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CONVERGENCE AND STABILITY RESULTS FOR ISHIKAWA ITERATIVE SCHEME IN CONVEX p-METRIC SPACES

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Abstract. In this paper, we introduce the concept of convex $p-$metric space as a generalization of convex metric space, and we prove convergence and stability results of Ishikawa type, and Picard hybrid iterative sequence for quasi- contractive operator in the setting convex $p-$metric space. As a result of our analysis, the Picard and Mann schemes emerge as corollaries. An example is given to verify the main results.

Keywords: convex $p-$metric space; convergence; Ishikawa iteration scheme; Mann iteration scheme; Picard hybrid iterative sequences; stability.

2020 AMS Subject Classification: 47H10.

1. Introduction and Preliminaries

In 1922, Banach [1] introduced the use of Picard iteration as a simple yet powerful method for approximating fixed points. In recent decades, many researchers have focused on extending this classical result to broader contexts within generalized metric spaces [2,3]. In particular, Czerwik [9] introduced the concept of b-metric spaces by weakening the coefficient of the triangle inequality and generalized Banach's contraction principle to these spaces. After that,

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several interesting fixed point results in b-metric spaces were studied by many authors. Ref. [1, 7, 21] are some works in this line of research.

In 2020, Parvaneh and Ghoncheh [18] introduced the notion of an extended metric space p-metric space as a new generalization of the concept of b-metric space where the constant s is replaced by a continuous function $\Omega(t)$.

Definition 1.1 [18]. Let X be a non empty set. A mapping $\tilde{d}: X\times X\to[0,+\infty)$ is a *p-metric* if there exists a strictly increasing continuous function $\Omega:[0,\infty)\to[0,\infty)$ with $t\le\Omega(t)$ for all $t\ge 0$ such that for all $x,y,z\in X$:

(i) $\tilde{d}(x,y)=0$ if and only if $x=y$;

(ii) $\tilde{d}(x,y)=\tilde{d}(y,x)$;

(iii) $\tilde{d}(x,y)\le\Omega(\tilde{d}(x,z)+\tilde{d}(z,y))$.

The pair $(X,\tilde{d})$ is called a *p-metric space.*

Remark 1.2 [18]. The class of p-metric spaces is larger than the class of b-metric spaces introduced by [7] since $\Omega(x)=s(x)$, where s is a constant with $s\ge 1$ and studied by several authors including [13], [14].

(i) A b-metric space is a $p-$metric when $\Omega(x)=s(x)$.

(ii) it is metric if $\Omega(x)=x$.

However, there have been a few attempts to introduce the structure of convexity outside linear spaces. Kirk [4 , 5], Penot [6] and Takahashi [25] for example, presented notions of convexity for sets in metric spaces. Even in the more general setting of topological spaces there is the work of Liepin's [15] and Taskovic [26].

Takahashi [25] introduced a general notion of convexity structure and a convex metric space, that gave rise to what is referred to convex metric spaces, and formulated fixed point theorems for nonexpansive mappings in the convex metric space.

In 1970, Takahashi [25] introduced the concept of convexity in metric spaces (X,d) as follows:

Definition 1.3 [17]. Let (X,d) be a metric space and $I=[0,1]$. A continuous function $W: X\times X\times[0,1]\to X$ is said to be a convex structure on X if for all $x,y,u\in X$ and $\lambda\in I=[0,1]$,

$$d(u,W(x,y;\lambda)) \leq \lambda d(u,x)+(1-\lambda)d(u,y).$$

Definition 1.4 [25]. A metric space together with a convex structure W is known as *convex metric space* and is denoted by (X,d,W).

Definition 1.5 [23]. Let $(X,\tilde{d})$ be a $p-$metric space. Then a sequence $\{x_n\}$ in X is said to be

(i.) $p-$*convergent* if and only if there exists $x \in X$ such that $\tilde{d}(x_n,x) \to 0$ as $n \to \infty \lim_{n\to\infty} x_n = x$ that is, $\lim_{n\to\infty} x_n = x$.

