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FUZZY GRADE *L*-CORDIAL LABELING OF GRAPHS

R. PONRAJ[1,*], A. SUBRAMANIAN[1,†], A.M.S. RAMASAMY[2]

[1] Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627412, Tamilnadu, India (Affiliated to Manonmaniam Sundaranar University, Tirunelveli– 627 012, Tamilnadu, India)

[2] Department of Mathematics, Pondicherry University, Pondicherry-605 014, India

Abstract. In this paper we introduce a new labeling called fuzzy grade L-cordial labeling of graphs. The results of our investigation concerning fuzzy grade L-cordial labeling of star, bistar, path, cycle, complete, wheel and $K_2 + mK_1$ graphs for certain grade are presented.

Keywords: path; cycle; star; bistar; complete graph; wheel graph.

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1. Introduction

In this paper we consider a finite, simple and undirected connected graph. The number of vertices of a graph G is called the order of G and it is denoted by p. The number of edges of a graph G is called the size of G and it is denoted by q. The first research paper on graph theory was published by Leonhard Euler. The concept of graph labeling was introduced by Rosa [14]. Attention was focused upon graceful related labeling of certain graphs in [2, 3, 18]. For a detailed survey on graph labeling, we refer the book of Gallian [6]. Lucky edge labeling of new graphs

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[†]Research Scholar, Reg. No. 241212312042.

[*]Corresponding author

E-mail address: ponrajmaths@gmail.com

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was discussed by Nagarajan and G. Priyadharsini in [11]. The central theme in [10] was the examination of square difference labeling of some special graphs. The notion of cordial labeling was first introduced by Cahit [5]. Abdel et. al [1] carried out a study on certain varieties of equivalent cordial labeling of graphs. Vector basis (1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)-cordial labeling was discussed in [12] in respect of generalized friendship graph, tadpole graph and gear graph. Cordial related labeling was studied in [4, 13, 16, 17] by considering various types of graphs. For the general terminology and standard notations of graphs, we follow Harary [9], Klir and Folger [8] and Klir and Yuan [7].

In the recent times, new techniques are developed for the labeling of the vertices and edges of a graph. In this paper we provide a new labeling technique by using a fuzzy grade set. The concept of fuzzy set was introduced by Zadeh [19] in the year 1965. It finds applications in several diversified areas such as decision making, pattern recognition, image processing, anti-skid braking system, transmission system, etc. A fuzzy set is a pair (X,μ) where X is a universal set and $\mu : X \to [0,1]$ is a membership function taking values on a closed, real interval. Motivated by the pioneering work of Zadeh, we introduce a new graph labeling method employing a grade set of fuzzy set and investigate some graphs such as star, bistar, path, cycle, complete, wheel and $K_2 + mK_1$ for this labeling technique.

2. FUZZY GRADE L-CORDIAL GRAPHS

Definition 2.1. *Let $G = (V,E)$ be a graph. Choose a set $L \subseteq [0,1]$. L is called a grade set. Define $f : V(G) \to L$. For each edge uv of G, assign the grade $f(u)+ f(v)$ if $f(u)+ f(v) \leq 1$ and $(f(u)+ f(v))-1$ if $f(u)+ f(v) > 1$. An edge of G with grade greater than or equal to $\frac{1}{2}$ is called the satisfactory edge and the number of satisfactory edges is denoted by Ψ_{fs}. An edge with grade less than $\frac{1}{2}$ is called the unsatisfactory edge and the number of unsatisfactory edges is denoted by Ψ_{fus}. Then f is called a fuzzy grade L-cordial labeling of G if $|V_i(f)-V_j(f)| \leq 1$, where $i, j \in L$. $V_x(f)$ denotes the number of vertices with grade x and $|\Psi_{fs} - \Psi_{fus}| \leq 1$. A graph G which admits a fuzzy grade L-cordial labeling is called a fuzzy grade L-cordial graph.*

