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A STUDY ON HILBERT'S INTEGRAL INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, we presented as follows

Let f, g, h, $k \geq 0$, *h is homogeneous and symmetric of degree* λ *and* $F(x,y) = 1 - k(x) + k(y) \geq 0$. *Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)\left|\ln\frac{x}{y}\right|^\mu}{h(x,y)}\,dxdy\right)^4$$

$$\leq \left(\left(C\int_0^\infty x^{1-\lambda}\,f^2(x)\,dx\right)^2 - \left(\int_0^\infty (Ck(x)-C(x))\,x^{1-\lambda}f^2(x)\,dx\right)^2\right)$$

$$\times\left(\left(C\int_0^\infty x^{1-\lambda}\,g^2(x)\,dx\right)^2 - \left(\int_0^\infty (Ck(x)-C(x))\,x^{1-\lambda}g^2(x)\,dx\right)^2\right),$$

where

$$C = \int_0^\infty \frac{t^a\,|\ln t|^\mu}{h(1,t)}\,dt, \quad C(x) = \int_0^\infty \frac{k(xt)\,t^a\,|\ln t|^\mu}{h(1,t)}\,dt\,,$$

provided the integrals on the RHS do exists.

Some other special cases are also deduced.

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1. Introduction

If $f,\ g \geq 0,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\int\limits_0^\infty f^p(x)\,dx<\infty,\ \ 0<\int\limits_0^\infty g^q(x)\,dx<\infty,$ *then*

$$\int\limits_0^\infty\int\limits_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \leq \frac{\pi}{\sin(\pi/p)}\left(\int\limits_0^\infty f^p(x)\,dx\right)^{1/p}\left(\int\limits_0^\infty g^q(x)\,dx\right)^{1/q} \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Many mathematicians presented generalizations or new kinds of Hardy-Hilbert's inequality (1). Hardy inequality is very important in analysis theory and applications, it has been absorbing much interest of analysis see ([1], [2]) .

Very recently P. X. Ying and G. Mingzhe (see [3]) proved the following new kind

Theorem 1.1. *Let* $f(x)$ *be a real function. If* $0<\int\limits_0^\infty f^2(x)\,dx<\infty$, *then*

$$\left(\int\limits_0^\infty\int\limits_0^\infty \frac{f(x)\,f(y)}{x+y}dxdy\right)^2 < \pi^2\left(\left(\int\limits_0^\infty f^p(x)\,dx\right)^2-\left(\int\limits_0^\infty \omega(x)\,f^p(x)\,dx\right)^2\right), \tag{2}$$

where $\omega(x)=\frac{1}{1+\sqrt{x}}-\frac{1}{1+x}$.

2. Lemmas

The following lemma are needed for our aim

Lemma 2.1. *Let* $S(x,y)$ *be symmetric. Then*

$$\int\limits_0^\infty\int\limits_0^\infty S(x,y)\,F(x,y)\,dxdy=\int\limits_0^\infty\int\limits_0^\infty S(x,y)\,dxdy \tag{3}$$

where $F(x,y)=1-k(x)+k(y)$.

Proof.

$$\int_0^\infty\int_0^\infty S(x,y)\,F(x,y)\,dxdy$$

$$= \int_0^\infty\int_0^\infty S(x,y)\,(1-k(x)+k(y))\,dxdy$$

$$= \int_0^\infty\int_0^\infty S(x,y)\,dxdy - \int_0^\infty\int_0^\infty S(x,y)\,k(x)\,dxdy + \int_0^\infty\int_0^\infty S(x,y)\,k(y)\,dxdy$$

$$= \int_0^\infty\int_0^\infty S(x,y)\,dxdy - \int_0^\infty\int_0^\infty S(x,y)\,k(x)\,dxdy + \int_0^\infty\int_0^\infty S(y,x)\,k(x)\,dydx$$

$$= \int_0^\infty\int_0^\infty S(x,y)\,dxdy - \int_0^\infty\int_0^\infty S(x,y)\,k(x)\,dxdy + \int_0^\infty\int_0^\infty S(x,y)\,k(x)\,dxdy$$

$$= \int_0^\infty\int_0^\infty S(x,y)\,dxdy.$$

Lemma 2.2[3].

$$\int_0^\infty \frac{du}{(1+xu^2)(1+u^2)} = \frac{\pi}{1+\sqrt{x}}. \tag{4}$$

Lemma 2.3[4]. *Let* $0<\alpha<1$, s *is a non-negative integer. Then*

$$\int_0^\infty \frac{t^{\alpha-1}}{1+t}\left(\ln\frac{1}{t}\right)^{2s} dt = (2s)!\left(\xi(2s+1,\alpha)+\xi(2s+1,1-\alpha)\right), \tag{5}$$

where $\xi(s,\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(\alpha+k)^s}$.

