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RESOLUTION OF THE FRACTIONAL DERIVATIVES SEEPAGE FLOW EQUATION IN $(1+N)D$ POROUS MEDIA

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Abstract. The seepage flow equation describes the movement of groundwater through porous media like soil and rock and complexities arise when dealing with non-uniform soil conditions, unsaturated flow, or transient flow situations. In this paper, we use the Adomian decomposition method to solve fractional derivatives seepage flow equation in porous media in $1+N$ dimension. We give a solution as function series and we prove the convergence of the given series. Also, we give some properties of the fractional derivatives and the Adomian decomposition method and highlight the advantages of fractional approaches over classical methods.

Keywords: Adomian decomposition method; fractional differential equations; seepage flow; approximation.

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1. Introduction

The concept of fractional derivatives has been introduced for the first time by Lacroix in 1819 but it gained significant attention in recent years due to its ability to model various physical and engineering phenomena more accurately than classical integer-order derivatives. These

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fractional order calculus appears in fields of diffusion phenomena, combustion process, viscoelasticity and geophysics. In addition, they have applications in electrochemistry, flow of magnetic particles in blood and the formation of suspension in medicine [2, 21].

Since the fractional calculus extends the traditional differentiation and integration to non-integer orders, this provides a new tool for describing memory and hereditary properties of materials and processes, [14]. An important application of fractional derivatives is the study of diffusion, viscoelasticity and fluid dynamics, [8, 15]. In the flow modeling domain, fractional derivatives have been employed to enhance the understanding of seepage flow.

Seepage flow equations describe the process of liquid or gas slowly leaking or flowing through small openings or porous materials. in this paper we will consider the following equation

$$\frac{1}{v}u_t = \sum_{i=1}^{N} D_{x_i}^{\alpha} u \tag{1}$$

which refers to the flow of a fluid through a porous medium. The equation (1) simplifies the problem by considering a specific interaction between water and the porous medium. And so, the seepage flow equation is very important in designing dams, levees and other earthen structures in order to ensure stability and mainly prevent excessive water leakage. In the engineering domain, the seepage flow equations are used in understanding soil consolidation and settlement. The traditional models of seepage flow, based on integer-order derivatives, often fail to detect anomalous transport behaviors in heterogeneous porous media. But by incorporating fractional derivatives, these models can more better describe complex flow dynamics and help to clarify long-range interactions, [4, 16]. The seepage flow equations appear in hydrogeology, petroleum engineering, and environmental sciences [3]. They describe the movement of fluids through porous media and the influencing groundwater management. They also govern oil recovery and contaminant transport [24]. It is pointed out, in [25], that the inclusion of fractional derivatives in the seepage flow equation provides a refined framework for analyzing non-Fickian transport and anomalous diffusion patterns. These lead to more accurate predictions and improve decision-making in practical applications [25].

The resolution of the seepage flow equation attract many researchers and has evolved through

two approaches, analytical and numerical. Adding fractional derivatives to seepage flow equations has been extensively studied in order to find analytical solutions. In [12], the author develop an approximate analytical solution for fractional derivatives seepage flow in porous media in $2D$ using variational iteration method. Later, the same equation in $2D$ was studied in [10] and, in [18], in $4D$ porous media by the Adomian decomposed method (ADM). In addition, in [26], the authors proposed a nonlinear model of water flux and hydraulic gradient using fractional derivatives, offering an analytic solution that aligns closely with experimental data for flow in low-permeability media. These studies collectively underscore the significance of fractional derivatives in accurately modeling seepage flow across various spatial dimensions.
In this paper, we give an approximating analytical solution of the fractional derivatives seepage flow equation in $(1+N)D$ using (ADM) and highlight its methodology. More precisely, in section 2 we give some definitions and results of fractional derivatives. Section 3 is devoted to explain the methodology of the Adomian decomposition method. In section 4, the Analytical solution of the seepage flow equation in higher dimension is given.