Remark 2.2. *A unit fraction is of the form $\frac{1}{n}$ where n is a natural number. Using unit fractions, we construct an example presented below.*

Example 2.3. *A fuzzy grade* $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$*-cordial labeling is given in Figure 1.*

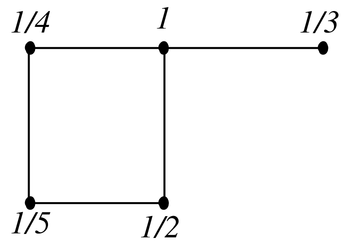

Figure 1

Here $V_1(f) = 1, V_{\frac{1}{2}}(f) = 1, V_{\frac{1}{3}}(f) = 1, V_{\frac{1}{4}}(f) = 1$ *and* $V_{\frac{1}{5}}(f) = 1$.
On computation, we obtain $\Psi_{fs} = 2$ *and* $\Psi_{fus} = 3$. *Hence the above graph G is a fuzzy grade* $L = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$*-cordial graph.*

3. Fuzzy Grade $L = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{p}\}$

Consider the fuzzy grade $L = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{p}\}$, where p is the order of G. With respect to the grade set L, we establish certain results below.

Theorem 3.1. *The star* $K_{1,n}$ *is a fuzzy grade L-Cordial if and only if* $n \leq 5$.

Proof. Let $K_{1,n}$ be the star graph. Let $V(K_{1,n}) = \{u, u_i \mid 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uu_i \mid 1 \leq i \leq n\}$. Then $K_{1,n}$ has $n+1$ vertices and n edges.

Case (i): $n \leq 5$

A fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+1}\}$-cordial labeling of $K_{1,n}$ is given in the following Table 1.

n	u	u_1	u_2	u_3	u_4	u_5
1	1	$\frac{1}{2}$				
2	1	$\frac{1}{3}$	$\frac{1}{2}$			
3	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$		
4	$\frac{1}{5}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	
5	$\frac{1}{6}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

Table 1

Case (ii): $n \geq 6$

Suppose f is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n+1}\}$-cordial labeling for $K_{1,n}$.

When $f(u) = 1$, we have $\Psi_{fs} = 1$ and $\Psi_{fus} = n-1$.

Hence $|\Psi_{fs} - \Psi_{fus}| = |2-n| > 1$ since $n \geq 6$.

When $f(u) = \frac{1}{2}$, we get $\Psi_{fs} = n$ and $\Psi_{fus} = 0$.

Therefore $|\Psi_{fs} - \Psi_{fus}| = |n| > 1$.

When $f(u) = \frac{1}{3}$, we obtain $\Psi_{fs} = 4$ and $\Psi_{fus} = n-4$.

So $|\Psi_{fs} - \Psi_{fus}| = |8-n| > 1$.

When $f(u) = \frac{1}{4}$, we have $\Psi_{fs} = 2$ and $\Psi_{fus} = n-2$.

Consequently $|\Psi_{fs} - \Psi_{fus}| = |4-n| > 1$.

Now

$$\begin{aligned} \frac{1}{m} + \frac{1}{n} < \frac{1}{2} &\Leftrightarrow \frac{m+n}{mn} < \frac{1}{2} \\ &\Leftrightarrow 2(m+n) < mn \end{aligned} \tag{1}$$

From (1), it follows that the sum of any two numbers in the set $\{\frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{p}\}$ is always $< \frac{1}{2}$. Therefore $\Psi_{fus} \geq n = 3$, which is a contradiction. Hence the theorem. □

Theorem 3.2. *The Bistar graph $B_{n,n}$ is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2n+2}\}$-cordial for all n.*

