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GLOBAL DYNAMICS OF A RATIO-DEPENDENT COMMENSALISM MODEL WITH NON-SELECTIVE HARVESTING AND PARTIAL CLOSURE FOR THE POPULATIONS

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Abstract. Populations of species in ecosystems subjected to harvesting often face the problem of extinction of one or more species. We propose and analyze the global dynamics of a ratio-dependent commensalism model with non-selective harvesting and partial closure of populations. In addition to positivity and uniform boundedness, we discuss the persistence and permanence of the system. The conditions for the existence, local, and global stability of all four equilibrium points were studied. Additionally, the analytical results showed a negative correlation between the fraction of stock available for harvesting and the equilibrium densities of the species. This implies a positive correlation between the partial closure for the populations and equilibrium densities of the species. We performed numerical simulations to validate the theoretical results. It was found that for at least 80% of the stock available for harvesting (or a partial closure of at most 20%), both species could be driven to extinction within a finite time. On the other hand, for a sustainable ecosystem where both species coexist in a stable equilibrium, at most 35% of stock available for harvesting (or a partial closure of at least 65%) should be implemented.

Keywords: partial closure; commensalism model; ratio-dependent functional response; non-selective harvesting; global stability; numerical simulations.

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1. INTRODUCTION

Predation, a dominant theme in ecology, is a type of symmetrical relationship in which one species loses in terms of population density while the other species gains. However, a lesser-known asymmetric relationship where a species (the commensal) gains while the other (the host) does not lose or gain is known as commensalism. A classical example of such a relationship is the interaction between the remora fish and the shark. The remora rides on the back of the shark and feeds on leftover food from the latter. Numerous authors [1–15] have recently conducted studies on the dynamics of commensalism models. Mougi [1] asserts that asymmetric interactions like commensalism and amensalism tended to give more stability to community dynamics than symmetric interactions like mutualism and competition. The local and global dynamic behaviours of an obligate commensal symbiosis system incorporating a Crowley-Martin response function were studied by Xu et al. [2]. They also investigated the conditions that lead to the extinction and permanence of the system. In [3], the authors analyzed a density-dependent commensalism model with a non-monotonic functional response to ascertain the suitable conditions for the global attractivity of the equilibrium points of the system. A Lotka Volterra commensalism model in the presence of additive Allee effect was examined by He et al. [4]. In addition to exploring the local and global stability of the system, they showed the impact of the Allee effect on the final densities of the species. Xie et al. [5] discussed a two-species commensalism model with the Allee effect on the second species and a Beddington-DeAngelis functional response. It was revealed that, with a weak enough Allee effect, the coexistence equilibrium was globally stable. A commensalism system incorporating a non-monotonic response system was studied by Chen and Wu [6]. Their analysis of the model indicates the existence of a unique positive equilibrium point, which is globally stable. The authors in [7] studied a commensalism system with a density-dependent birth rate and Merdan-type Allee effect. They demonstrated that a strong weak Allee effect increases the time taken for the system to reach equilibrium.

Wu and Li [8] proposed a commensalism model with ratio-dependent functional response and where one party could not independently survive. They discovered that, in the autonomous case, the two species could coexist in a stable equilibrium if the rate of commensalism exceeds the intrinsic death rate of the commensal species. Chen [9] extended the model in [8] by incorporating an Allee effect in the host species. Numerical simulations of the system indicated that, with an increase in Allee effect, the time taken for the commensal to go extinct is hardly impacted. However, the host species takes a much longer time to reach a steady state. In [10], the authors investigated the global dynamics of a predator-prey system with ratio-dependent response function. They showed that, in general, the local asymptotic stability of the positive equilibrium point does not guarantee its global asymptotic stability. A study on the qualitative properties of a ratio-dependent predator-prey model was undertaken by Xiao and Ruan [11]. The main focus of the study was on the dynamics of the system around the origin in the interior of the positive quadrant. It was shown that the origin exists and it is a nonhyperbolic equilibrium point.

The impact of commensalism on a Lotka-Volterra model with nonlinear harvesting of the commensal species was studied in [12]. In addition to investigating the local and global dynamics of the system, a threshold value of the commensalism rate that will ensure the stable coexistence of the two species was determined. Srinivas et al. [13] discussed the proportional harvesting of the commensal in a two-species commensal-host system. They analytically and numerically determined the local and global dynamics of the stable equilibrium point. In [14], a commensalism model with a Michaelis-Menten type harvesting on the commensal was examined. The study explored the conditions for the achievement of stability and persistence of the coexistence equilibrium point. A commensalism system with non-selective Michaelis-Menten type harvesting was studied by Zhu et al. [15]. It was revealed that the species could go extinct, partially survive or stably coexist depending on suitable conditions. Ibrahim and Sarwardi [16] examined the global dynamics of a generalist predator-prey model with Holling type IV functional response and harvesting of both species. Along with conducting stability analysis of the system, they discussed the level of harvesting that sustained the coexistence of the populations. In [17], the authors analyzed the global stability properties of a predator-prey model with two marine areas comprising a fishing zone and a no fishing zone.

A study by Lin [18] on a commensalism system with non-monotonic response function and non-selective harvesting in the presence of a partial closure demonstrated that, whenever the positive equilibrium point exists, it is globally asymptotically stable. A predator-prey model with variable refuge was studied in [19] to assess the role of the refuge on the stability of the system. In [20], a Beddington-DeAngelis predator-prey fishery model with non-selective harvesting and partial closure for the populations was examined. After performing local and global analysis of the system, an optimal harvesting scheme was found for the fishery. Ibrahim [21] studied a predator-prey model with dynamic harvesting effort and partial closure for the fish stock. The global dynamics of the coexistence equilibrium point as well as the optimal size of the closure were determined. In a recent study, Chen [22] investigated a harvested amensalism model with a Holling type I functional response and partial closure for the populations. Stimulated by the reviewed works, we propose and studied a novel commensalism model with a ratio-dependent functional response and non-selective harvesting incorporating a partial closure for the populations.

The following depicts the organization of this paper. Formulation of the mathematical model is presented in Section 2. In Section 3, the qualitative properties of the system in terms of positivity and boundedness are discussed. The stability analysis of the equilibrium points of the system are carried out in Section 4. This comprises determining the nonnegative equilibrium points of the system and subjecting each point to local and global analyses. The persistence and permanence of the system are examined in Section 5. To validate the theoretical results of the study, numerical simulations are carried out in Section 6 while the last section, Section 7, is devoted to the summary and conclusions.