The object of this paper is to present the following general result

3. Main results

Theorem 3.1. *Let f, g, h, $k \geq 0$, h is homogeneous and symmetric of degree λ and $F(x,y) = 1 - k(x) + k(y) \geq 0$. Then*

$$\begin{aligned}
&\left(\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)\left|\ln\frac{x}{y}\right|^\mu}{h(x,y)}\,dxdy\right)^4 \\
\leq\ & \left(\left(C\int_0^\infty x^{1-\lambda}\,f^2(x)\,dx\right)^2 - \left(\int_0^\infty (Ck(x)-C(x))\,x^{1-\lambda}f^2(x)\,dx\right)^2\right) \\
&\times\left(\left(C\int_0^\infty x^{1-\lambda}\,g^2(x)\,dx\right)^2 - \left(\int_0^\infty (Ck(x)-C(x))\,x^{1-\lambda}g^2(x)\,dx\right)^2\right) \qquad (6)
\end{aligned}$$

where

$$C = \int_0^\infty \frac{t^a\,|\ln t|^\mu}{h(1,t)}\,dt, \qquad C(x) = \int_0^\infty \frac{k(xt)\,t^a\,|\ln t|^\mu}{h(1,t)}\,dt\,,$$

provided the integrals on the RHS do exists.

Proof.

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\mu f(x)\,g(y)F(x,y)}{h(x,y)}dxdy \\
=\ & \int_0^\infty\int_0^\infty \frac{\left|\ln\left(\frac{y}{x}\right)\right|^{\frac{\mu}{2}} f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} \times \frac{\left|\ln\left(\frac{x}{y}\right)\right|^{\frac{\mu}{2}} g(y)\sqrt{F(x,y)}}{\sqrt{h(x,y)}}dxdy \\
\leq\ & \left(\int_0^\infty\int_0^\infty \frac{\left|\ln\left(\frac{y}{x}\right)\right|^\mu f^2(x)F(x,y)}{h(x,y)}dxdy\right)^{1/2}\left(\int_0^\infty\int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\mu g^2(y)F(x,y)}{h(x,y)}dxdy\right)^{1/2} \\
=\ & \sqrt{M}\sqrt{N}.
\end{aligned}$$

$$\begin{aligned} M &= \int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{y}{x}\right)\right|^{\frac{\mu}{2}} f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} \left(\frac{y}{x}\right)^{\alpha/2} \times \frac{\left|\ln\left(\frac{x}{y}\right)\right|^{\mu} f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} \left(\frac{x}{y}\right)^{\alpha/2} dxdy \\ &\leq \left(\int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{y}{x}\right)\right|^{\mu} f^2(x)F(x,y)}{h(x,y)} \left(\frac{y}{x}\right)^{\alpha} dxdy\right)^{1/2} \\ &\times \left(\int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^{\mu} f^2(x)F(x,y)}{h(x,y)} \left(\frac{x}{y}\right)^{\alpha} dxdy\right)^{1/2} \\ &= \sqrt{M_1}\sqrt{M_2}. \end{aligned}$$

$$\begin{aligned} M_1 &= \int_0^\infty \int_0^\infty \frac{\left|\ln\left(\frac{y}{x}\right)\right|^{\mu} f^2(x)\left(1-k(x)+k(y)\right)}{h(x,y)} \left(\frac{y}{x}\right)^{\alpha} dxdy \\ &= \int_0^\infty \left|\ln\left(\frac{y}{x}\right)\right|^{\mu} f^2(x)\left(1-k(x)\right) \int_0^\infty \frac{(y/x)^\alpha}{h(x,y)} dydx \\ &\quad + \int_0^\infty \left|\ln\left(\frac{y}{x}\right)\right|^{\mu} f^2(x) \int_0^\infty \frac{k(y)\,(y/x)^\alpha}{h(x,y)} dydx \\ &= \int_0^\infty x^{1-\lambda}\left(1-k(x)\right) f^2(x) \int_0^\infty \frac{u^\alpha \left|\ln u\right|^\mu}{h(1,u)} dudx \\ &\quad + \int_0^\infty x^{1-\lambda} f^2(x) \int_0^\infty \frac{u^\alpha \left|\ln u\right|^\mu k(xu)}{h(1,u)} dudx \\ &= C\int_0^\infty x^{1-\lambda} f^2(x)\,dx - \int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} f^2(x)\,dx. \end{aligned}$$

Similarly,

$$M_2 = C\int_0^\infty x^{1-\lambda} f^2(x)\,dx + \int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} f^2(x)\,dx.$$

Therefore

$$M^2 = \left(C \int_0^\infty x^{1-\lambda} f^2(x)\, dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} f^2(x)\, dx \right)^2,$$

and

$$N^2 = \left(C \int_0^\infty x^{1-\lambda} g^2(x)\, dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} g^2(x)\, dx \right)^2,$$

Applying Lemma 2.1 to have

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)\, g(y) \left| \ln \frac{x}{y} \right|^\mu}{h(x,y)}\, dxdy \right)^4 = \left(\int_0^\infty \int_0^\infty \frac{f(x)\, g(y) \left| \ln \frac{x}{y} \right|^\mu \; F(x,y)}{h(x,y)}\, dxdy \right)^4$$

$$\leq \left(\left(C \int_0^\infty x^{1-\lambda} f^2(x)\, dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} f^2(x)\, dx \right)^2 \right)$$

$$\times \left(\left(C \int_0^\infty x^{1-\lambda} g^2(x)\, dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} g^2(x)\, dx \right)^2 \right).$$