(ii.) $p-$*Cauchy* if and only if $\tilde{d}(x_n,x_m) \to 0$ as $n,m \to \infty$.

(iii.) *p-complete* if every $p-$*Cauchy* sequence in X is *p-convergent*.

Proposition 1.6 [23]. In a p-metric space $(X,\tilde{d})$ such that $\Omega(0)=0$,

1. a *p-convergent* sequence has a unique limit.
2. each *p-convergent* sequence is $p-$*Cauchy*.
3. generally, a $p-$metric is not continuous.

Convergence of fixed points iterative schemes in convex metric spaces has been the subject of research in fixed point theory for sometime now. Several authors studied convergence of fixed points iterative schemes in convex metric spaces Beg [2], Sastry *et al.* [23], Olatinwo [17].

Definition 1.7 [4]. Let $(X,\tilde{d},W)$ be a convex metric space and $T: X \to X$ is a self mapping,

Suppose that $F(T)=\{p \in X : Tp=p\}$ is the set of fixed point of T.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative scheme involving T which is defined by:

$$x_{n+1}=f(Tx_n), \qquad n=0,1,2,..$$

where $x_0 \in X$ is the initial approximation and $f(Tx_n)$ is some function having convex structure such that , $\alpha_n \in [0,1]$.

Suppose that $\{x_n\}$ converges to a fixed point p of T .

Let $\{y_n)_{n=0}^{\infty}\} \subset X$ and set

$$\varepsilon_n = d(y_{n+1}, f(Ty_n)), \;\; n=0,1,2,..$$

Then the iterative scheme is said to T-stable or stable with respect to T if and only if $\lim_{n\to\infty} y_n = p$.

The stability theory of fixed point iteration schemes results established in metric space, normed linear spaces and Banach space settings are available in the literature. Several authors whose contributions are of Immense worth in the study of stability of the fixed point iterative procedures are Berinde [3-5], Bosede and Rhoades [26], Harder and Hicks [9], Jachymski [12], Osilike and Udomene [19], Ostrowski [20], and Rhoades [27].

Lemma 1.8 [5]. Let δ be a real number such that $0 \leq \delta < 1$ and $\{\varepsilon_n\}_{n=0}^{\infty}$ is a sequence of positive number such that $\lim_{n\to\infty} \varepsilon_n = 0$ then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying:

$$u_{n+1} \leq u_n + \varepsilon_n, \qquad n = 0, 1, 2, \ldots$$

then

$$\lim_{n\to\infty} u_n = 0.$$

Now we present some of the iterative schemes in convex spaces.

Definition 1.9. Let $(X, \tilde{d}, W)$ be a convex p-metric space and $T : X \to X$ be a self mapping of X. The sequence $\{x_n\}_{n=0}^{\infty}$ is Picard-type iterative scheme if

$$\left\{ x_{n+1} = W(0, Tx_n; 0), \qquad n = 0, 1, 2, \ldots \right. \tag{1.1}$$

Definition 1.10. Let $(X, \tilde{d}, W)$ be a convex p-metric space and $T : X \to X$ be a self mapping of X. We say that the sequence $\{x_n\}_{n=0}^{\infty}$ is a Mann -type iterative scheme if

$$\left\{ x_{n+1} = W(x_n, Tx_n; \alpha_n), \qquad n = 0, 1, 2, \ldots \right. \tag{1.2}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$.

Definition 1.11. Let $(X, \tilde{d}, W)$ be a convex p-metric space and $T : X \to X$ be a self mapping of X. We say the sequence $\{x_n\}_{n=0}^{\infty}$ is a Ishikawa type iterative scheme if

$$\begin{cases} x_{n+1} = W(x_n, Ty_n; \alpha_n) \\ y_n = W(x_n, Tx_n; \beta_n) \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.3}$$

$\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$.