2. MODEL FORMULATION

In this study, we propose the following ratio-dependent commensalism model with non-selective harvesting and partial closure for the populations:

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1-\dfrac{x}{K}\right)+\dfrac{bxy}{x+ay}-mq_1Ex \\ \dfrac{dy}{dt} = sy\left(1-\dfrac{y}{L}\right)-mq_2Ey\ , \end{cases} \tag{1}$$

with initial conditions:

$$x(0)=x_0>0 \quad \text{and} \quad y(0)=y_0>0\,.$$

The variables $x(t)$ and $y(t)$ represent the population densities at time t of the commensal and host species respectively. It is assumed that the populations grow logistically and are subjected to non-selective harvesting with the parameter m $(0 \leq m < 1)$ denoting the fraction of stock available for harvesting. The partial closure for the populations is represented by $(1-m)$. Thus, the population densities of commensal and host species closed to harvesting are respectively $(1-m)x$ and $(1-m)y$. The rest of the parameters in System (1) are described in Table 1.

TABLE 1. Description of parameters

Parameter	Description
r	Intrinsic growth rate of commensal species
s	Intrinsic growth rate of host species
K	Carrying capacity of commensal species
L	Carrying capacity of host species
a	Handling time coefficient
b	Rate of commensalism
E	Non-selective harvesting effort of the species
q_1	Catchability coefficient of commensal species
q_2	Catchability coefficient of host species

It is worthy of note that the ratio-dependent functional response, which depends on both species, is preferred in this study as it usually elicits more complex system dynamics than functional responses depending on only one species, like the Holling type responses [23]. Further, the terms mq_1Ex and mq_2Ey depict the partial harvesting of the commensal and host species respectively. In the special case where $m = 0$, full closure (or no harvesting) of the populations is enforced.

3. POSITIVITY AND BOUNDEDNESS

In this section, we establish a compact region that will be positively invariant in which the system will be well-posed ecologically and mathematically. Given the initial positive conditions, the theorem below ensures that all the solutions of System (1) admit non-negative values.

Theorem 3.1. *Starting from the interior of the first quadrant of the xy-plane, the solutions* $x(t)$ *and* $y(t)$ *of System* (1) *will remain in the quadrant.*

Proof. For the initial conditions $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$, the first equation of System (1) can be written as

$$x(t) = x_0 \exp\left\{\int_0^t \left[r\left(1 - \frac{x(n)}{K}\right) + \frac{by(n)}{x(n) + ay(n)} - mq_1E\right] dn\right\}.$$

Similarly, the second equation of the system is presented as

$$y(t) = y_0 \exp\left\{\int_0^t \left[s\left(1 - \frac{y(n)}{L}\right) - mq_2E\right] dn\right\}.$$

Hence all the solutions of the system are non-negative since they are confined to only the first quadrant of xy-plane. □

We proceed to show the uniform boundedness of System (1) in the theorem below:

Theorem 3.2. *For* $\varepsilon > 0$, $\eta = \frac{K}{4r}\left(r + \frac{b}{a} + \varepsilon - mq_1E\right)^2 + \frac{L}{4s}(s + \varepsilon - mq_2E)^2$, *and* $K_{max} = K\left(1 + \frac{b}{ar}\right)$, *the set*

$$\rho = \left\{(x,y) \in \mathbb{R}_+^2 : 0 \leq x \leq K_{max},\ 0 \leq y \leq L,\ 0 \leq x + y \leq \frac{\eta}{\varepsilon}\right\}$$

is positively invariant and bounded for all solutions starting from the interior of the first quadrant.

Proof. When we apply the comparison theorem of differential equations to the first equation of System (1), we have

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) + \frac{bxy}{x + ay} - mq_1Ex \\ &\leq rx\left(1 - \frac{x}{K}\right) + \frac{b}{a}x \\ &\leq x\left(r + \frac{b}{a} - \frac{rx}{K}\right).\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} x(t) \leq K\left(1 + \frac{b}{ar}\right) := K_{\max}.$$

Similar application of the comparison theorem to the second equation of the system shows

$$\begin{aligned}\frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right)-mq_2Ey\\ &\leq sy\left(1-\frac{y}{L}\right).\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} y(t) \leq L.$$

Let $\psi(t)=x(t)+y(t)$, and suppose there exists a positive constant ε. Then

$$\begin{aligned}\frac{d\psi}{dt}+\varepsilon\psi &= \frac{dx}{dt}+\frac{dy}{dt}+\varepsilon x+\varepsilon y\\ &\leq \left(r+\frac{b}{a}+\varepsilon-mq_1E\right)x-\frac{rx^2}{K}+(s+\varepsilon-mq_2E)y-\frac{sy^2}{L}\\ &\leq \frac{K}{4r}\left(r+\frac{b}{a}+\varepsilon-mq_1E\right)^2-\frac{r}{K}\left[x-\frac{K}{2r}\left(r+\frac{b}{a}+\varepsilon-mq_1E\right)\right]^2\\ &+\frac{L}{4s}(s+\varepsilon-mq_2E)^2-\frac{s}{L}\left[y-\frac{L}{2s}(s+\varepsilon-mq_2E)\right]^2\\ &\leq \frac{K}{4r}\left(r+\frac{b}{a}+\varepsilon-mq_1E\right)^2+\frac{L}{4s}(s+\varepsilon-mq_2E)^2\\ &\leq \eta.\end{aligned}$$

Thus, $\frac{d\psi}{dt}+\varepsilon\psi\leq\eta$. Employing Gronwall's theorem, we have

$$0\leq\psi(t)\leq\frac{\eta}{\varepsilon}(1-e^{-\varepsilon t})+\psi(0)e^{-\varepsilon t}.$$

Therefore,

$$\limsup_{t\to\infty}\psi(t)\leq\frac{\eta}{\varepsilon}.$$

Hence, all solutions of System (1) with initial positive values lie in the compact region ρ. □

4. EQUILIBRIUM POINTS AND STABILITY ANALYSIS

We present four feasible equilibrium points of System (1) and their stability, which indicates that the system may go to extinction, the species may partially survive, or the two species could coexist in a steady state. Letting $P(x,y)=rx\left(1-\frac{x}{K}\right)+\frac{bxy}{x+ay}-mq_1Ex$ and $Q(x,y)=sy\left(1-\frac{y}{L}\right)-mq_2Ey$, assume that $P(0,0)=0$. This establishes the continuity of $P(x,y)$ as well as $Q(x,y)$ in the first quadrant of the xy-plane.

4.1. Equilibrium Points.

The equilibrium points of System (1) and the conditions for their existence are presented.