Proof. Let $B_{n,n}$ be a bistar graph. Let $V(B_{n,n}) = \{u, v, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i \mid 1 \leq i \leq n\}$. Then $B_{n,n}$ has $2n+2$ vertices and $2n+1$ edges.
Assign the grade $\frac{1}{2}$ and $\frac{1}{2n+2}$ to the vertices u and v respectively. Next assign the grade $\frac{1}{2n+1}, \frac{1}{2n}, \ldots, \frac{1}{n+2}$ to the vertices $v_1, v_2, \ldots, v_n$ respectively. Finally assign the grade $1, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$ to the vertices $u_1, u_2, \ldots, u_n$ respectively. Then $\Psi_{fs} = n+1$ and $\Psi_{fus} = n$. Therefore f is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2n+2}\}$-cordial labeling of $B_{n,n}$. □

Theorem 3.3. *The path P_n is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$-cordial if and only if $n \leq 10$.*

Proof. Let P_n be the path $u_1, u_2, \ldots, u_n$.

Case (i): $n \leq 10$

A fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$-cordial labeling of P_n is given in the following Table 2.

n	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
1	1									
2	1	$\frac{1}{2}$								
3	$\frac{1}{2}$	1	$\frac{1}{3}$							
4	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$						
5	1	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$					
6	1	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$				
7	1	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$			
8	1	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{8}$		
9	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{9}$	
10	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{10}$

Table 2

Case (ii): $n > 10$

Suppose f is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$-cordial labeling for P_n.

It is seen that the maximum number of satisfactory edges contributed from vertex grade $\frac{1}{2}$ is 2. Subsequently the maximum number of satisfactory edges contributed from vertex grade $\frac{1}{3}$ is 2 (since the possible options are $\frac{1}{3}+\frac{1}{4}$, $\frac{1}{3}+\frac{1}{5}$ and $\frac{1}{3}+\frac{1}{6}$). As in Theorem 3.1. the sum of any numbers in the set $\{\frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}\}$ is always less than $\frac{1}{2}$. Hence the maximum number of satisfactory edges is 4. That is $\Psi_{fs} = 4$. Next we note that the maximum number of unsatisfactory edges in the path P_n is $(n-1)-4 = n-5$. That is $\Psi_{fus} = n-5$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |4-(n-5)| = |1-n| > 1$ (since $n > 10$), which is a contradiction. Hence the theorem. □

Theorem 3.4. *The cycle C_n is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$-cordial if and only if $n \leq 9$.*

Proof. Let C_n be the cycle $u_1u_2\dots u_nu_1$. The cycles C_n, $n = 3,5,6,7,8,9$ are fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$-cordial as in Theorem 3.3. It is seen that C_4 is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$-cordial as indicated by Figure 2.

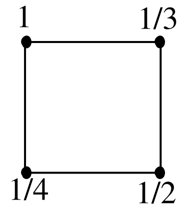

Figure 2

Next assume that $n > 9$. As in Theorem 3.3, we have $\Psi_{fs} = 4$ and $\Psi_{fus} = n-4$.
Therefore $|\Psi_{fs} - \Psi_{fus}| = |4-(n-4)| = |8-n| > 1$ (since $n > 9$), which leads to a contradiction. Hence the theorem. □

Theorem 3.5. *The complete graph K_n is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\}$-cordial if and only if $n \leq 3$.*

Proof. Clearly K_1 is a fuzzy grade $\{1\}$-cordial graph, K_2 is a fuzzy grade $\{1, \frac{1}{2}\}$-cordial graph by Theorem 3.3. and K_3 is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial graph by Theorem 3.4. Next assume that $n > 3$. The edges with vertex have grade 1 contribute one satisfactory edge. An edge for which one vertex has grade $\frac{1}{2}$ and the grade of another vertex is not equal to 1 contributes $n-2$ satisfactory edges. An edge with one vertex having grade $\frac{1}{3}$ and the other vertex having a grade not equal to 1 or $\frac{1}{2}$ contributes 3 satisfactory edges, the possible options being $\frac{1}{3}+\frac{1}{4}$, $\frac{1}{3}+\frac{1}{5}$ and $\frac{1}{3}+\frac{1}{6}$. We see that the edges with grades $\{\frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}\}$ are unsatisfactory edges. Consequently we have $\Psi_{fs} = 1+n-2+3 = n+2$ and $\Psi_{fus} = \binom{n}{2} - (n+2)$. Therefore