2. Preliminaries

In this section, we give some definitions and properties of the fractional derivatives, for more details we can refer to [6].
In 1819, Lacroix induce and define the fraction derivative of the function $\phi(\zeta) = \zeta^\beta$ as

$$D^\alpha \phi(\zeta) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \zeta^{\beta-\alpha}$$

where β an α are fractional numbers and the function Γ is given by

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt. \tag{2}$$

Later, in 1832, Liouville set

$$D^\alpha e^{\beta\zeta} = \beta^\alpha e^{\beta\zeta} \tag{3}$$

as the extension to the natural derivative of the exponential function. When a function f has a series expansion

$$f(\zeta) = \sum_{i=0}^{\infty} c_i e^{\beta_i \zeta},$$

Liouville define

(4) $$D^{\alpha} f(\zeta) = \sum_{i=0}^{\infty} c_i \beta_i^{\alpha} e^{\beta_i \zeta}.$$

The equation (4) is called the first formula of Liouville. In addition, he extend it to define the fractional derivative of order α to the function $g(\zeta) = \zeta^{-\beta}$, $\zeta > 0$, for a positive fractional number β, in the following way. From (2), we have

$$\Gamma(\beta)\zeta^{-\beta} = \int_0^{\infty} \zeta^{-\beta} t^{\beta-1} e^{-t} dt.$$

Therefore, by setting $u = t\zeta^{-1}$ we obtain

(5) $$\Gamma(\beta)\zeta^{-\beta} = \int_0^{\infty} u^{\beta-1} e^{-u\zeta} du.$$

Hence,

$$\Gamma(\beta) D^{\alpha} \zeta^{-\beta} = \int_0^{\infty} u^{\beta-1} D^{\alpha}\left(e^{-u\zeta}\right) du.$$

Then, from (3) we have

$$\begin{aligned}\Gamma(\beta) D^{\alpha} \zeta^{-\beta} &= \int_0^{\infty} u^{\beta-1} (-1)^{\alpha} u^{\alpha} e^{-u\zeta} du \\ &= (-1)^{\alpha} \int_0^{\infty} u^{\beta+\alpha-1} e^{-u\zeta} du.\end{aligned}$$

As in the equation (5), we get

$$\Gamma(\beta) D^{\alpha} \zeta^{-\beta} = (-1)^{\alpha} \Gamma(\beta+\alpha) \zeta^{-\beta-\alpha}.$$

We deduce that

(6) $$D^{\alpha} \zeta^{-\beta} = (-1)^{\alpha} \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \zeta^{-\beta-\alpha}.$$

The equation (6) is called the second formula of Liouville.

In [5, 7, 12], we find an inverse of the Liouville's derivatives, the Riemann-Liouville's fractional integrate for a function F given by

$$I^{\alpha} F(\zeta) = \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta - t)^{\alpha-1} F(t) dt.$$

Next, we use the fact that for $\alpha > 0$ and $\lambda > -1$, we have

(7) $$I^{\alpha} \zeta^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} \zeta^{\lambda+\alpha}$$

3. The Adomian Decomposition Method

The Adomian method was introduced for the first time, in [1], to solve Burgers' equation in $(1+1)$ dimension. In this section, we will explain the methodology of the method in order to investigate it in the next section and solve the seepage flow fractional partial differential equation in $(1+n)$ dimension, $n \geq 1$. We can refer to [9, 19, 20, 23] to more details.
For a partial differential equation of the form

$$\mathscr{N}(u) = f, \tag{8}$$

we first decompose it in the form

$$\mathscr{L}(u) + \mathscr{R}(u) = f, \tag{9}$$

where $\mathscr{L}$ is the linear part of the operator $\mathscr{N}$. We suppose the $\mathscr{L} \neq 0$. Let l a term of the linear operator $\mathscr{L}$, the lower order derivative term for example, and denote l^{-1} its inverse. So, the equation (9) can be written as

$$l(u) + H(u) = f. \tag{10}$$

By using the initial and the boundary conditions of the given problem (8), the equation (10) is simplified under the form

$$u = h - l^{-1}(H(u)). \tag{11}$$

Now, if we set

$$u(\zeta) = \sum_{k=0}^{\infty} u_k(\zeta), \tag{12}$$

the equation (11) gives

$$\sum_{k=0}^{\infty} u_k(\zeta) = f(\zeta) - l^{-1}\Big(H(\sum_{k=0}^{\infty} u_n(\zeta))\Big). \tag{13}$$

If we have from the initial conditions that $u_0(\zeta) = f(\zeta)$, we can determine the components $\{u_k\}_{k\geq 1}$ by the formula

$$u_k = -l^{-1}(H(u_{k-1})) \tag{14}$$

4. Analytical Solution of the Seepage Flow Equation

In this paragraph, we will solve the following fractional seepage flow equation

$$\frac{1}{\nu}\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial^{\alpha} u}{\partial x_i^{\alpha}} \tag{15}$$