1. The trivial equilibrium point, which shows mutual extinction of the populations, always exists and is given by $A_0 = (0,0)$.
2. The axial equilibrium point representing the host-free scheme is denoted by $A_1 = (\bar{x}, 0)$, where
$$\bar{x} = K\left(1 - \frac{mq_1E}{r}\right) \quad \text{exists when} \quad m < \frac{r}{q_1E}.$$
3. The axial equilibrium point signifying the commensal-free scenario is given by $A_2 = (0, \hat{y})$, where
$$\hat{y} = L\left(1 - \frac{mq_2E}{s}\right) \quad \text{exists when} \quad m < \frac{s}{q_2E}.$$
4. The unique interior equilibrium point showing the coexistence of species is depicted by $A^* = (x^*, y^*)$, where

$$\begin{cases} r\left(1 - \dfrac{x^*}{K}\right) + \dfrac{by^*}{x^* + ay^*} - mq_1E & = 0 \\ s\left(1 - \dfrac{y^*}{L}\right) - mq_2E & = 0 \end{cases} \tag{2}$$

Applying Descartes' rule of signs and solving System (2) simultaneously guarantees a unique positive solution

$$x^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad y^* = L\left(1 - \frac{mq_2E}{s}\right),$$

where

$$\begin{aligned} A &= rs, \\ B &= arL(s - mq_2E) - sK(r - mq_1E), \quad \text{and} \\ C &= KL(amq_1E - ar - b)(s - mq_2E) < 0, \end{aligned}$$

provided that

$$m < \min\left\{\frac{ar + b}{aq_1E}, \frac{s}{q_2E}\right\}.$$

4.1.1. *The impact of the parameter m.* To investigate the influence of the parameter m on the coexistence equilibrium, we differentiate System (2) with respect to m [24] to obtain the following:

$$\begin{cases} -\dfrac{r}{K}\dfrac{dx^*}{dm}+\left[(x^*+ay^*)b\dfrac{dy^*}{dm}-by^*\left(\dfrac{dx^*}{dm}+a\dfrac{dy^*}{dm}\right)\right](x^*+ay^*)^{-2}-q_1E=0 \\ -\dfrac{s}{L}\dfrac{dy^*}{dm}-q_2E=0 \end{cases}$$

Solving simultaneously for $\frac{dx^*}{dm}$ and $\frac{dy^*}{dm}$, we obtain

$$\begin{cases} \dfrac{dx^*}{dm}=-\dfrac{bKLx^*q_2E+(x^*+ay^*)^2sKq_1E}{rs(x^*+ay^*)^2+bsKy^*} \\ \dfrac{dy^*}{dm}=-\dfrac{q_2EL}{s} \end{cases}$$

Clearly, $\frac{dx^*}{dm}<0$ and $\frac{dy^*}{dm}<0$. This implies that any increase in the fraction of stock available for harvesting will result in a decrease in the equilibrium densities of the commensal and host species. In other words, there is a negative correlation between the fraction of stock available for harvesting and the equilibrium densities of the species.

Equivalently, an increase in the partial closure for the populations will lead to a corresponding increase in the equilibrium densities of both species. This signifies that there is a positive correlation between the partial closure for the populations and the equilibrium densities of the species.

4.2. Local stability.

To obtain conditions for each of the equilibria of System (1) to be locally asymptotically stable, we first compute the Jacobian matrix for the system and use it to find the eigenvalues of each equilibrium point. The local stability of each equilibrium point is established if each eigenvalue corresponding to that equilibrium point has its real part to be negative. The Jacobian matrix for the system is computed as follows:

$$\mathrm{J}(x,y)=\begin{bmatrix} r\left(1-\dfrac{2x}{K}\right)+\dfrac{by}{(x+ay)}-\dfrac{bxy}{(x+ay)^2}-mq_1E & \dfrac{bx}{(x+ay)}-\dfrac{abxy}{(x+ay)^2} \\ 0 & s\left(1-\dfrac{2y}{L}\right)-mq_2E \end{bmatrix}.$$

Theorem 4.1. *The mutual extinction equilibrium* $A_0=(0,0)$ *exists for System* (1) *and is a locally asymptotically stable node when* $m>\max\left\{\frac{ar+b}{aq_1E},\frac{s}{q_2E}\right\}$. *Otherwise, it is unstable.*

Proof. The Jacobian matrix at $A_0 = (0,0)$ cannot be evaluated because of issues relating to singularity at the origin. Thus, the local stability of A_0 cannot be directly studied. However, it will be shown in Theorem 4.5 that the equilibrium point is globally asymptotically stable. Therefore, the mutual extinction equilibrium point is locally stable since global stability is a sufficient condition for local stability. □

Theorem 4.2. *The host-free equilibrium point $A_1 = (\bar{x},0)$ is a locally asymptotically stable node when $\frac{s}{q_2E} < m < \frac{r}{q_1E}$, Otherwise it is unstable.*

Proof. Evaluating the Jacobian matrix at $A_1 = (\hat{x},0)$ gives

$$\mathrm{J}(\bar{x},0) = \begin{bmatrix} mq_1E - r & b \\ 0 & s - mq_2E \end{bmatrix}.$$

Since this gives an upper triangular matrix, the eigenvalues are $\lambda_1 = mq_1E - r$ and $\lambda_2 = s - mq_2E$. This shows that the equilibrium point A_1 is locally asymptotically stable for the stated condition in the theorem because $\lambda_1 < 0$ and $\lambda_2 < 0$. Any feasible condition aside this will make the equilibrium point unstable. □

Theorem 4.3. *The commensal-free equilibrium point $A_2 = (0,\hat{y})$ is a locally asymptotically stable node when $\frac{ar+b}{aq_1E} < m < \frac{s}{q_2E}$, otherwise it is unstable.*

Proof. When the Jacobian matrix of the system is evaluated at $A_2 = (0,\hat{y})$, it gives

$$\mathrm{J}(0,\hat{y}) = \begin{bmatrix} r + \dfrac{b}{a} - mq_1E & 0 \\ 0 & mq_2E - s \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = r + \dfrac{b}{a} - mq_1E -$ and $\lambda_2 = mq_2E - s$. Thus, the equilibrium point A_2 is locally asymptotically stable since the stated condition in the theorem makes both eigenvalues negative. Any other feasible condition will make the equilibrium point unstable. □

Theorem 4.4. *The unique coexistence equilibrium point $A^* = (x^*, y^*)$ is a locally asymptotically stable node whenever it exists.*

