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COMMUTING MAP ON SEMIGROUP OF BINARY OPERATIONS

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Abstract. This paper explores the algebraic structure of the magma monoid $(\mathscr{M}(S), \triangleleft)$, where $\mathscr{M}(S)$ comprises all binary operations on S. We investigate the notion of the center of the monoid and introduce a commuting map τ_* for each element $* \in \mathscr{M}(S)$. Furthermore, we characterize the commuting maps for all commutative elements in $\mathscr{M}(S)$.

Keywords: magma monoids; binary operations; semigroup; center; locally zero-groupoids; commuting maps.

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1. Introduction

A magma is a mathematical structure consisting of a set S together with a binary operation $\odot$ defined on it, denoted as $(S, \odot)$. Following established notation, $\mathscr{M}(S)$ represents the collection of all binary operations on S. Historically, this concept has been known by other names, including "binar" and "groupoid". The operation $\triangleleft$, first introduced in [13], has been the subject of extensive research by numerous authors, including those in [7], [14], [4], and [3]. We provide a concise overview of the operation's definition and essential properties below:

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Definition 1.1. An operation $\triangleleft$ on $\mathscr{M}(S)$ is defined by the following rule:

$$(*\triangleleft\circ)(i,j) = \circ(*(i,j),*(j,i))$$

for all $i,j \in S$ and all $*,\circ \in \mathscr{M}(S)$. Furthermore, the notations $*\triangleleft\circ$ and $\triangleleft(*,\circ)$ are used synonymously.

Remark 1.2. *In defining the operation $\triangleleft$, we encountered at least four plausible alternatives. We selected one of these options without loss of generality. The four definitions differ solely in the order of the operations' inputs. They yield operations on $\mathscr{M}(S)$ that are pairwise isomorphic or anti-isomorphic to one another. Specifically, for any $*_1, *_2 \in \mathscr{M}(S)$ and $i, j \in S$:*

(1) $(*_1 \triangleleft *_2)(i,j) = *_2(*_1(i,j), *_1(j,i))$
(2) $(*_1 \triangleleft *_2)(i,j) = *_2(*_1(j,i), *_1(i,j))$
(3) $(*_1 \triangleleft *_2)(i,j) = *_1(*_2(i,j), *_2(j,i))$
(4) $(*_1 \triangleleft *_2)(i,j) = *_1(*_2(j,i), *_2(i,j))$

Notation 1.3. *We define two binary operations, π_1 and π_2, on the set S. The operation π_1 projects onto the first component, mapping (i,j) to i for all $i,j \in S$. This operation serves as the identity element with respect to $\triangleleft$. Conversely, the operation π_2 projects onto the second component, mapping (x,y) to y for all $x,y \in S$. π_1 and π_2 are referred to as left zero-semigroup and right zero-semigroup, respectively, in the literature.*

Theorem 1.4. *For any set S, the magma monoid $(\mathscr{M}(S),\triangleleft)$ is a non-commutative monoid with π_1 serving as its identity element.*

Proof. See Theorem 2.1 in [3] or Theorem 2 in [13]. □

We define the magma monoid resulting from endowing $\mathscr{M}(S)$ with the operation $\triangleleft$ as $(\mathscr{M}(S),\triangleleft)$. This construct has been extensively explored in various works, including [13], [7], [14], [15], [17], [18], [5], and [8].

Building on the construction in [2], [5]examined the set of all graphs with vertex set S and introduced graph-induced operations, such as One-Value and Two-Value graph magmas which are closed under the operation, $\triangleleft$. These graph-induced operations have significant applications in amenable bases over infinite-dimensional algebras([1], [6]).

The concept of distributivity hierarchy graphs for a set was first introduced in [12], encompassing left, right, and two-sided distributivity. A subsequent study [4] explored various combinatorial aspects of these hierarchy graphs. This research was largely motivated by the algebraic underpinnings of tropical geometry, which relies on the distributive properties of addition over max or min operations [19]. Distributivity plays a crucial role in the study of nearrings [20] and serves as a fundamental concept in the theory of left-braces, skew-braces, trusses, and related generalizations [10], [9], [16].

