sobolev spaces, Georgian Math. J. 29 (2022), 45–54.

[21] S. Heidari, A. Razani, Multiple solutions for a class of nonlocal quasilinear elliptic systems in Orlicz–sobolev spaces, Bound. Value Probl. 2021 (2021), 22.

[22] R. Landes, V. Mustonen, On the existence of weak solutions for quasilinear parabolic initial-boundary value problems, Proc. R. Soc. Edinburgh Sect. A 89 (1981), 217-237.

[23] R. Landes, V. Mustonen, A strongly nonlinear parabolic initial-boundary value problem, Ark. Mat. 25 (1987), 29-40.

[24] R. Landes, V. Mustonen, On parabolic initial-boundary value problems with critical growth for the gradient, Ann. Inst. Henri Poinc. C, 11 (1994), 135–158.

[25] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of truncations, Ann. Mat. Pura Appl. 177 (1999), 143–172.

[26] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Gauthiers-Villars, 1969.

[27] J. Musielak, Modular spaces and Orlicz spaces, Lecture Notes in Math. 1034 (1983).

[28] J. Robert, Inéquations variationnelles paraboliques fortement non lineaires, J. Math. Pures Appl. 53 (1974) 299-321.

[29] M. Sidi El Vally, Strongly nonlinear elliptic problems in Musielak-Orlicz-Sobolev spaces, Adv. Dyn. Syst. Appl. 8 (2013), 115–124.

[30] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura. Appl. 146 (1987), 65–96.