Proof. Since the point (x^*, y^*) satisfies System (2), the Jacobian can be simplified to

$$\mathrm{J}(x^*, y^*) = \begin{bmatrix} -\dfrac{rx^*}{K} - \dfrac{bx^*y^*}{(x^*+ay^*)^2} & \dfrac{bx^{*2}}{(x^*+ay^*)^2} \\ 0 & -\dfrac{sy^*}{L} \end{bmatrix}.$$

The eigenvalues of the matrix are $\lambda_1 = -\frac{rx^*}{K} - \frac{bx^*y^*}{(x^*+ay^*)^2} < 0$ and $\lambda_2 = -\frac{sy^*}{L} < 0$. Hence, the coexistence equilibrium point is locally asymptotically stable whenever it exists. □

4.3. **Global stability.**

The global behavior of the equilibrium points is discussed in this section. The various conditions for the points to be globally stable are established and proved. This is done by constructing an appropriate Lyapunov function to analyze each equilibrium point along the trajectories of System (1), but for the unique coexistent equilibrium, the Dulac-Bendixson criterion is applied.

Theorem 4.5. *Assuming that* $m > \max\left\{\frac{ar+b}{aq_1E}, \frac{s}{q_2E}\right\}$ *holds, then the mutual extinction equilibrium point* $A_0 = (0,0)$ *is globally asymptotically stable.*

Proof. Let

$$V_0(x,y) = x + y$$

be a Lyapunov function. Then differentiating V_0 with respect to time t along the trajectories of System (1) gives

$$\begin{aligned} \frac{dV_0}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} \\ &= rx\left(1 - \frac{x}{K}\right) + \frac{bxy}{x+ay} - mq_1Ex + sy\left(1 - \frac{y}{L}\right) - mq_2Ey \\ &\le \left(r + \frac{b}{a} - mq_1E\right)x - \frac{r}{K}x^2 + (s - mq_2E)y - \frac{s}{L}y^2. \end{aligned}$$

By Lyapunov's theorem on stability, A_0 is globally asymptotically stable since the derivative is negative definite under the condition stated in the theorem. □

Theorem 4.6. *The host-free equilibrium point* $A_1 = (\bar{x}, 0)$ *is globally asymptotically stable when* $\frac{s+b}{q_2E} < m < \frac{r}{q_1E}$.

Proof. It is worth noting that at the point A_1, $r\left(1-\frac{\bar{x}}{K}\right)-mq_1E=0$. We define a positive definite function:

$$V_1(x,y)=\left(x-\bar{x}-\bar{x}\ln\frac{x}{\bar{x}}\right)+y\,.$$

The time derivative of V_1 along the trajectories of the system gives

$$\begin{aligned}\frac{dV_1}{dt}&=\left(1-\frac{\bar{x}}{x}\right)\frac{dx}{dt}+\frac{dy}{dt}\\&=(x-\bar{x})\left[r\left(1-\frac{x}{K}\right)+\frac{by}{x+ay}-mq_1E\right]+sy\left(1-\frac{y}{L}\right)-mq_2Ey\\&=(x-\bar{x})\left[r\left(1-\frac{x}{K}\right)+\frac{by}{x+ay}-r\left(1-\frac{\bar{x}}{K}\right)\right]+sy\left(1-\frac{y}{L}\right)-mq_2Ey\\&=-\frac{r}{K}(x-\bar{x})^2-\frac{b\bar{x}y}{x+ay}+\frac{bxy}{x+ay}+(s-mq_2E)y-\frac{s}{L}y^2\\&\leq-\frac{r}{K}(x-\bar{x})^2-\frac{b\bar{x}y}{x+ay}+(b+s-mq_2E)y-\frac{s}{L}y^2\,.\end{aligned}$$

Under the condition stated in the theorem, the derivative is negative definite. Thus, by Lyapunov's theorem on stability, A_1 is globally asymptotically stable. □

Theorem 4.7. *The commensal-free equilibrium point $A_2=(0,\hat{y})$ is globally asymptotically stable whenever it is locally stable.*

Proof. Note that at the point A_2, $s\left(1-\frac{\hat{y}}{L}\right)-mq_2E=0$. Consider the following Lyapunov function:

$$V_2(x,y)=x+\left(y-\hat{y}-\hat{y}\ln\frac{y}{\hat{y}}\right)\,.$$

Differentiating V_2 along the trajectories of the system produces

$$\begin{aligned}\frac{dV_2}{dt}&=\frac{dx}{dt}+\left(1-\frac{\hat{y}}{y}\right)\frac{dy}{dt}\\&=rx\left(1-\frac{x}{K}\right)+\frac{bxy}{x+ay}-mq_1Ex+(y-\hat{y})\left[s\left(1-\frac{y}{L}\right)-mq_2E\right]\\&=rx\left(1-\frac{x}{K}\right)+\frac{bxy}{x+ay}-mq_1Ex+(y-\hat{y})\left[s\left(1-\frac{y}{L}\right)-s\left(1-\frac{\hat{y}}{L}\right)\right]\\&\leq\left(r+\frac{b}{a}-mq_1E\right)x-\frac{r}{K}x^2-\frac{s}{L}(y-\hat{y})^2\,.\end{aligned}$$

Under the condition of local stability, the derivative is negative definite. Therefore, by Lyapunov's theorem on stability, A_2 is globally asymptotically stable. □

Theorem 4.8. *For all solutions initiating in the compact region ρ, the coexistence equilibrium point $A^* = (x^*, y^*)$ is globally asymptotically stable whenever it exists.*

Proof. Since

$$P(x,y) = rx\left(1-\frac{x}{K}\right) + \frac{bxy}{x+ay} - mq_1Ex \quad \text{and}$$
$$Q(x,y) = sy\left(1-\frac{y}{L}\right) - mq_2Ey,$$

consider a positive Dulac function

$$\phi(x,y) = \frac{1}{xy}.$$

Then

$$\begin{aligned}\Gamma(x,y) &= \frac{\partial}{\partial x}(\phi P) + \frac{\partial}{\partial y}(\phi Q)\\ &= \frac{\partial}{\partial x}\left[\frac{r}{y}\left(1-\frac{x}{K}\right) + \frac{b}{x+ay} - \frac{mq_1E}{y}\right] + \frac{\partial}{\partial y}\left[\frac{s}{x}\left(1-\frac{y}{L}\right) - \frac{mq_2E}{x}\right]\\ &= -\frac{r}{yK} - \frac{b}{(x+ay)^2} - \frac{s}{xL}.\end{aligned}$$