In this paper, we build upon the work of Fayoumi [7], who characterized the centers of the semigroup $(\mathscr{M}(S), \triangleleft)$. Specifically, we introduce a commuting map τ_* for each element $* \in \mathscr{M}(S)$ where $\circ \in \tau_*$ implies that $*$ commutes with $\circ$. Our contributions include showing that every center element is a unit in $\mathscr{M}(S)$, that $\langle \tau_* \rangle$ forms a submonoid of $(\mathscr{M}(S), \triangleleft)$, and characterizing the elements of the commuting maps for all commutative binary operations in $\mathscr{M}(S)$.

2. Preliminaries

2.1. Number representation of the operations on a finite set.

Binary operations on a finite set S can be viewed as representations of numbers in base $|S|$, where applying the operation to a specific input pair corresponds to selecting the relevant entry from the operation's representation, as determined by the input pair's positional value. This concept was first introduced in [4]. Consider the set $S = \{0, 1\}$, a representation of a generic two-element set. To illustrate, let's represent addition modulo 2 using a table, but with rows and columns labeled in descending order, contrary to the conventional approach.

+	1	0
1	0	1
0	1	0

Reading the table entries row by row, from top to bottom and left to right, yields the sequence 0110. This sequence corresponds to the binary representation of the decimal number 6.

Example 2.1. Operations in $M(S)$ for $S = \{0, 1\}$. Each operation is named with the number between 0 and 15 whose binary representation is given by the entries of the table read from the top.

0	1	0
1	0	0
0	0	0

1	1	0
1	0	0
0	0	1

2	1	0
1	0	0
0	1	0

3	1	0
1	0	0
0	1	1

4	1	0
1	0	1
0	0	0

5	1	0
1	0	1
0	0	1

6	1	0
1	0	1
0	1	0

7	1	0
1	0	1
0	1	1

8	1	0
1	1	0
0	0	0

9	1	0
1	1	0
0	0	1

10	1	0
1	1	0
0	1	0

11	1	0
1	1	0
0	1	1

12	1	0
1	1	1
0	0	0

13	1	0
1	1	1
0	0	1

14	1	0
1	1	1
0	1	0

15	1	0
1	1	1
0	1	1

Example 2.2. For $n = 3$ operations on the set $S = \{0,1,2\}$ can be numbered between 0 and $3^9 - 1 = 19,682$. First, the three by three table representing operation 58, considering that $(58)_3 = 000002011$, is as follows:

58	2	1	0
2	0	0	0
1	0	0	2
0	0	1	1

8229	2	1	0
2	1	0	2
1	0	2	1
0	2	1	0

On the other hand, addition modulo 3 corresponds to number $8,229$. So the rule $1+2 = 0(\mod 3)$ translates into *the fifth digit of* $8,229$ *in base* 3 *is* 0, while $2+0 = 2(\mod 3)$ holds because the sixth digit in the ternary representation of $8,229$ is 2.

2.2. Units in $(\mathscr{M}(S), \triangleleft)$**.** To deepen our understanding of the magma monoid, we seek to characterize its units in a general framework. To this end, we introduce a crucial function (originally presented in [11]) that plays a pivotal role in this characterization.

Definition 2.3. For each set S with binary operations $*$ and $\circ$ in the magma monoid, define a map $(* \times \circ \zeta)$ that takes a pair of elements (x,y) in $S \times S$ to a new pair $(x * y, y \circ x)$ in $S \times S$

In light of Definition 2.3 $* \lhd \circ$ can be interpreted as the composition of the operations $\circ$ and $* \times *\zeta$.

Theorem 2.4. *Given $* \in \mathscr{M}(S)$, $*$ is a unit in the magma monoid if, and only if the map $* \times *\zeta$ is bijective.*

Proof. See Theorem 3.5 in [3] ☐

Example 2.5. For $n = 4$ the following operations are units in $(\mathscr{M}(S), \lhd)$:

$\star_1$	3	2	1	0
3	2	3	1	3
2	2	1	1	0
1	0	3	0	2
0	0	2	1	3

$\star_2$	3	2	1	0
3	1	3	0	3
2	2	0	3	2
1	1	1	3	1
0	0	0	2	2

$\star_3$	3	2	1	0
3	3	1	0	2
2	3	1	0	2
1	3	1	0	2
0	3	1	0	2

Using the insights from Definition 2.3 and Theorem 2.4, we can now determine the cardinality of the set of units in the magma monoid, denoted by $\mathscr{U}$, as presented in Theorem 2.6.