where ν is a positive parameter, $\alpha \in (0,1)$ and $u = u(t,x) = u(t,x_1,\dots,x_N)$ a reel function depending on $(1+N)$ variables. If $x = (x_1,x_2,\dots,x_N)$, we denote
$\widehat{x}^i = (\widehat{x}_1,\widehat{x}_2,\dots,\widehat{x}_N)$ such that $\widehat{x}_j = x_j$ for all $j \neq i$ and $\widehat{x}_i = 0$. We want to find a solution of the equation (15) satisfying

$$u(0,x) = 1 + \sum_{j=1}^{N} e^{x_j} \tag{16}$$

$$u(t,\widehat{x}_i) = e^t + \sum_{j=1}^{N} e^{\widehat{x}_j} \tag{17}$$

$$u_0(t,x) = e^t + 1 + \sum_{j=2}^{N} e^{x_j}. \tag{18}$$

Let us choose $l = D_{x_1}^{\alpha} = \dfrac{\partial^{\alpha}}{\partial x_i^{\alpha}}$. The equation (15) is equivalent to

$$l(u) = \frac{1}{\nu} D_t u - \sum_{i=2}^{n} D_{x_i}^{\alpha} u \tag{19}$$

The inverse of the operator l is given by (7) and so,

$$u = u_0 + I_{x_1}^{\alpha}\Big(\frac{1}{\nu} D_t u - \sum_{i=2}^{n} D_{x_i}^{\alpha} u\Big).$$

Now, we set

$$u(t,x) = \sum_{k=0}^{\infty} u_k(t,x)$$

and for $k \geq 0$,

$$u_{k+1}(t,x) = I_{x_1}^{\alpha}\Big(\frac{1}{\nu} D_t u_k(t,x) - \sum_{i=2}^{n} D_{x_i}^{\alpha} u_k(t,x)\Big).$$

More precisely,

$$u_0(t,x) = e^t + 1 + \sum_{j=2}^{N} e^{x_j},$$

$$u_1(t,x) = I_{x_1}^{\alpha}\Big(\frac{1}{\nu} D_t u_0(t,x) - \sum_{i=2}^{n} D_{x_i}^{\alpha} u_0(t,x)\Big)$$

and so

$$u_1(t,x) = I_{x_1}^{\alpha}\Big(\frac{1}{\nu}e^t - \sum_{i=2}^{n} e^{x_i}\Big).$$

Therefore, from (7) we get

$$u_1(t,x) = \frac{1}{\nu}\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}e^t - \frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=2}^{n} e^{x_i}.$$

Now, since $I_{x_1}^{\alpha} x_1^{\alpha} = \dfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} x_1^{2\alpha}$, we have

$$\begin{aligned} u_2(t,x) &= I_{x_1}^{\alpha}\Big(\frac{1}{\nu}\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}e^t - \frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=2}^{n} e^{x_i}\Big) \\ &= \frac{1}{\nu}\frac{x_1^{2\alpha}}{\Gamma(2\alpha+1)}e^t - \frac{x_1^{2\alpha}}{\Gamma(2\alpha+1)}\sum_{i=2}^{n} e^{x_i}. \end{aligned}$$

By recurrence, we have

$$u_k(t,x) = \frac{1}{\nu}\frac{x_1^{k\alpha}}{\Gamma(k\alpha+1)}e^t - \frac{x_1^{k\alpha}}{\Gamma(k\alpha+1)}\sum_{i=2}^{n} e^{x_i}.$$

We get

$$\begin{aligned} u(t,x) =& 1 + e^t + \sum_{j=2}^{N} e^{x_j} \\ &+ \sum_{k=1}^{\infty} \frac{x_1^{k\alpha}}{\Gamma(k\alpha+1)}\Big(\frac{1}{\nu}e^t - \sum_{j=2}^{N} e^{x_j}\Big). \end{aligned} \tag{20}$$

That is,

$$\begin{aligned} u(t,x) =& 1 + \Big(1 + \frac{1}{\nu}\sum_{k=1}^{\infty}\frac{x_1^{k\alpha}}{\Gamma(k\alpha+1)}\Big)e^t \\ &+ \Big(\sum_{k=1}^{\infty}\frac{x_1^{k\alpha}}{\Gamma(k\alpha+1)}\Big)\sum_{j=2}^{N} e^{x_j}. \end{aligned} \tag{21}$$

We remark that by taking $\alpha = 1$, we find the exact solution

$$u(t,x) = 1 + \frac{1}{\nu}(\nu - 1 + e^{x_1})e^t + \sum_{j=2}^{N} e^{x_j + x_1}.$$

5. Conclusion

In this paper, we investigate the Adomian decomposition method (ADM) to solve the fractional seepage flow partial differential equation in $(1+N)$ dimension, $N \geq 1$. This method is very efficient and powerful analytical technique to solve linear and nonlinear differential and fractional differential equations. The (ADM) is particularly useful for handling equations where traditional numerical methods may be complex or inefficient. This method consists to decompose the solution of a differential equation into an infinite series and determines each term of the series without requiring linearization or perturbation. We can think to use it to integral equations and make some comparison with other numerical methods as transform Method and variational iteration method [22] or use the (ADM) method for different type of fractional differential equations as in [11] guiding with the important applications and uses of fractional differential equations in real engineering field, [13].