Since $\Gamma(x,y)$ does not change sign in the compact region ρ, then by the Bendixson-Dulac criterion the system has no limit cycles and hence A^* is globally asymptotically stable. □

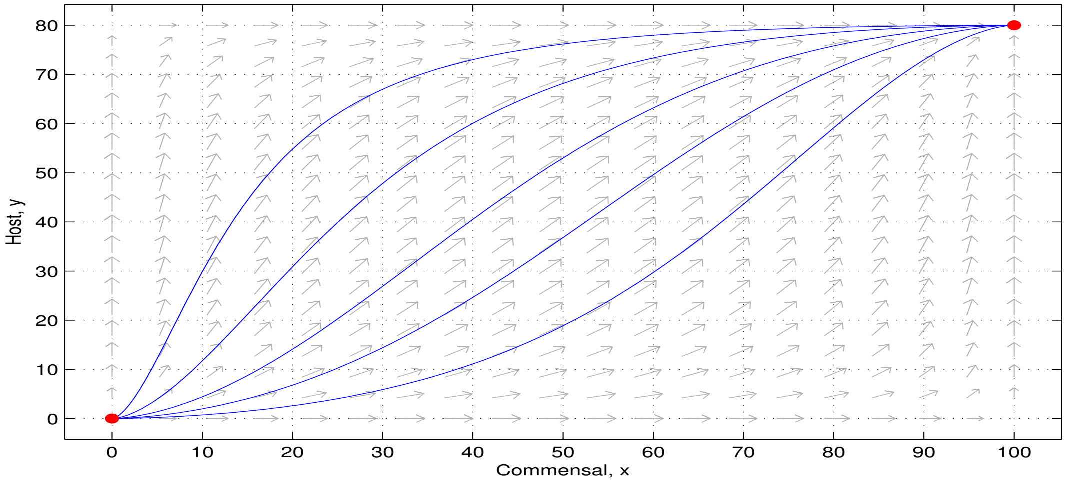

FIGURE 1. Phase portrait for the scenario $m = b = 0$

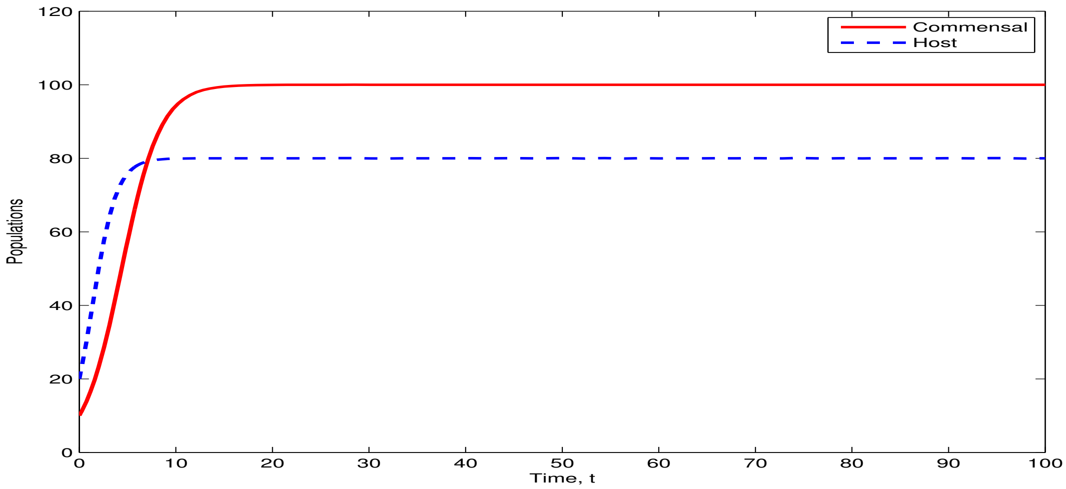

FIGURE 2. Simulation for the scenario $m = b = 0$

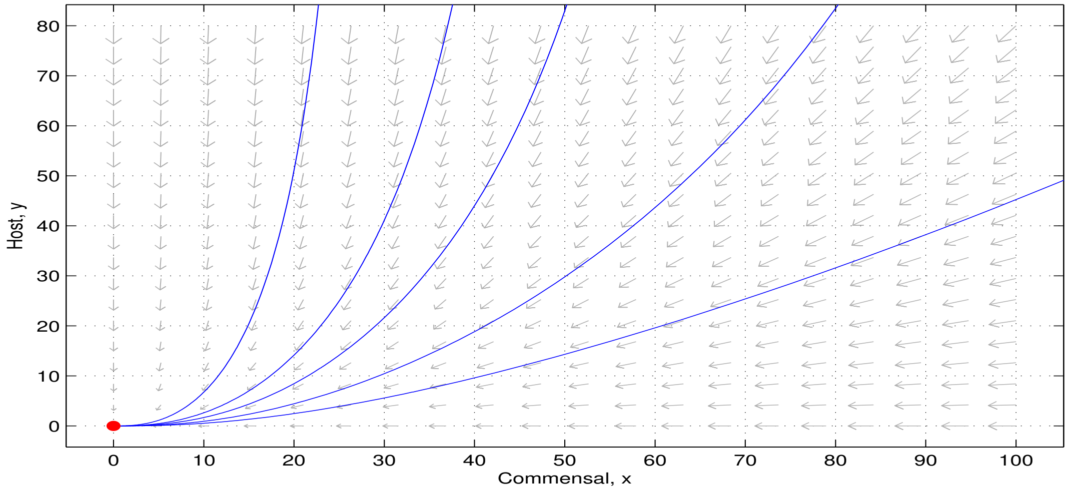

FIGURE 3. Phase portrait of the mutual extinction equilibrium point $A_0 = (0,0)$ with $m = 0.8$

5. PERSISTENCE AND PERMANENCE

Dynamical systems have important concepts relating to them, and two of such concepts, persistence and permanence, are discussed in this section.

5.1. Persistence. The long-term survival of species of a given initial population density in an ecosystem is termed as persistence. In mathematical terms, the system should not possess omega-limit points on the boundary of the non-negative state space for strictly positive solutions. The following theorem gives the conditions for the system to be uniformly persistent.