Definition 1.12. Let $(X, \tilde{d}, W)$ be a convex p-metric space and $T : X \to X$ is a self-mapping of X. We say the sequence $\{x_n\}_{n=0}^{\infty}$ is a Picard -hybrid type iterative sequence if

$$\begin{cases} x_{n+1} = W(0, Ty_n; 0) \\ y_n = W(0, Tx_n; 0) \quad n = 0, 1, 2, ... \end{cases} \tag{1.4}$$

In 1999, Osilike and Udomene [19] defined a more general definition of quasi contractive operator.

Definition 1.13 [19]. Let there exist δ, L satisfying $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx). \tag{1.5}$$

In 2003, a more general definition was introduced by Imoru and Olatinwo [10].

Definition 1.14 [10]. Let there exist δ satisfying $\delta \in [0,1)$ and a increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + \varphi(d(x, Tx)). \tag{1.6}$$

2. Main Results

In this section, we introduce the concept of convex p-metric spaces, and prove some theorems on convergence and stability of Ishikawa [11] and Mann [16] iteration for quasi-contractive operator in convex p-metric spaces. To prove the main result we need following modified definitions and proposition:

Proposition 2.1. Let there exist δ, L satisfying $\delta \in [0,1)$ and $L \geq 0$ such that

$$\tilde{d}(Tx, Ty) \leq \delta \tilde{d}(x, y) + L\tilde{d}(x, Tx) \tag{2.1}$$

$\forall x, y \in X$ and $L \geq 0$, $\delta \in [0,1)$.

Proposition 2.2. Let there exists a constant $0 \leq \delta < 1$ and a monotonically increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that for all $x, y \in X$,

$$\tilde{d}(Tx, Ty) \leq \delta \tilde{d}(x, y) + \varphi \tilde{d}(x, Tx). \tag{2.2}$$

Definition 2.3. Let $(X, \tilde{d})$ be a p-metric space with $I = [0,1]$ and $\Omega : [0,\infty) \to [0,\infty)$ the strictly increasing continuous function. A continuous function $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X, if for each $x,y,u \in X$ and $\lambda \in [0,1)$

$$\tilde{d}(u,W(x,y;\lambda)) \leq \Omega\Big(\lambda \tilde{d}(u,x)+(1-\lambda)\tilde{d}(u,y)\Big). \tag{2.3}$$

Then $(X,\tilde{d},W)$ is said to be a convex $p-$metric space.

Definition 2.4. Let $(X,\tilde{d},W)$ be a convex $p-$metric space. A nonempty subset C of X is said to be convex if $W(x,y,\lambda) \in C$ for all $(x,y;\lambda) \in C \times C \times I$.

Theorem 2.5. Let K be a nonempty closed convex-p metric space X, and let $T : K \to K$ be a self-mapping satisfying

$$\tilde{d}(Tx,Ty) \leq \delta \tilde{d}(x,y)+\varphi(\tilde{d}(x,Tx)),$$

where $0 \leq \delta < 1$ and $\phi : [0,\infty) \to [0,\infty)$ is continuous with $\varphi(0)=0$.

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by the Ishikawa iteration:

$$\begin{cases} y_n = W(x_n,Tx_n,\beta_n), \\ x_{n+1} = W(x_n,Ty_n,\alpha_n), \end{cases}$$

where $\{\alpha_n\},\{\beta_n\} \subset [0,1]$ are sequences of positive numbers such that $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to the fixed point of T.