Theorem 2.6. *For a set S of size n, the cardinality of the set of unit elements in $\mathscr{M}(S)$, $\mathscr{U}$, is given by $\binom{n}{2}! \cdot n! \cdot 2^{\binom{n}{2}}$.*

Example 2.7. For $n = 2$, $|\mathscr{U}| = \binom{2}{2}!2!2^{\binom{2}{2}} = 4$

For $n = 3$, $|\mathscr{U}| = \binom{3}{2}!3!2^{\binom{3}{2}} = 144$

A step-by-step procedure for obtaining the inverses of the units is outlined in [8].

2.3. Constant and Unique Square Binary Operations.

Definition 2.8.
(1) A binary operation $\star$ on a set S is called a unique square (or constant diagonal) operation if, for all elements a and b in S, the result of $a \star a$ is always equal to $b \star b$.
(2) A binary operation $*$ on a set S is commutative if the order of its operands does not change the result, i.e., $a * b = b * a$ for all $a, b \in S$.
(3) For each element i in set S, there exists a binary operation c_i in $\mathscr{M}(S)$, known as the constant operation induced by i, such that for all x, y in S, $c_i(x, y)$ always equals i. The

collection of all constant operations on S is denoted by $\mathscr{K}$ (or $\mathscr{K}(S)$ when specifying the set).

(4) A binary operation $\star \in \mathscr{M}(S)$ is said to be idempotent if $\star \triangleleft \star = \star$ i.e $\star^2 = \star$

Example 2.9. For $|S| = 3$, consider the following three commutative unique square binary operations:

δ_1	2	1	0
2	0	2	0
1	2	0	2
0	0	2	0

δ_2	2	1	0
2	1	0	2
1	0	1	1
0	2	1	1

δ_3	2	1	0
2	1	1	1
1	1	1	1
0	1	1	1

Remark 2.10. *A commutative unique square binary operation, $*$ is idempotent if and only $*$ is a constant operation. From Example 2.9, only the operation δ_3 is idempotent.*

Proposition 2.11. *The following three sets are all ideals in $(\mathscr{M}(S), \triangleleft)$:*

(1) The set Δ of all unique square(constant diagonals) binary operations on S,

(2) The set χ of all commutative binary operations on S,

(3) The set $\mathscr{K}$ of all constant operations on S.

Proof. (1) Suppose $\circ \in \Delta$ satisfies $a \circ a = A$ for all $a \in S$. Then for any $* \in \mathscr{M}(S)$ and $s \in S$, we have:

$$s(* \triangleleft \circ)s = (s * s) \circ (s * s) = A$$

This demonstrates that Δ is a left ideal. Furthermore:

$$s(\circ \triangleleft *)s = (s \circ s) * (s \circ s) = A * A$$

which implies $\circ \triangleleft * \in \Delta$, showing that Δ is also a right ideal.

(2) For every $\circ \in \chi$, $* \in \mathscr{M}(S)$, and $a, b \in S$, we have:

$a(\circ \triangleleft *)b = (a \circ b) * (b \circ a)$

$b(\circ \triangleleft *)a = (b \circ a) * (a \circ b)$

Since $\circ$ is commutative, the right-hand sides coincide, implying that χ is a right ideal of $(\mathscr{M}(S), \triangleleft)$. A similar argument shows that χ is also a left ideal.

(3) The result follows since, for every $* \in \mathscr{M}(S)$, we have:

$$* \lhd c_i = c_i$$

$$c_i \lhd * = c_{i*i}$$

□

The following proposition examines a specific property of constant operations. The ideal characterization of $\mathscr{K}$ established in Proposition 2.11 will be instrumental in this examination.

Proposition 2.12. *Given $\circ \in \mathscr{M}(S)$, $\circ \in \mathscr{K}$ if and only if for all $* \in \mathscr{M}(S)$, $* \lhd \circ = \circ$.*

2.4. Center of the magma monoid, $(\mathscr{M}(S), \lhd)$.

Definition 2.13. The center $\mathscr{Z}(S)$ of $(\mathscr{M}(S), \lhd)$ is defined as the collection of elements $* \in \mathscr{M}(S)$ such that

$$* \lhd \circ = \circ \lhd *$$

for all $\circ \in \mathscr{M}(S)$.