$$\begin{aligned}
|\Psi_{fs} - \Psi_{fus}| &= \left| n+2-\binom{n}{2}+(n+2) \right| \\
&= \left| 2n+4-\frac{n(n-1)}{2} \right| \\
&= \left| \frac{4n+8-n^2+n}{2} \right| \\
&= \left| \frac{5n-n^2+8}{2} \right| > 1 \quad \text{(since } n > 3\text{)}
\end{aligned}$$

This leads to a contradiction. Hence the theorem. □

Theorem 3.6. *The graph $K_2 + mK_1$ is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m+2}\}$-cordial for all m.*

Proof. Let $V(K_2) = \{u, v\}$ and $V(mK_1) = \{w_1, w_2, \ldots, w_m\}$. Assign the grade $\frac{1}{2}, 1$ respectively to the vertices u, v and then assign the grade $\{\frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{m+2}\}$ to the remaining vertices

$w_1, w_2, \ldots, w_m$. On compuation we get $m+1$ satisfactory edges and m unsatisfactory edges. Therefore $|\Psi_{fs} - \Psi_{fus}| = 1$. Hence the theorem. □

Theorem 3.7. *The wheel graph W_n is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}\}$-cordial if and only if $n \neq 3$.*

Proof. Let $W_n = C_n + K_1$. Let C_n be the cycle $u_1u_2 \ldots u_nu_1$ and $V(K_1) = \{u\}$. Since $W_3 = C_3 + K_1 \simeq K_4$, the result is following from Theorem 3.5.

Case (i): $n = 4, 5$

Clearly W_4 is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$-cordial from the figure 3 and W_5 is a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$-cordial graphs from the figure 4.

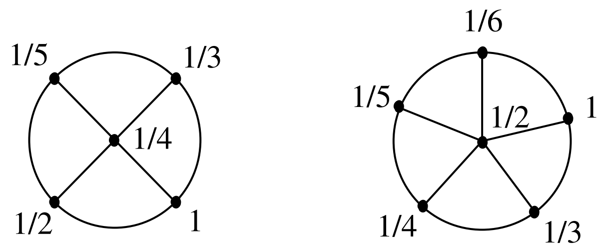

Figure 3 Figure 4

Case (ii): $n \geq 6$

Assume the grade $\frac{1}{2}$ to the vertex u. Next assign the grade $\frac{1}{n+1}, \frac{1}{3}, \frac{1}{n}, \frac{1}{4}, \frac{1}{n-1}, \frac{1}{5}, \ldots, 1$ respectively to the remaining vertices $u_1, u_2, \ldots, u_n$. Clearly $\Psi_{fs} = \Psi_{fus} = n$. Hence the theorem. □

4. Fuzzy Grade $L = \{1\}$

Observation 4.1. *There do not exist fuzzy grade $\{1\}$-cordial graphs other than K_1 (Since $\Psi_{fs} = q$ and $\Psi_{fus} = 0$).*

5. Fuzzy Grade $L = \{1, \frac{1}{2}\}$

Observation 5.1. *There do not exist fuzzy grade $\{1, \frac{1}{2}\}$-cordial graphs except K_1 and K_2.*

6. Fuzzy Grade $L = \{1, \frac{1}{2}, \frac{1}{3}\}$

Theorem 6.1. *The star $K_{1,n}$ is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial if and only if $n \in \{1, 2, 3, 4, 5, 6, 7, 9\}$.*

Proof. Let $K_{1,n}$ be the star graph. Let $V(K_{1,n}) = \{u, u_i \mid 1 \le i \le n\}$ and $E(K_{1,n}) = \{uu_i \mid 1 \le i \le n\}$. Then $K_{1,n}$ has $n+1$ vertices and n edges. A fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial labeling of $K_{1,n}$, $n \in \{1,2,3,4,5,6,7,9\}$ is given in the Table 3.

n	u	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	1	$\frac{1}{2}$								
2	1	$\frac{1}{2}$	$\frac{1}{3}$							
3	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$						
4	$\frac{1}{3}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$					
5	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$				
6	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
7	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		
9	1	1	1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 3

Assume that $n > 7$ and $n \neq 9$.