4. Applications

Corollary 4.1. *Let $f, h, k \geq 0$, h is homogeneous and symmetric of degree λ and $F(x,y) = 1 - k(x) + k(y) \geq 0$. Then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)\, f(y) \left| \ln \frac{x}{y} \right|^\mu}{h(x,y)}\, dxdy \right)^2$$

$$\leq \left(\left(C \int_0^\infty x^{1-\lambda} f^2(x)\, dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} f^2(x)\, dx \right)^2 \right) \qquad (7)$$

where

$$C = \int_0^\infty \frac{t^a \left| \ln t \right|^\mu}{h(1,t)}\, dt, \qquad C(x) = \int_0^\infty \frac{k(xt)\, t^a \left| \ln t \right|^\mu}{h(1,t)}\, dt\,,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from theorem 3.1 by putting $g = f$.

Corollary 4.2. **Let** *$f, h, k \geq 0$, h is homogeneous and symmetric of degree λ and $F(x,y) = 1 - k(x) + k(y) \geq 0$. Then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)\, f(y) \left|\ln \frac{x}{y}\right|^\mu}{(x+y)^\lambda}\, dxdy\right)^2 \leq \left(\left(C\int_0^\infty x^{1-\lambda}\, f^2(x)\, dx\right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} f^2(x)\, dx\right)^2\right) \quad (8)$$

where

$$C = \int_0^\infty \frac{t^a \left|\ln t\right|^\mu}{h(1,t)}\, dt, \quad C(x) = \int_0^\infty \frac{k(xt)\, t^a \left|\ln t\right|^\mu}{h(1,t)}\, dt\,,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from corollary 4.1 by putting $h\,(x,y) = (x+y)^\lambda$.

Corollary 4.3. Let *$f, h, k \geq 0$, h is homogeneous and symmetric of degree λ and $F(x,y) = 1 - k(x) + k(y) \geq 0$. Then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)\, f(y) \left|\ln \frac{x}{y}\right|^\mu}{x^\lambda + y^\lambda}\, dxdy\right)^2 \leq \left(\left(C\int_0^\infty x^{1-\lambda}\, f^2(x)\, dx\right)^2 - \left(\int_0^\infty (Ck(x) - C(x))\, x^{1-\lambda} f^2(x)\, dx\right)^2\right) \quad (9)$$

where

$$C = \int_0^\infty \frac{t^a \left|\ln t\right|^\mu}{h(1,t)}\, dt, \quad C(x) = \int_0^\infty \frac{k(xt)\, t^a \left|\ln t\right|^\mu}{h(1,t)}\, dt\,,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from corollary 4.1 by putting $h\,(x,y) = x^\lambda + y^\lambda$.

Corollary 4.4 [3]. *Let $f, g \geq 0$. Then*

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{h(x,y)}dxdy\right)^4$$

$$\leq \quad \pi^4\left(\left(\int_0^\infty x^{1-\lambda} f^2(x)\,dx\right)^2 - \left(\int_0^\infty \omega(x)\,f^2(x)\,dx\right)^2\right)$$

$$\times\left(\left(\int_0^\infty x^{1-\lambda} g^2(x)\,dx\right)^2 - \left(\int_0^\infty \omega(x)\,g^2(x)\,dx\right)^2\right) \qquad (10)$$

provided the integrals on the RHS do exists.

Proof. The proof follows from theorem 3.1 via Lemma 2.2, by putting

$$h(x,y) = x+y, \quad a = -1/2, \quad \lambda = 1, \quad \mu = 0, \quad k(x) = 1/(1+x),$$

as follows

$$C = \int_0^\infty \frac{u^{-1/2}}{1+u}du = 2\int_0^\infty \frac{du}{1+u^2} = \pi.$$

$$C(x) = \int_0^\infty \frac{k(xu)\,u^{-1/2}}{1+u}du = \int_0^\infty \frac{k(xu^2)}{1+u^2}du = \int_0^\infty \frac{du}{(1+xu^2)(1+u^2)} = \frac{\pi}{1+\sqrt{x}},$$

and

$$Ck(x) - C(x) = \pi\left(\frac{1}{1+x} - \frac{1}{1+\sqrt{x}}\right).$$

Corollary 4.5. *Let $f, g \geq 0,\ 0 < \alpha < 1$, s is a non-negative integer, Then*

$$\left(\int_0^\infty\int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\mu f(x)\,f(y)}{h(x,y)}\,dxdy\right)^2 \leq C^2\int_0^\infty f^2(x)\,dx\int_0^\infty g^2(x)\,dx, \qquad (11)$$

where

$$C = (2s)!\,(\xi(2s+1,\alpha) + \xi(2s+1,1-\alpha)),$$

provided the integrals on the RHS do exists.

Proof. The proof follows from Theorem 3.1 via Lemma 2.3 by putting

$$h(x,y) = x+y, \quad a = \alpha-1, \quad \mu = 2s, \quad k(x) = 1, \ \lambda = 1.$$

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