Proof. Let p be the fixed point of T. Then from the iteration, we have

$$\tilde{d}(x_{n+1},p) = \tilde{d}(W(x_n,Ty_n;\alpha_n),p).$$

By the convexity property of the p-metric, we obtain:

$$\tilde{d}(x_{n+1},p) \leq \Omega\left((1-\alpha_n)\tilde{d}(x_n,p)+\alpha_n\tilde{d}(Ty_n,p)\right). \tag{2.4}$$

Applying the contractive condition

$$\tilde{d}(Ty_n,p) \leq \delta\tilde{d}(y_n,p)+\varphi(\tilde{d}(y_n,Ty_n)). \tag{2.5}$$

From (2.4) and (2.5), we get:

$$\tilde{d}(x_{n+1},p) \leq \Omega\left((1-\alpha_n)\tilde{d}(x_n,p)+\alpha_n\delta\tilde{d}(y_n,p)+\alpha_n\phi(\tilde{d}(y_n,Ty_n))\right). \tag{2.6}$$

Now, estimate $\tilde{d}(y_n,p)$. Since $y_n = W(x_n,Tx_n,\beta_n)$, we use convexity again:

$$\tilde{d}(y_n,p) \leq \Omega\left((1-\beta_n)\tilde{d}(x_n,p)+\beta_n\tilde{d}(Tx_n,p)\right). \tag{2.7}$$

By the contractive condition,

$$\tilde{d}(Tx_n, p) \leq \delta \tilde{d}(x_n, p) + \varphi(\tilde{d}(x_n, Tx_n)). \tag{2.8}$$

Substituting (2.8) into (2.7):

$$\tilde{d}(y_n, p) \leq \Omega\left([(1-\beta_n) + \beta_n \delta]\, \tilde{d}(x_n, p) + \beta_n \varphi(\tilde{d}(x_n, Tx_n))\right). \tag{2.9}$$

Now, substitute (2.9) into (2.6):

(2.10)

$$\tilde{d}(x_{n+1}, p) \leq \Omega\left([1-\alpha_n(1-\delta(1-\beta_n+\beta_n\delta))]\, \tilde{d}(x_n, p) + \alpha_n \beta_n \varphi(\tilde{d}(x_n, Tx_n)) + \alpha_n \varphi(\tilde{d}(y_n, Ty_n))\right).$$

Let $u_n = \tilde{d}(x_n, p)$. If ϕ is continuous with $\phi(0) = 0$ and if $\tilde{d}(x_n, Tx_n) \to 0$, $\tilde{d}(y_n, Ty_n) \to 0$, then the last two terms in (2.9) vanish as $n \to \infty$.

Since $0 \leq \delta < 1$ and $\sum \alpha_n = \infty$, we can apply Lemma 1.8 to conclude:

$$\lim_{n \to \infty} \tilde{d}(x_n, p) = 0.$$

That is, $x_n \to p$ strongly.

Now apply the exponential product we obtain

$$\prod_{k=0}^{n} \Omega[1-\alpha_k(1-\delta)]\tilde{d}(x_0, p) \quad \leq \quad e^{-\sum_{k=0}^{n} \Omega(\alpha_k(1-\delta)\,\tilde{d}(x_0,p))}.$$

Thus,

$$\tilde{d}(x_{n+1}, p) \leq e^{-\sum_{k=0}^{n} \Omega[\alpha_k(1-\delta)\tilde{d}(x_0,p)]}.$$

Hence the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Corollary 2.6. Let K be a non empty closed convex-p metric spaces X and $T : K \to K$ be a self mapping satisfying

$$\tilde{d}(Tx, Ty) \leq \delta \tilde{d}(x, y) + \varphi(\tilde{d}(x, Tx))$$

Let $\{x_n\}_n^{\infty}$ be a sequence defined by the Mann iterative scheme (1.2) and $\{x_0\} \in X$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive number in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

The proof of Corollary 2.6 is similar to that of Theorem 2.5. This ends the proof $\square$.