Case (i): $n \equiv 0 \pmod 3$

Let $n = 3t$, $t > 0$.

Subcase (i): $f(u) = 1$

When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t$, then $\Psi_{fs} = 2t$ and $\Psi_{fus} = t$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |t| > 1$.

Subcase (ii): $f(u) = \frac{1}{3}$

When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t$, then $\Psi_{fs} = 2t$ and $\Psi_{fus} = t$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |t| > 1$.

Case (ii): $n \equiv 1 \pmod 3$

Let $n = 3t+1$, $t > 2$.

Subcase (i): $f(u) = 1$

When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t+1$, then $\Psi_{fs} = t-1+t+1 = 2t$ and $\Psi_{fus} = t+1$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |2t-(t+1)| = |t-1| > 1$.

When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t+1$, then $\Psi_{fs} = t+t = 2t$ and $\Psi_{fus} = t+1$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |2t-(t+1)| = |t-1| > 1$.

When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t$, then $\Psi_{fs} = t+t+1 = 2t+1$ and $\Psi_{fus} = t$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |2t+1-(t)| = |t+1| > 1$.

Subcase (ii): $f(u) = \frac{1}{3}$

When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t$, then $\Psi_{fs} = t+1+t-1 = 2t$ and $\Psi_{fus} = t+1$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |2t-(t+1)| = |t-1| > 1$.

When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t+1$, then $\Psi_{fs} = t+t = 2t$ and $\Psi_{fus} = t+1$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |2t-(t+1)| = |t-1| > 1$.

When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t+1$, then $\Psi_{fs} = t+t+1 = 2t+1$ and $\Psi_{fus} = t-1$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |2t+1-(t-1)| = |t+2| > 1$.

Case (iii): $n \equiv 2 \pmod 3$

When $f(u) = \frac{1}{2}$, then clearly $\Psi_{fs} = n$ and $\Psi_{fus} = 0$. Therefore $|\Psi_{fs} - \Psi_{fus}| = |n| > 1$.

Hence the theorem. □

Theorem 6.2. *The complete graph K_n is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial if and only if $n \leq 3$.*

Proof. Let K_n be the complete graph. It is seen that K_1, K_2 and K_3 are fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial graphs as evidenced from Figures 5, 6 and 7 respectively.

Figure 5

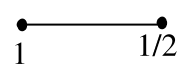

Figure 6

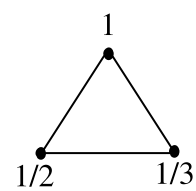

Figure 7

Now let us deal with the situation of $n > 3$. We have to consider three cases as described below.

Case (i): $n \equiv 0 \pmod 3$

Let $n = 3t$, $t > 1$.

Assign the grades $1, \frac{1}{2}$ and $\frac{1}{3}$ to t vertices each. Then $\Psi_{fs} = \binom{t}{2} + \binom{t}{2} + \binom{t}{2} + t^2 + t^2 = \frac{3t(t-1)}{2} + 2t^2$. Hence we obtain $\Psi_{fus} = t^2$.

$$|\Psi_{fs} - \Psi_{fus}| = \left|\frac{3t^2 - 3t + 2t^2}{2}\right|$$
$$= \left|\frac{5t^2 - 3t}{2}\right| > 1$$

Case (ii): $n \equiv 1 \pmod 3$

Let $n = 3t+1$, $t > 1$. In this case the following types are occurring.