Building on Fayoumi's work in [7] we summarize the key findings on $\mathscr{Z}(S)$ in Theorem 2.14 below.

Theorem 2.14. *Given $* \in (\mathscr{M}(S), \lhd)$ the following conditions hold:*

(1) If $* \in \mathscr{Z}(S)$ then $*(x,x) = x$ for all $x \in S$.

(2) For $x \neq y$ in S, $* \in \mathscr{Z}(S)$ if and only if $*(x,y)$ project onto either the first component (x) or project onto the second component (y).

The elements of $\mathscr{Z}(S)$ are referred to as locally-zero groupoids in [7].

Example 2.15. For $|S| = 4$, the following operations are elements of $\mathscr{Z}(S)$

$*_1$	3	2	1	0
3	3	3	1	3
2	2	2	2	2
1	3	1	1	0
0	0	0	1	0

$*_2$	3	2	1	0
3	3	2	3	3
2	3	2	2	2
1	1	1	1	1
0	0	0	0	0

$*_3$	3	2	1	0
3	3	3	1	3
2	2	2	2	0
1	3	1	1	1
0	0	2	0	0

Remark 2.16. *For $|S| = n$, the number of elements in $\mathscr{Z}(S)$ is $2^{\binom{n}{2}}$.*

Proposition 2.17. *The set $\mathscr{Z}(S)$ is a subset of $\mathscr{U}$ i.e., every center is a unit. Moreover, If $\circ \in \mathscr{Z}(S)$ then $\circ$ is self-invertible.*

Proof. From Definition 2.3 and Theorem 2.4, it is obvious that every element in the center is a unit.

We show that for all $i, j \in S$, $(* \lhd *)(i, j) = \pi_1(i, j)$.

Given $* \in \mathscr{Z}(S)$, let $a, b \in S$ such that $a \neq b$. We consider two cases:

Case One: For $a, b \in S$, let $*(a,b)$ project onto the first element, i.e., $*(a,b) = a$ and $*(b,a) = b$. Then

$$(* \lhd *)(a,b) = *(*(a,b), *(b,a)) = *(a,b) = a$$

and

$$(* \lhd *)(b,a) = *(*(b,a), *(a,b)) = *(b,a) = b$$

Case Two: For $a, b \in S$, let $*(a,b)$ project onto the second element, i.e., $*(a,b) = b$ and $*(b,a) = a$. Then

$$(* \lhd *)(a,b) = *(*(a,b), *(b,a)) = *(b,a) = a$$

and

$$(* \lhd *)(b,a) = *(*(b,a), *(a,b)) = *(a,b) = b$$

Thus, $(* \lhd *)(a,b) = \pi_1(a,b)$. □

3. Main Results

3.1. Commuting Maps.

Definition 3.1. Let $*$ be an element of $\mathscr{M}(S)$. We define a map $\tau_* : \mathscr{M}(S) \to \mathscr{M}(S)$, called the commuting map, such that for any $\circ \in \mathscr{M}(S)$, $\tau_*(\circ) = * \lhd \circ = \circ \lhd *$.

By Definition 3.1, $\langle \pi_* \rangle$ encompasses all binary operations in $\mathscr{M}(S)$ that commute with $*$.

Example 3.2. (1) The subsemigroup generated by τ_{π_1} is equal to $\mathscr{M}(S)$, because π_1 commutes with every element $\circ$ in $\mathscr{M}(S)$, i.e., $\pi_1 \lhd \circ = \circ = \circ \lhd \pi_1$ for all $\circ \in \mathscr{M}(S)$.

(2) We have $\langle \tau_{\pi_2} \rangle = \mathscr{M}(S)$, since π_2 satisfies the property $\pi_2 \triangleleft \circ = \circ^{op} = \circ \triangleleft \pi_2$ for all $\circ \in \mathscr{M}(S)$, where the opposite operation $\circ^{op}$ is defined by $\circ^{op}(a,b) = \circ(b,a)$. This property implies that π_2 acts as an involution on $\mathscr{M}(S)$, and consequently, the subsemigroup generated by τ_{π_2} is equal to $\mathscr{M}(S)$.