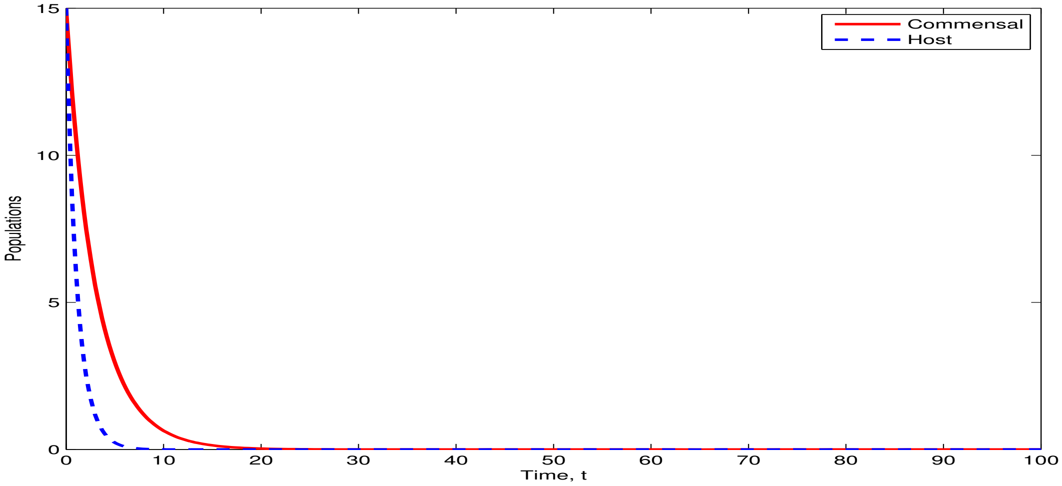

FIGURE 4. Simulation for the mutual extinction equilibrium point $A_0 = (0,0)$ with $m = 0.8$

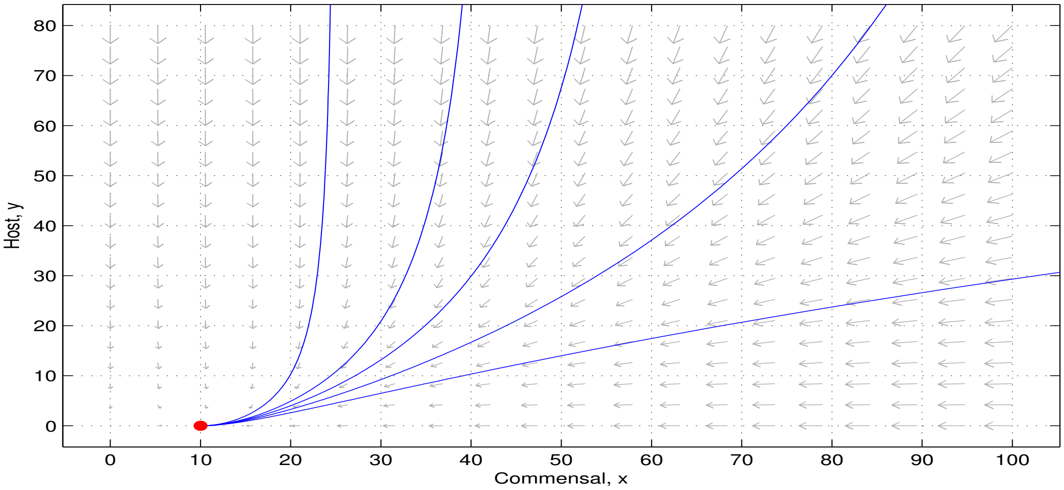

FIGURE 5. Phase portrait of the host-free equilibrium point $A_1 = (10,0)$ with $m = 0.45$

Theorem 5.1. *System* (1) *is uniformly persistent provided the following sufficient condition holds:*

$$m < \min\left\{\frac{ar+b}{aq_1E}, \frac{s}{q_2E}\right\}. \tag{3}$$

Proof. Let an average Lyapunov function be given by $\psi(x,y) = xy$. Clearly, $\psi(x,y) > 0$ for all (x,y) in the interior of the first quadrant and $\psi(x,y) = 0$ on the boundary of the quadrant.

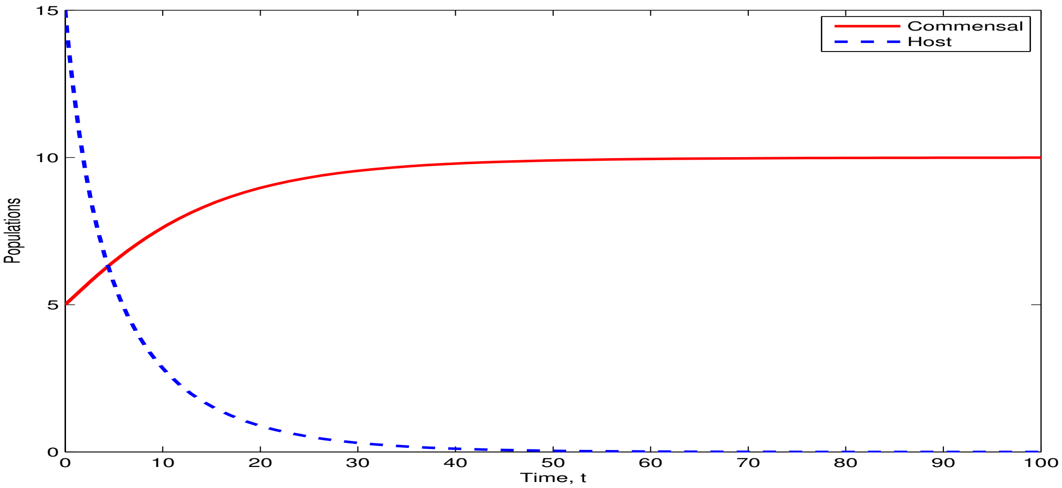

FIGURE 6. Simulation for the host-free equilibrium point $A_1 = (10, 0)$ with $m = 0.45$

Suppose $\psi'(x,y) = \frac{d}{dt}\psi(x,y)$. Then

$$\begin{aligned}\varpi(x,y) &= \frac{\psi'(x,y)}{\psi(x,y)} = \frac{1}{x}\frac{dx}{dt} + \frac{1}{y}\frac{dy}{dt} \\ &= \left[r\left(1-\frac{x}{K}\right) + \frac{by}{x+ay} - mq_1E\right] + \left[s\left(1-\frac{y}{L}\right) - mq_2E\right].\end{aligned}$$

Considering only the boundary equilibrium points A_0, A_1 and A_2, we assume that $\varpi(A_0) > 0$ since only A^* is stable when the stated condition holds. Thus,

$$\begin{aligned}\varpi(A_1) &= s - mq_2E\,, \\ \varpi(A_2) &= r + \frac{b}{a} - mq_1E\,.\end{aligned}$$

It is clear that $\varpi(A_1) > 0$ and $\varpi(A_2) > 0$ whenever the condition is satisfied. Hence the system is uniformly persistent. $\square$

5.2. Permanence.

From the ecological perspective, when the populations of species in an ecosystem continue to exist regardless of time, it is referred to as permanence. In mathematical terms, when the solutions of a system remain positive for all time, the system is considered to be permanent.