Type (I): When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t+1$.

Then $\Psi_{fs} = \binom{t}{2} + \binom{t+1}{2} + \binom{t}{2} + t^2 + t(t+1) = t(t-1) + \frac{t(t+1)}{2} + 2t^2 + t$. Therefore $\Psi_{fus} = t^2 + t$.

$$|\Psi_{fs} - \Psi_{fus}| = \left| \frac{2t^2 - 2t + t^2 + t + 2t^2}{2} \right|$$
$$= \left| \frac{5t^2 - t}{2} \right| > 1$$

Type (II): When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t$.

Then $\Psi_{fs} = \binom{t}{2} + \binom{t+1}{2} + \binom{t}{2} + t^2 + t(t+1) = t(t-1) + \frac{t(t+1)}{2} + 2t^2 + t$. So $\Psi_{fus} = t^2 + t$.

$$|\Psi_{fs} - \Psi_{fus}| = \left| \frac{2t^2 - 2t + t^2 + t + 2t^2}{2} \right|$$
$$= \left| \frac{5t^2 - t}{2} \right| > 1$$

Type (III): When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t$.

Then $\Psi_{fs} = \binom{t}{2} + \binom{t}{2} + \binom{t+1}{2} + t^2 + t + t^2 + t = t(t-1) + \frac{t(t+1)}{2} + 2t^2 + 2t$. Hence $\Psi_{fus} = t^2$.

$$|\Psi_{fs} - \Psi_{fus}| = \left| \frac{2t^2 - 2t + t^2 + t + 2t^2 + 4t}{2} \right|$$
$$= \left| \frac{5t^2 + 3t}{2} \right| > 1$$

Case (iii): $n \equiv 2 \pmod 3$

Let $n = 3t+2$, $t > 1$. In this case the following three types occur.

Type (I): When $V_1(f) = t$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t+1$.

Then $\Psi_{fs} = 2\binom{t+1}{2} + \binom{t+1}{2} + t^2 + t + (t+1)^2 = t(t+1) + \frac{t(t-1)}{2} + 2t^2 + 3t + 1$. Hence we obtain $\Psi_{fus} = t^2 + t$.

$$|\Psi_{fs} - \Psi_{fus}| = \left| \frac{2t^2 + 2t + t^2 - t + 2t^2 + 4t + 2}{2} \right|$$
$$= \left| \frac{5t^2 + 5t + 2}{2} \right| > 1$$

Type (II): When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t$ and $V_{\frac{1}{3}}(f) = t+1$.

Then $\Psi_{fs} = \binom{t}{2} + 2\binom{t+1}{2} + 2t^2 + 2t$ and $\Psi_{fus} = (t+1)^2$.

Therefore $|\Psi_{fs} - \Psi_{fus}| = \left| \frac{5t^2 + t - 2}{2} \right| > 1$.

Type (III): When $V_1(f) = t+1$, $V_{\frac{1}{2}}(f) = t+1$ and $V_{\frac{1}{3}}(f) = t$.
Then $\Psi_{fs} = 2\binom{t+1}{2} + \binom{t}{2} + (t+1)^2 + t^2 + t$ and $\Psi_{fus} = t^2 + t$.
Therefore $|\Psi_{fs} - \Psi_{fus}| = \left|\frac{5t^2+5t+2}{2}\right| > 1$
Hence the theorem. □

Theorem 6.3. *The path P_n is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial for all n.*

Proof. Let P_n be the path $u_1u_2\dots u_n$. We note that P_1, P_2 and P_3 are fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial of graphs as depicted in Figures 8, 9 and 10 respectively.

Figure 8

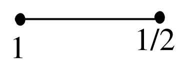

Figure 9

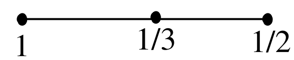

Figure 10

Now assume that $n > 3$. The following three cases arise.