Theorem 2.7. Let K be a non empty closed convex-p metric spaces X and $T : K \to K$ be a self mapping satisfying

$$\tilde{d}(Tx,Ty) \leq \delta\tilde{d}(x,y) + \varphi(\tilde{d}(x,Tx)).$$

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by the Picard hybrid iterative scheme (1.4) and $x_0 \in X$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. Let p be the fixed point of T. Then, from (1.2), we have

$$\tilde{d}(x_{n+1},p) \;\; = \;\; \tilde{d}(W(0,Ty_n;0), \;\; p).$$

By convexity property

$$\tilde{d}(x_{n+1},p) \;\; = \;\; \Omega[(1-0)\tilde{d}(0,p) + 0\tilde{d}(Ty_n,p)].$$

By contraction,

$$\tilde{d}(Ty_n,Tp) \leq \delta\tilde{d}(y_n,p) + \varphi(\tilde{d}(y_n,Ty_n)),$$

By convexity property,

$$\tilde{d}(x_{n+1},p) \leq \Omega(\tilde{d}(Ty_n,p)).$$

Also, for $\tilde{d}(W(0,Tx_n;0)$,

$$\begin{aligned}
\tilde{d}(y_n,p) \;\; &= \;\; \tilde{d}(W(0,Tx_n;0), \\
\tilde{d}(W(0,Tx_n;0) \;\; &\leq \;\; (1-0)\tilde{d}(0,Tx_n) + 0\tilde{d}(Tx_n,p),
\end{aligned}$$

By contraction,

$$\tilde{d}(Tx_n,Tp) \leq \delta\tilde{d}(,x_np) + \varphi(\tilde{d}(x_n,Tx_n))$$

substitute into $\tilde{d}(x_{n+1},p)$

$$\tilde{d}(x_{n+1},p) \;\; \leq \;\; \delta\tilde{d}(y_n,p) + \varphi(\tilde{d}(y_n,Ty_n)).$$

But $y_n = Tx_n$

$$\begin{aligned}
\tilde{d}(x_{n+1},p) \;\; &\leq \;\; \delta\tilde{d}(Tx_n,p) + \varphi(\tilde{d}(y_n,Ty_n)). \\
&\leq \;\; \delta(\delta\tilde{d}(,x_np) + \varphi(\tilde{d}(x_n,Tx_n))) + \varphi(\tilde{d}(y_n,Ty_n)), \\
&\leq \;\; \delta^2\tilde{d}(x_n,p) + \delta\varphi(\tilde{d}(x_n,Tx_n))) + \varphi(\tilde{d}(y_n,Ty_n))),
\end{aligned}$$

$$\vdots$$

$$\tilde{d}(x_{n+1},p) \leq \delta^n \tilde{d}(x_0,p)+\sum_{k=0}^{n}\delta^k\varphi(\tilde{d}(x_k,Tx_k)),$$

as $\delta^n \to 0$ as $n\to\infty$ and $\sum_{k=0}^{n}\delta^k\varphi(\tilde{d}(x_k,Tx_k))$ then

$$\lim_{n\to\infty}\tilde{d}(x_n,p)=0,$$

$$\lim_{n\to\infty}x_n=0. \qquad \square.$$

Since the Picard hybrid scheme reduces to the standard Picard iteration when $\alpha_n=0$ and $\beta_n=0$ the proof follows from the Banach fixed-point theorem [1] under the contractive condition. Thus, the sequence $\{x_n\}$ converges strongly to the unique fixed point p of T.

Corollary 2.8 Let K be a non empty closed convex p subset of a convex metric spaces X and $T:K\to K$ be a self mapping satisfying

$$\tilde{d}(Tx,Ty)\leq\delta\tilde{d}(x,y)+\varphi(\tilde{d}(x,Tx))$$

Let $\{x_n\}_n^\infty$ be a sequence defined by the Picard iterative scheme (1.1) and $\{x_0\}\in X$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of positive number in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

The proof of Corollary 2.8 is similar to that of Theorem 2.7. This ends the proof $\square$.