Theorem 5.2. *Given any solutions $x(t)$ and $y(t)$ of System* (1), *suppose $l = L\left(1 - \frac{mq_2E}{s}\right)$ and $k = K\left(1 + \frac{bl-(K_{max}+aL)mq_1E}{r(K_{max}+aL)}\right)$ are positive constants independent of the initial conditions. Then* *the system is considered permanent if the following conditions hold:*

$$k \leq \liminf_{t\to\infty} x(t) \leq \limsup_{t\to\infty} x(t) \leq K_{max}\,,$$

$$l \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq L\,.$$

Proof. When we apply the comparison theorem of differential equations to the second equation of System (1), we get

$$\begin{aligned}\frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right) - mq_2Ey\\ &= y\left(s-\frac{sy}{L}-mq_2E\right)\,.\end{aligned}$$

Thus,

$$\liminf_{t\to\infty} y(t) \geq L\left(1-\frac{mq_2E}{s}\right) := l\,.$$

Hence, l is positive provided that $m < \frac{s}{q_2E}$.

From the first equation of the system, we have

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right) + \frac{bxy}{x+ay} - mq_1Ex\\ &\geq rx\left(1-\frac{x}{K}\right) + \frac{bl}{K_{\max}+aL}x - mq_1Ex\\ &\geq x\left(r-\frac{rx}{K}+\frac{bl}{K_{\max}+aL}-mq_1E\right)\,.\end{aligned}$$

Therefore,

$$\liminf_{t\to\infty} x(t) \geq K\left(1+\frac{bl-(K_{\max}+aL)mq_1E}{r(K_{\max}+aL)}\right) := k\,.$$

Hence, k is positive provided that $m < \min\left\{\frac{r(K_{\max}+aL)+bl}{(K_{\max}+aL)q_1E}, \frac{s}{q_2E}\right\}$. □

6. NUMERICAL SIMULATIONS

We perform numerical simulations to validate the theoretical findings on the system. Of particular interest is the role played by the fraction of stock available for harvesting (or the partial closure) on the extinction or coexistence of the species. The parameter values used in the simulations are as follows: $r = 0.5$, $s = 0.8$ ($s = 1.4$ for A_2 only), $K = 100$, $L = 80$, $a = 1$, $b = 0.05$, $E = 10$, $q_1 = 0.1$ and $q_2 = 0.2$.

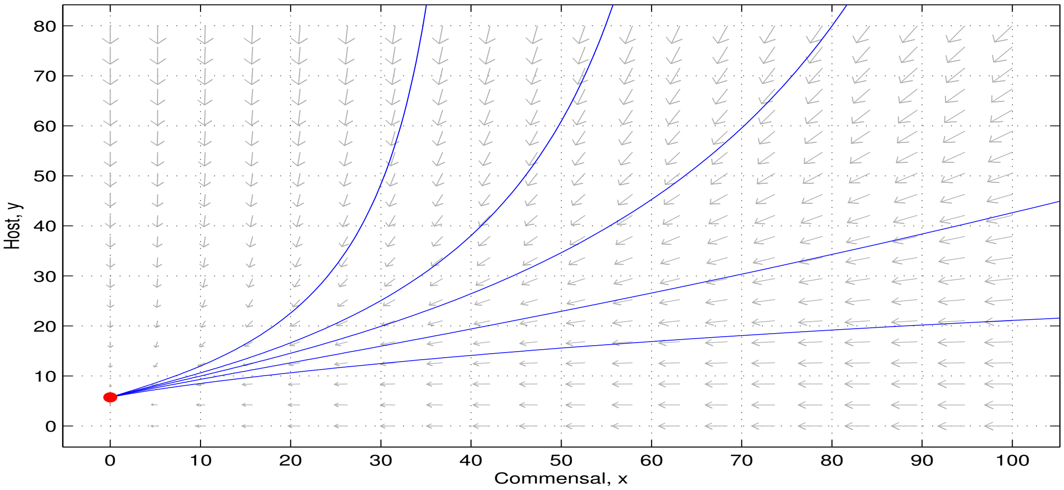

FIGURE 7. Phase portrait of the commensal-free equilibrium point $A_2 = (0, 5.71)$ with $m = 0.65$

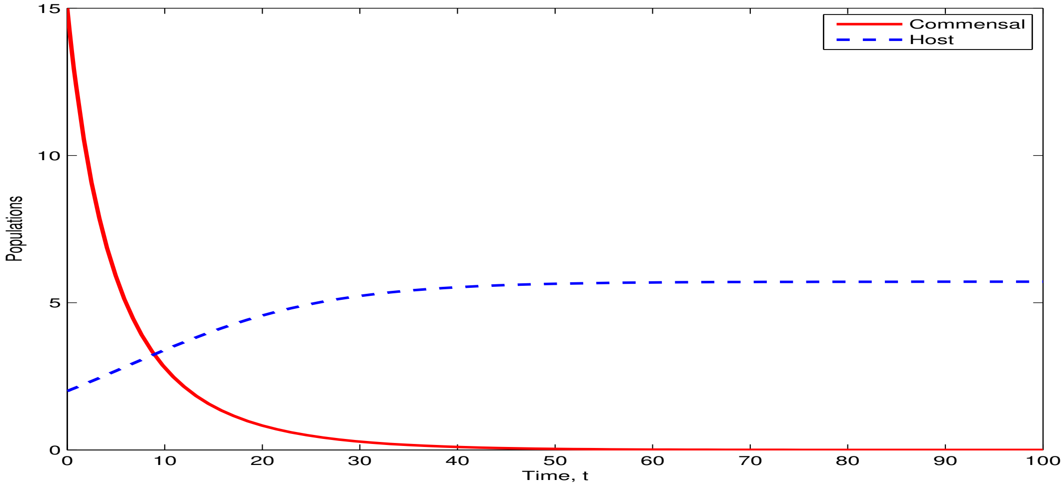

FIGURE 8. Simulation for the commensal-free equilibrium point $A_2 = (0, 5.71)$ with $m = 0.65$

From Figure 1, it is seen that when $m = 0$ and $b = 0$, the populations for both the commensal and the host attain their carrying capacities. Thus, each population increases till it reaches the carrying capacity because it obeys the law of logistic growth.