(3) If $* \in \mathscr{Z}(S)$ then $\langle \tau_* \rangle = \mathscr{M}(S)$.

(4) Let c_i be a constant binary operation. Then, by Proposition 3.7 the subsemigroup generated by τ_{c_i} is given by:

$$\langle \tau_{c_i} \rangle = \{\circ \in \mathscr{M}(S) : \circ(i,i) = i \text{ for some } i \in S\}.$$

(5) If $* \in \mathscr{M}(S)$ is a commutative binary operation then by Proposition 2.11, $* \triangleleft \circ$ is also commutative; however, $\circ$ might not necessarily be contained within the subsemigroup generated by τ_*.

Proposition 3.3. *Let $\circ_1, \circ_2, * \in \mathscr{M}(S)$. Suppose $\circ_1, \circ_2$ commutes with the operation $*$ i.e. $\circ_1, \circ_2 \in \tau_*$. Then, their composition $\circ_1 \triangleleft \circ_2$ also commutes with $*$.*

Proof. Let $\circ_1, \circ_2$ be elements of the subsemigroup generated by the commuting map of $*$. By definition:

$$* \triangleleft \circ_1 = \circ_1 \triangleleft *, \ * \triangleleft \circ_2 = \circ_2 \triangleleft *.$$

Now we show that $\circ_1 \triangleleft \circ_2$ commutes with $*$:

$$* \triangleleft (\circ_1 \triangleleft \circ_2) = (* \triangleleft \circ_1) \triangleleft \circ_2 \qquad = (\circ_1 \triangleleft *) \triangleleft \circ_2 \ = \circ_1 \triangleleft (* \triangleleft \circ_2) \qquad = (\circ_1 \triangleleft \circ_2) \triangleleft *.$$

Therefore $\circ_1 \triangleleft \circ_2 \in \tau_*$ □

The above proposition implies that the set τ_* is invariant under the operation $\triangleleft$, meaning that it is closed under this operation. Since the identity element π_1 belongs to τ_*, $\langle \tau_* \rangle$ is a submonoid of $(\mathscr{M}(S), \triangleleft)$.

The submonoids generated by τ_* for all binary operations $*$ on the set S with 2 elements are presented in Example 3.4. Notably, the center of the semigroup $(\mathscr{M}(2), \triangleleft)$, contains exactly two operations: **10** and **12**.

Example 3.4. For $S = \{0,1\}$, we present all the subsemigroups generated by the commuting map for each of the elements (See Example 2.1 for the number representations):

(1) $\langle \tau_{10} \rangle = \langle \tau_{12} \rangle = \mathscr{M}(2)$
(2) $\langle \tau_{\mathbf{0}} \rangle = \{$**0, 2, 4, 6, 8, 10, 12, 14** $\}$
(3) $\langle \tau_{\mathbf{1}} \rangle = \{$**1, 10, 12, 14** $\}$
(4) $\langle \tau_{\mathbf{2}} \rangle = \{$**0, 2, 4, 8, 10, 12**$\}$
(5) $\langle \tau_{\mathbf{3}} \rangle = \{$**3, 5, 10, 12** $\}$
(6) $\langle \tau_{\mathbf{4}} \rangle = \{$**0, 2, 4, 8, 10, 12**$\}$
(7) $\langle \tau_{\mathbf{5}} \rangle = \{$**3, 5, 10, 12** $\}$
(8) $\langle \tau_{\mathbf{6}} \rangle = \{$**0, 6, 10, 12** $\}$
(9) $\langle \tau_{\mathbf{7}} \rangle = \{$ **7, 8, 10, 12**$\}$
(10) $\langle \tau_{\mathbf{8}} \rangle = \{$**0, 2, 4, 7, 8, 10, 12** $\}$
(11) $\langle \tau_{\mathbf{9}} \rangle = \{$ **9, 10, 12, 15** $\}$
(12) $\langle \tau_{\mathbf{11}} \rangle = \{$ **10, 11, 12, 13, 14, 15** $\}$
(13) $\langle \tau_{\mathbf{13}} \rangle = \{$ **10, 11, 12, 13, 14, 15** $\}$
(14) $\langle \tau_{\mathbf{14}} \rangle = \{$**0, 1, 11, 13, 14, 15** $\}$
(15) $\langle \tau_{\mathbf{15}} \rangle = \{$**8, 9, 10, 11, 12, 13, 14, 15**$\}$