Case (i): $n \equiv 0 \pmod 4$

Assign the grade 1 to the vertices $u_1, u_3, \dots, u_{\frac{n}{2}+1}$. Next assign the grade $\frac{1}{3}$ to the vertices $u_2, u_4, \dots, u_{\frac{n}{2}}$ and the last vertex u_n.

Subcase (i): $n \equiv 0 \pmod 3$

Assign the grade 1 to the vertices $u_{\frac{n}{2}+2}, u_{\frac{n}{2}+3}, \dots, u_{\frac{n}{2}+\frac{n}{3}-\frac{n}{4}}$. Next assign the grade $\frac{1}{3}$ to the vertices $u_{\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+1}, u_{\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+2}, \dots, u_{\frac{n}{2}+2\frac{n}{3}-2\frac{n}{4}-2}$. Then we have assign the grade $\frac{1}{2}$ to the vertices $u_{\frac{n}{2}+2\frac{n}{3}-2\frac{n}{4}-1}, u_{\frac{n}{2}+2\frac{n}{3}-2\frac{n}{4}}, \dots, u_{n-1}$.

Subcase (ii): $n \equiv 1 \pmod 3$

Assign the grade 1 to the vertices $u_{\frac{n}{2}+2}, u_{\frac{n}{2}+3}, \dots, u_{\frac{n}{2}+\frac{n-1}{3}-\frac{n}{4}}$. Next assign the grade $\frac{1}{3}$ to the vertices $u_{\frac{n}{2}+\frac{n-1}{3}-\frac{n}{4}+1}, u_{\frac{n}{2}+\frac{n-1}{3}-\frac{n}{4}+2}, \dots, u_{\frac{n}{2}+2\frac{n-1}{3}-2\frac{n}{4}-1}$. Then assign the grade $\frac{1}{2}$ to the vertices $u_{\frac{n}{2}+2\frac{n-1}{3}-2\frac{n}{4}}, u_{\frac{n}{2}+2\frac{n-1}{3}-2\frac{n}{4}+1}, \dots, u_{n-1}$.

Subcase (iii): $n \equiv 2 \pmod 3$

As in subcase (ii) we get a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial labeling of P_n.

Case (ii): $n \equiv 1 \pmod 4$

As in case (i), assign the grade to the vertices $u_1, u_2, \dots, u_{n-1}$. Finally assign the grade $\frac{1}{2}$ to the vertex u_n.

Case (iii): $n \equiv 2 \pmod 4$

As in case (i), assign the grade to the vertices $u_1, u_2, \dots, u_{n-1}$. Finally assign the grade $\frac{1}{3}$ to the vertex u_n.

Case (iv): $n \equiv 3 \pmod 4$

As in case (i), assign the grade to the vertices $u_1, u_2, \ldots, u_{n-1}$. Finally assign the grade 1 to the vertex u_n. Hence the theorem. □

Corollary 6.4. *The cycle C_n is fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial for all $n \geq 3$.*

Proof. Let C_n be the cycle $u_1 u_2 \ldots u_n u_1$. The vertex labeling given in Theorem 6.1 is also a fuzzy grade $\{1, \frac{1}{2}, \frac{1}{3}\}$-cordial labeling of C_n, $n \geq 3$. □

7. CONCLUSION

In this paper, we have introduced the concept of a new labeling called a fuzzy grade L-cordial labeling. We have presented our results on the fuzzy grade L-cordial labeling behaviour of certain standard graphs such as star, bistar, path, cycle, wheel, complete graph and $K_2 + mK_1$ graph. Concerning a graph of order p, we have described our study for some grades namely $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p}\}$, $\{1\}$, $\{1, \frac{1}{2}\}$ and $\{1, \frac{1}{2}, \frac{1}{3}\}$. The introduction of a new technique of graph labeling brings out certain significant mathematical properties involved in the concerned situation. Our study leads to the open problem of the investigation of several other families of graphs for the existence of fuzzy grade L-cordial labeling.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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