It can be seen that without any harvesting and at zero rate of commensalism, there is a nodal source at (0, 0) from which the trajectories initiate and a nodal sink at (100, 80) into which the trajectories terminate.

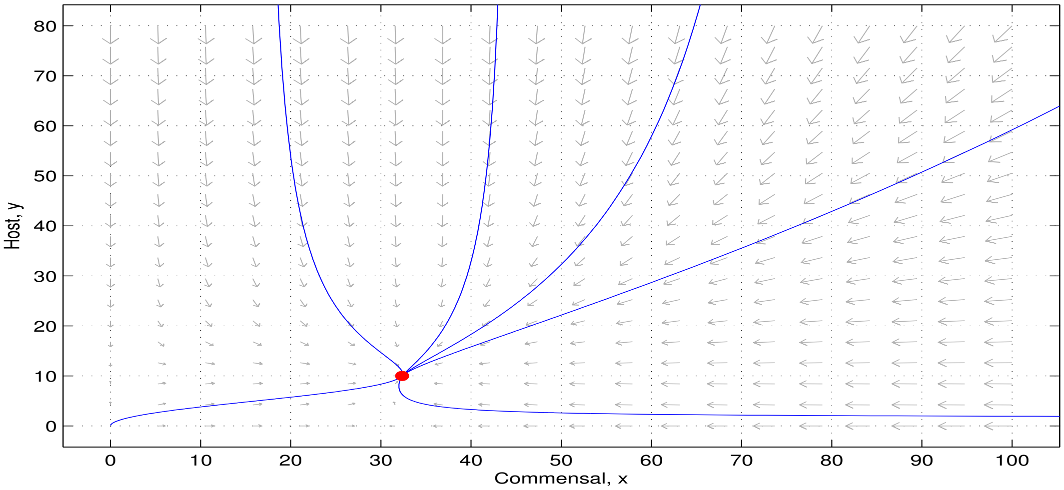

FIGURE 9. Phase portrait of the coexistence equilibrium point $A^* = (32.36, 10)$ with $m = 0.35$

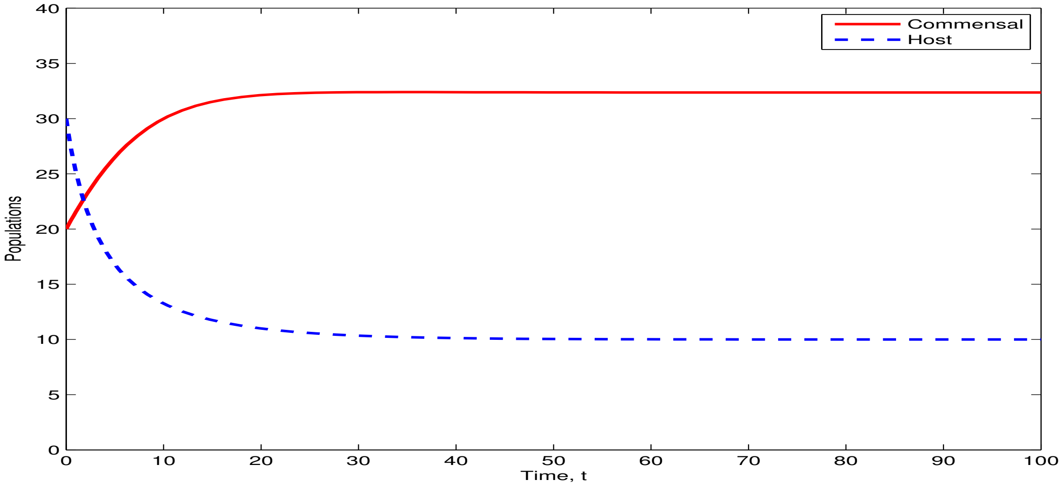

FIGURE 10. Simulation for the coexistence equilibrium point $A^* = (32.36, 10)$ with $m = 0.35$

Figure 2 shows that even if we start with any positive population size for the commensal and the host, in the absence of harvesting and commensalism, the populations will converge to 100 and 80 respectively, which are their carrying capacities. The indication is that, no matter the minimum positive population size for either the commensal or the host, they will grow to their respective carrying capacities if there is no harvesting nor commensalism.

The phase portrait in Figure 3 shows the global asymptotic stability of the mutual extinction equilibrium point, where all trajectories converge to the point. Figure 4 indicates that when the available stock for harvesting is about 80% of each population, the system will collapse. This implies that a 20% partial closure for the populations could drive both species to extinction.

Figure 5 portrays the global asymptotic stability of the host-free equilibrium point. In Figure 6, when the available stock for harvesting is about 45% of each of the populations, the host cannot survive but the commensal does.

In Figure 7, the global asymptotic stability of the commensal-free equilibrium point is shown. Figure 8 indicates that with about 65% of stock available for harvesting for each population, the commensal is driven to extinction while the host persists. It is worth noting that the intrinsic growth rate of the host is given by $s = 1.4$.

Figure 9 depicts the global asymptotic stability of the coexistence equilibrium point. In Figure 10, when the available stock for harvesting is about 35% of each of the populations, the species coexist and are permanent. This implies that for a sustainable ecosystem, a 65% partial closure for the populations should be implemented.

7. CONCLUSIONS

In this paper, we studied a ratio-dependent commensalism model with non-selective harvesting and partial closure for the populations. To demonstrate the well-posedness of the model, positivity and uniform boundedness of the system was established. The equilibria of the system comprising the mutual extinction, host-free, commensal-free and coexistence equilibrium points were determined. Further, the sufficient conditions for the local and global stability of each equilibrium point were discussed. Employing the average Lyapunov approach, the system was found to be persistent while permanence of the system was established by showing that positive solutions of the system were bounded. In addition, the analytical results showed that the equilibrium density of each species is an decreasing function of the fraction of stock available for harvesting.

Performing numerical simulations to validate the theoretical findings, the phase portraits confirmed the asymptotic stability of the equilibrium points under suitable conditions. Further, for at least 80% of stock available for harvesting (or at most 20% closure for the populations) the two species could go extinct in finite time. For 45% of stock available for harvesting (or partial closure of 55%), the commensal persists while the host is driven to extinction. Similarly, for

65% of stock available for harvesting (or partial closure of 35%), the host persists while the commensal goes into extinction. For long-term sustainability of both species, at most 35% of stock available for harvesting (or at least 65% closure for the populations) should be implemented. This study provides the threshold values for extinction and conservation of the species, and thus could be helpful in the design of marine protected areas.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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