Proposition 3.5. *Let $\circ_1, \circ_2, * \in \mathscr{M}(S)$. Suppose $\circ_1$ and $\circ_2$ commute with the operation $*$ i.e. $\circ_1, \circ_2 \in \tau_*$. If $*$ is idempotent, then*

$$\tau_*(\circ_1) \triangleleft \tau_*(\circ_2) = \tau_*(\circ_1 \triangleleft \circ_2).$$

Remark 3.6. *If $*$ is a non-center unit element in $\mathscr{M}(S)$, the submonoid $\langle \tau_* \rangle$ may not be contained in $\mathscr{U}$. In particular, if $*(i,i) = i$, then by Proposition 3.7, c_i is in $\langle \tau_* \rangle$. Since c_i is not invertible, $\langle \tau_* \rangle$ cannot be a subgroup of $\mathscr{U}$.*

Proposition 3.7. *Given $* \in \mathscr{M}(S)$, the following statements hold:*

(1) If $* \in \mathscr{M}(S)$ is a constant operation, defined by $*(x,y) = i$ for all $x, y \in S$, then

$$\circ \in \langle \tau_* \rangle \Longleftrightarrow \circ(i,i) = i$$

(2) Two Unique square operations with different diagonal elements do not commute.

Proof. (1) Let c_i be a constant binary operation. Then, the subsemigroup generated by τ_{c_i} is given by:

$$\langle \tau_{c_i} \rangle = \{\circ \in \mathscr{M}(S) : \circ(i,i) = i \text{ for some } i \in S\}.$$

This follows from the fact that for any $\circ \in \mathscr{M}(S)$,

$$c_i \triangleleft \circ(x,y) = \circ(c_i(x,y), c_i(y,x)) = \circ(i,i).$$

Therefore, if $\circ(i,i) = i$, then we have:

$$c_i \triangleleft \circ = c_i = \circ \triangleleft c_i,$$

showing that τ_{c_i} generates the specified subsemigroup.

(2) Given two unique square binary operations $\circ, \star \in \mathscr{M}(S)$. Suppose $\star(i,i) = j$ for all $i \in S$ and $\circ(i,i) = k$ for all $i \in S$ with $j \neq k$. Then

$$(\star \triangleleft \circ)(i,i) = \star(\circ(i,i), \circ(i,i)) = \star(k,k) = j$$

and

$$(\circ \triangleleft \star)(i,i) = \circ(\star(i,i), \star(i,i)) = \circ(j,j) = k$$

Thus, $\star \triangleleft \circ \neq \circ \triangleleft \star$

□

Proposition 3.8. *Let* $* \in \mathscr{M}(S)$. *If* $\circ \in \langle \tau_* \rangle$, *then there are elements* i, j *in* S *for which* $*(i,i) = \circ(j,j)$.

Proof. Given $*, \circ \in \mathscr{M}(S)$, let $\circ \in \langle \tau_* \rangle$. Then, by definition, $* \triangleleft \circ = \circ \triangleleft *$. Assume, for the sake of contradiction, that for all $i, j \in S$, $*(i,i) \neq \circ(j,j)$. Suppose $*(x,x) = j$ and $\circ(x,x) = i$. Then:

$$(* \triangleleft \circ)(x,x) = \circ(*(x,x), *(x,x)) = \circ(j,j)$$

and

$$(\circ \triangleleft *)(x,x) = *(\circ(x,x), \circ(x,x)) = *(i,i)$$

Thus, there is a contradiction since $* \triangleleft \circ = \circ \triangleleft *$

□

Example 3.9. For $|S| = 3$, consider the following operations:

Case I: Non-Commuting Operations.

$*_1$	2	1	0
2	0	0	2
1	0	1	1
0	2	1	1

$*_2$	2	1	0
2	2	0	2
1	0	2	1
0	2	1	2

These operations do not commute with each other.