Theorem 2.9. Let $(X,\tilde{d},W)$ be a C complete convex p metric space and $T:X\to X$ be a self mapping satisfying

$$\tilde{d}(Tx,Ty)\leq\delta\tilde{d}(x,y)+\varphi(\tilde{d}(x,Tx))$$

Suppose that T has a fixed point p. For $x_0\in X$, let Ishikawa iterative scheme $\{x_n\}_{n=0}^{\infty}$ be defined by (1.3) where $\alpha_n,\ \beta_n\in[0,1]$ such that $0<\alpha\leq\alpha_n$. Then the Ishikawa iterative scheme is stable.

Proof. Suppose that $\{x_n\}_{n=0}^{\infty}\subseteq X$ is an arbitrary sequence in X and define

$$\varepsilon_n=\tilde{d}(y_{n+1},W(y_n,Tq_n,\alpha_n)$$

where

$$q_n = W(y_n, Ty_n, \alpha_n).$$

Let $\lim_{n\to\infty} \varepsilon_n = 0$. using the contractive condition we want to show that $\lim_{n=0\to\infty} y_n = p$.

$$\begin{aligned} \tilde{d}(y_{n+1}, p) &\leq \varepsilon_n + \tilde{d}(y_n, Tq_n, \alpha_n), p), \\ &\leq \varepsilon_n + (1-\alpha_n)\tilde{d}(y_n, p) + \alpha_n \tilde{d}(Tq_n, Tp), \end{aligned} \tag{2.9}$$

using the contractive condition

$$\tilde{d}(Tq_n, Tp) \leq \delta\tilde{d}(q_n, p) + \varphi(\tilde{d}(q_n, Tq_n)).$$

Using the triangle inequality

$$\tilde{d}(y_{n+1}, p) \leq \Omega(\tilde{d}(y_{n+1}, W(y_n Tq_n, \alpha_n) + \tilde{d}(W(y_n Tq_n, \alpha_n), p)),$$

using convex-p structure we have that

$$\tilde{d}(W(y_n Tq_n, \alpha_n), p) \leq \Omega[(1-\alpha_n)\tilde{d}(y_n, p) + \alpha_n \tilde{d}(Tq_n, Tp)], \tag{2.10}$$

substitute (2.10) into (2.9)

$$\begin{aligned} \tilde{d}(y_{n+1}, p) &\leq \varepsilon_n + (1-\alpha_n)\tilde{d}(y_n, p) + \alpha_n \tilde{d}(Tq_n, Tp), \\ &\leq \varepsilon_n + (1-\alpha_n)\tilde{d}(y_n, p) + \alpha_n \tilde{d}(q_n, p) \qquad (2.11). \end{aligned}$$

For the estimate of $\tilde{d}(q_n, p)$, we have

$$\begin{aligned} \tilde{d}(q_n, p) &= \tilde{d}(W(y_n, Ty_n, \beta_n), p), \\ &\leq \varepsilon_n + (1-\beta_n)\tilde{d}(y_n, p) + \beta_n \tilde{d}(Ty_n, Tp), \\ &\leq (1-\beta_n)\tilde{d}(y_n, p) + \beta_n \delta \tilde{d}(y_n, p), \end{aligned} \tag{2.12}$$

substitute (2.12) into (2.11)

$$\begin{aligned} \tilde{d}(y_{n+1}, p) &\leq \varepsilon_n + (1-\alpha_n)\tilde{d}(y_n, p) + \alpha_n[(1-\beta_n)\tilde{d}(y_n, p) + \beta_n \delta \tilde{d}(y_n, p)], \\ &\leq \varepsilon_n + [1-(1-\delta)\alpha_n - \alpha_n\beta_n\delta(1-\delta)]\tilde{d}(y_n, p). \end{aligned}$$