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] G. Adomian, Solutions of nonlinear P.D.E., Appl. Math. Lett. 11 (1998), 121–123.

[2] M. Ahmad, M.A. Imran and M. Nazar, Mathematical modeling of $(Cu-Al_2O_3)$ water based Maxwell hybrid nanofluids with Caputo-Fabrizio fractional derivative, Adv. Mech. Eng. 12 (2020), 1–11.

[3] J. Bear, Dynamics of fluids in porous media, American Elsevier Publishing Company, New York, 1972.

[4] D.A. Benson, R. Schumer, M.M. Meerschaert et al., Fractional Dispersion, Lévy Motion, and the MADE Tracer Tests, Transp. Porous Media 42 (2001), 211–240.

[5] L.M.B.C. Campos, On the solution of some simple fractional differential equations, Int. J. Math. Math. Sci. 13 (1990), 481–496.

[6] L. Debnath and D. Bhatta, Integral transforms and their applications, 3rd ed., CRC Press, Taylor and Francis Group, 2015.

[7] D. Delbosco, Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204 (1996), 609–612.

[8] K. Diethelm, The analysis of fractional differential equations, an application-oriented exposition using differential operators of Caputo type, Springer, 2004.

[9] J.S. Duan, R. Rach, D. Băleanu and A.M. Wazwaz, A review of the Adomian decomposition method and its applications to fractional differential equations, Commun. Frac. Calc. 3 (2012), 73–99.

[10] F.A. Faisal and M.A. Bashir, Application of Adomian Decomposition method to seepage flow derivatives in porous media using fractional calculus, Int. J. Sci. Res. Publ. 5 (2015), 1–6.

[11] P. Guo, The Adomian decomposition method for a type of fractional differential equations, J. Appl. Math. Phys. 7 (2019), 2459–2466.

[12] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Eng. 167 (1998), 57–68.

[13] C. Jadhav, T. Dale and S. Dhondge, A review on applications of fractional differential equations in engineering domain, Stat. Eng. Appl. 71 (2022), 7147–7166.

[14] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential equations, North-Holland Math. Stud. 204, Elsevier, 2006.

[15] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, 2010.

[16] R. Metzler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A Math. Gen. 37 (2004), 161–208.

[17] C. Milici, Introduction to differential equation and fractional equations, Timisoara, 2016.

[18] S.M. Mirgani, Comparative analysis of analytical solutions for seepage flow derivatives in 4D porous media, Eur. J. Pure Appl. Math. 18 (2025), 5685.

[19] S. Mungkasi and W. Ekaputra, Adomian decomposition method for solving initial value problems in vibration models, MATEC Web Conf. 159 (2018), 02007.

[20] A.A. Okeke, P. Tumba and J.J. Gambo, The use of Adomian decomposition method in solving second order autonomous and non-autonomous ordinary differential equations, Int. J. Math. Stat. Invention 7 (2019), 91–97.

[21] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.

[22] U.K. Shaun, T. Akter, M.R. Akter, S. Afroz and M.M. Miah, Numerical solution of composite fractional oscillation equation by fractional differential transform method and variational iteration method, MBSTU J. Sci. Technol. 10 (2024), 33–39.

[23] I. Sumiati, Sukono and A.T. Bon, Adomian decomposition method and the new integral transform, in: Proceedings of the 2nd African International Conference on Industrial Engineering and Operations Management, Harare, Zimbabwe, December 7-10, 2020.

[24] Y. Zhang, H. Sun and W. Chen, A review of time-space fractional diffusive transport: theories, modeling, and applications, Sci. China Technol. Sci. 58 (2015), 762–774.

[25] X. Zhang and F. Liu, A new fractional derivative model for simulating non-Fickian solute transport in porous media, Adv. Water Resour. 111 (2018), 328–340.

[26] H.W. Zhou, S. Yang, R. Wang and J.C. Zhong, Non-Darcian flow or fractional derivative?, arXiv:1806.00977, 2018.