Case 2: Commuting Operations.

$*_1$	2	1	0
2	2	2	2
1	2	2	2
0	2	2	2

$*_2$	2	1	0
2	2	0	2
1	0	1	1
0	2	1	0

These operations commute with each other.

Case 3: Non-Commuting Operations.

$*_1$	2	1	0
2	0	0	2
1	0	0	1
0	2	1	0

$*_2$	2	1	0
2	1	0	2
1	0	1	1
0	2	1	1

These operations do not commute with each other.

Proposition 3.10. *Let $* \in \mathscr{M}(S)$.Then $*^{op} \in \langle \tau_* \rangle$, for all $* \in \mathscr{M}(S)$, where $*^{op}(a,b) = *(b,a)$ for all $a,b \in S$.*

Proof. Given $*, *^{op} \in \mathscr{M}(S)$, we have for all $a,b \in S$:

$$(* \triangleleft *^{op})(a,b) = *^{op}(*(a,b), *(b,a))$$

$$= *(*(b,a), *(a,b))$$

$$= *(*^{op}(a,b), *^{op}(b,a))$$

$$= (*^{op} \triangleleft *)(a,b)$$

Therefore, $*^{op}$ commutes with $*$. Hence, $*^{op} \in \langle \tau_* \rangle$. □

Proposition 3.11. *Given commutative and unique square binary operations $*_1, *_2 \in \mathscr{M}(S)$, $*_1$ commutes with $*_2$ if and only if $*_1(i,i) = *_2(i,i) = j$ for all $i \in S$.*

Proof. Given commutative and unique square binary operations $*_1, *_2 \in \mathscr{M}(S)$ that commute with each other, i.e., $*_1 \lhd *_2 = *_2 \lhd *_1$. Assume for all $i \in S$, $*_1(i,i) = i$ and $*_2(i,i) = k$ and $i \neq k$. Then:

$$(*_1 \lhd *_2)(i,i) = *_2(*_1(i,i), *_1(i,i)) = *_2(i,i) = k$$

and

$$(*_2 \lhd *_1)(i,i) = *_1(*_2(i,i), *_2(i,i)) = *_1(k,k)$$

Since $*_1 \lhd *_2 = *_2 \lhd *_1$, we have $*_1(k,k) = k$. However, this contradicts the assumption that $*_1(i,i) = i$ for all $i \in S$, since $k \neq i$. □

Proposition 3.12. *Let $*, \circ \in \mathscr{M}(S)$ be commutative binary operations such that $*(i,i) = \circ(i,i) = i$ for all $i \in S$ and $* \neq \circ$. Then, $* \lhd \circ \neq \circ \lhd *$ i.e. $*$ and $\circ$ do not commute with each other.*

Proof. Let $*, \circ \in \mathscr{M}(S)$ be commutative binary operations such that for all $i \in S$ $*(i,i) = \circ(i,i) = i$. Choose $x, y \in S$ such that $*(x,y) = k$ and $\circ(x,y) = t$ and $t \neq k$. Then:

$$(* \lhd \circ)(x,y) = \circ(*(x,y), *(y,x)) = \circ(k,k) = k$$

and

$$(\circ \lhd *)(x,y) = *(\circ(x,y), \circ(x,y)) = *(t,t) = t$$

Since $* \lhd \circ \neq \circ \lhd *$ for $x, y \in S$, the operation $*$ and $\circ$ do not commute. □

4. CONCLUSION

In conclusion, this paper advances our understanding of the semigroup $(\mathscr{M}(S), \lhd)$ by building upon the work of Fayoumi([7]). We have introduced a new map τ_* for each element $* \in \mathscr{M}(S)$ offering new insights into commutativity of elements in $\mathscr{M}(S)$. Our key results show that $\langle \tau_* \rangle$ forms a submonoid of $(\mathscr{M}(S), \lhd)$ and provide a comprehensive characterization of commuting maps for all commutative binary operations in $\mathscr{M}(S)$. These findings enhance our understanding of the algebraic structure of $\mathscr{M}(S)$, laying the groundwork for future research in this area.

Conflict of Interests

The authors declare that there is no conflict of interests.

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