Note that $0 \leq [1-\alpha(1-\delta)) < 1$

Conversely, let $\lim_{n\to\infty} y_n = p$ then

$$\begin{aligned}\varepsilon_n &= \tilde{d}(y_{n+1}, W(y_n, Tq_n, \alpha_n),\\ &\leq \tilde{d}(y_{n+1}, p) + \tilde{d}(W(y_n, Tq_n, \alpha_n), p),\\ &\leq \tilde{d}(y_{n+1}, p) + (1-\alpha(1-\delta)\tilde{d}(y_n, p) \to 0,\end{aligned}$$

Since $\tilde{d}(y_n, p) \to 0, \quad$ as $n \to \infty$ we conclude that $\lim_{n\to\infty} \tilde{d}\{y_n\} = p$

Hence the $\{x_n\}$ is T- stable. $\quad\square$.

Corollary 2.10. Let $(X, \tilde{d}, W)$ be a C complete convex p metric space and $T : X \to X$ be a self mapping satisfying

$$\tilde{d}(Tx, Ty) \leq \delta\tilde{d}(x, y) + \varphi(\tilde{d}(x, Tx)).$$

Suppose that T has a fixed point p. For $x_0 \in X$, let Mann iterative scheme $\{x_n\}_{n=0}^{\infty}$ be defined by (1.2) where α_n, such that $0 < \alpha \leq \alpha_n$. Then the Mann iterative scheme is stable. $\quad\square$.

Theorem 2.12 . Let $(X, \tilde{d}, W)$ be a C complete convex p metric space and $T : X \to X$ be a self mapping satisfying

$$\tilde{d}(Tx, Ty) \leq \delta\tilde{d}(x, y) + \varphi(\tilde{d}(x, Tx))$$

Suppose that T has a fixed point p. For $x_0 \in X$, let Picard hybrid iterative scheme $\{x_n\}_{n=0}^{\infty}$ be defined by (1.4) Then the Picard hybrid iterative scheme is stable.

Proof. Suppose that $\{x_n\}_{n=0}^{\infty} \subseteq X$ is an arbitrary sequence in X and $\{x_n\}, \{x'_n\}$ are sequence with different initial values. Define error function as

$$\varepsilon_n = \tilde{d}(x_n, x'_n).$$

$$\tilde{d}(x_{n+1}, x'_{n+1}) = \tilde{d}(Ty_n, Ty'_n)$$

By contraction

$$\tilde{d}(Ty_n, Ty'_n) \leq \delta\tilde{d}(y_n, y'_n) + \varphi\tilde{d}(y_n, Ty_n),$$

since $y_n = W(0, Tx_n; 0) = Tx_n$

$$\tilde{d}(y_n.y'_n) = \tilde{d}(Tx_n, Tx'_n),$$

apply contraction

$$\tilde{d}(Tx_n, Tx'_n) \leq \delta\tilde{d}(x_n, x'_n) + \varphi\tilde{d}(x_n, Tx_n),$$

substitute into $\tilde{d}\,(x_{n+1}, x'_{n+1})$

$$\tilde{d}(x_{n+1}, x'_{n+1}) \leq \delta\tilde{d}(y_n, y'_n) + \varphi\tilde{d}(y_n, Ty_n),$$

but $y_n = Tx_n$,

$$\begin{aligned}\tilde{d}(x_{n+1}, x'_{n+1}) &\leq \delta\tilde{d}(Tx_n, Tx'_n) + \varphi\tilde{d}(y_n, Ty_n)\\ &\leq \delta(\delta\tilde{d}(x_n.x'_n) + \varphi\tilde{d}(x_n, Tx_n) + \varphi\tilde{d}(y_n, Ty_n).\end{aligned}$$

Expanding the recurrence iterative

$$\tilde{d}(x_n.x'_n) \leq \delta^n\tilde{d}(x_0.x'_0) + \sum \delta^k\varphi\tilde{d}(x_k, Tx_k)$$

as $\delta^n \to 0$ as $n \to \infty$ and $\sum_{k=0}^{n} \delta^k \quad \varphi\,(\tilde{d}(x_k, Tx_k))$ then

$$\lim_{n\to\infty} \tilde{d}(x_n, x'_n) = 0,$$

Hence, the Picard Hybrid iteration is T-stable. □

3. Numerical Example

Consider the real line $\mathbb{R}$ with $\tilde{d}(x,y) = |x-y|$. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = \frac{1}{2}x$. We verify that T satisfies the contractive condition: $\tilde{d}(Tx, Ty) \leq \delta\tilde{d}(x,y) + \phi(\tilde{d}(x, Tx))$, with $\delta = \frac{1}{2}$ and $\phi(t) = 0$.

$$\tilde{d}(Tx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| = \delta\tilde{d}(x,y),$$

which satisfies the condition with $\phi(0) \equiv 0$.

Applying the Picard Hybrid Iteration given by

$$\begin{cases} y_n = Tx_n = \frac{1}{2}x_n, \\ x_{n+1} = Ty_n = \frac{1}{2}y_n = \frac{1}{4}x_n. \end{cases}$$

Starting with $x_0 = 1$, we compute:

$$\begin{aligned} x_0 &= 1, \\ x_1 &= \frac{1}{4}x_0 = \frac{1}{4}, \\ x_2 &= \frac{1}{4}x_1 = \frac{1}{16}, \\ x_3 &= \frac{1}{4}x_2 = \frac{1}{64}, \\ x_4 = \frac{1}{4}x_3 = \frac{1}{256}, &\quad \text{and so on.} \end{aligned}$$

Table of Iterates obtained

n	x_n	$\tilde{d}(x_n,0)$
0	1	1
1	$1/4$	0.25
2	$1/16$	0.0625
3	$1/64$	0.015625
4	$1/256$	0.00390625
5	$1/1024$	0.0009765625

Convergence: We know from the theorem that

$$\tilde{d}(x_{n+1},p) \leq \Omega\left(\delta^2\tilde{d}(x_n,p) + \delta\varphi\tilde{d}(x_n,Tx_n) + \varphi\tilde{d}(Tx_n,T^2x_n)\right),$$

from Definition 1.14, this becomes: $\tilde{d}(x_{n+1},p) \leq \Omega\left(\delta^2\tilde{d}(x_n,p)\right) = \Omega\left(\frac{1}{4}\tilde{d}(x_n,p)\right)$.

Since Ω satisfies $\Omega(0) = 0$, we conclude:

$$\tilde{d}(x_n,p) \to 0 \quad \text{as } n \to \infty,$$

Therefore, $x_n \to p = 0$ strongly.

To verify the T-stablity of the Picard-hybrid iterative scheme

Let $T(x) = \frac{1}{2}x$, we consider the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = \frac{1}{2}x$. Using the mapping above the Picard Hybrid iterative scheme is

$$x_{n+1} = Ty_n = \frac{1}{2}y_n = \frac{1}{4}x_n.$$

Let $p = 0$ be the unique fixed point of T, and $\{x_n\}$ be the sequence starting from $x_0 = 1$, and let $\{z_n\}$ be a perturbed sequence starting from $z_0 = 1+\varepsilon$, where ε is a small change. Then, the iterates are

$$x_n = \left(\frac{1}{4}\right)^n, \qquad z_n = \left(\frac{1}{4}\right)^n (1+\varepsilon).$$

The distance between the two sequences is given by

$$\tilde{d}(x_n, z_n) = |x_n - z_n| = \left|\left(\frac{1}{4}\right)^n - \left(\frac{1}{4}\right)^n (1+\varepsilon)\right| = \left(\frac{1}{4}\right)^n |\varepsilon|.$$

As $n \to \infty$, we observe: $\tilde{d}(x_n, z_n) = |\varepsilon| \left(\frac{1}{4}\right)^n \to 0$.

Conflict of Interests: The authors declare that there is no conflict of interests.

CONFLICT OF INTERESTS

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