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COMPARISON OF LEAPFROG AND SINGLE TERM HAAR WAVELET SERIES METHOD TO SOLVE THE SECOND ORDER LINEAR SYSTEM WITH SINGULAR-A

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Abstract: In this paper presents a comparison of Leapfrog and single-term Haar wavelet series (STHW) method to solve the second order linear system with singular-A. The results obtained using Leapfrog method and the STHW methods are compared with the exact solutions of the second order linear system with singular-A. It is observed that the result obtained using Leapfrog method is closer to the true solutions of the problems. Error graphs for the numerical results and exact solutions are presented in a graphical form to highlight the efficiency of this STHW.

Keywords: Haar wavelet; single-term Haar wavelet series (STHW); Leapfrog method; singular systems; system with singular-A.

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1. Introduction

Many of the real world problems that arise in the studies of mechanical vibrations, electrical circuits, planetary motions, etc., can be formulated as second order differential equations of

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the form

$$\ddot{y}(x) = f(x, y, \dot{y})$$

with initial condition $y(x_0) = y_0$ and $\dot{y}(x_0) = \dot{y}_0$. Singular systems are being applied to solve a variety of problems involved in various disciplines of science and engineering. They are applied to analyze neurological events and catastrophic behavior and they also provide a convenient form for the dynamical equations of large scale interconnected systems. Further, singular systems are found in many areas such as constrained mechanical systems, fluid dynamics, chemical reaction kinetics, simulation of electrical networks, electrical circuit theory, power systems, aerospace engineering, robotics, aircraft dynamics, neural networks, neural delay systems, network analysis, time series analysis, system modeling, social systems, economic systems, biological systems etc. [1, 2, 3, 7-14].

Wazwaz (1994) published a paper on modified Runge-Kutta formula based on a variety of means of third order. Murugesan *et al.* (1999,2000,2001) have analyzed different second-order systems and multivariable linear systems via RK method based on centroidal mean, and also, they extended RK formulae based on variety of means to solve system of IVPs. A second order linear system with singular-A of the form

$$K\ddot{x}(t) = A\dot{x}(t) + Bx(t) + Cu(t)$$

with initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ where K is an $n \times n$ matrix, A is an $n \times n$ singular matrix and B and C are $n \times p$ constant matrices respectively. $x(t)$ is an n-state vector and $u(t)$ is the p-input control vector. This system with singular-A has many aspects and applications.

STHW can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. S. Sekar and team of his researchers [7 - 13] introduced the STHW to study the time-varying nonlinear singular systems, analysis of the differential equations of the sphere, to study on CNN based hole-filter template design, analysis of the singular and stiff delay systems and nonlinear singular systems from fluid dynamics, numerical investigation of nonlinear volterra-hammerstein integral equations, to study on periodic and oscillatory problems, and numerical solution of nonlinear problems in the calculus of variations. In this paper, we consider the system with singular-A to solve by using the Leapfrog method. The results are compared with STHW method and with exact solution of the problem.

2. Leapfrog method

In mathematics Leapfrog integration is a simple method for numerically integrating differential equations of the form $\ddot{x} = F(x)$, or equivalently of the form $\dot{v} = F(x), \dot{x} \equiv v$, particularly in the case of a dynamical system of classical mechanics. Such problems often take the form $\ddot{x} = -\nabla V(x)$, with energy function $E(x,v) = \frac{1}{2}|v|^2 + V(x)$, where V is the potential energy of the system. The method is known by different names in different disciplines. In particular, it is similar to the Velocity Verlet method, which is a variant of Verlet integration. Leapfrog integration is equivalent to updating positions $x(t)$ and velocities $v(t) = \dot{x}(t)$ at interleaved time points, staggered in such a way that they 'Leapfrog' over each other. For example, the position is updated at integer time steps and the velocity is updated at integer-plus-a-half time steps.

Leapfrog integration is a second order method, in contrast to Euler integration, which is only first order, yet requires the same number of function evaluations per step. Unlike Euler integration, it is stable for oscillatory motion, as long as the time-step Δt is constant, and $\Delta t \le 2/w$. In Leapfrog integration, the equations for updating position and velocity are

$$\begin{aligned} x_i &= x_{i-1} + v_{i-1/2}\Delta t, \\ a_i &= F(x_i) \\ v_{i+1/2} &= v_{i-1/2} + a_i \Delta t, \end{aligned}$$

where x_i is position at step $i, v_{i+1/2}$, is the velocity, or first derivative of x, at step $i + 1/2, a_i = F(x_i)$ is the acceleration, or second derivative of x, at step i and Δt is the size of each time step. These equations can be expressed in a form which gives velocity at integer steps as well. However, even in this synchronized form, the time-step Δt must be constant to maintain stability.

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} a_i \Delta t^2,$$
$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1})\Delta t.$$

One use of this equation is in gravity simulations, since in that case the acceleration depends only on the positions of the gravitating masses, although higher order integrators (such as Runge–Kutta methods) are more frequently used. There are two primary strengths to Leapfrog integration when applied to mechanics problems. The first is the time-reversibility of the Leapfrog method. One can integrate forward *n* steps, and then reverse the direction of integration and integrate backwards *n* steps to arrive at the same starting position. The second strength of Leapfrog integration is its symplectic nature, which implies that it conserves the (slightly modified) energy of dynamical systems. This is especially useful when computing orbital dynamics, as other integration schemes, such as the Runge-Kutta method, do not conserve energy and allow the system to drift substantially over time.

3. Error terms

The local error in position of the this integrator is $O(\Delta t^4)$as described above, and the local error in velocity is $O(\Delta t^2)$. The global error in position, in contrast, is $O(\Delta t^2)$and the global error in velocity is $O(\Delta t^2)$. These can be derived by noting the following:

$$error(x(t_0 + \Delta t)) = O(\Delta t^4)$$

and

$$x(t_0 + 2\Delta t) = 2x(t_0 + \Delta t) - x(t_0) + \Delta t^2 x''(t_0 + \Delta t) + O(\Delta t^4)$$

Therefore:

$$error(x(t_0 + 2\Delta t)) = 2error(x(t_0 + \Delta t)) + O(\Delta t^4) = 3O(\Delta t^4)$$

Similarly:

$$error(x(t_0 + 3\Delta t)) = 6O(\Delta t^4)$$
$$error(x(t_0 + 4\Delta t)) = 10O(\Delta t^4)$$
$$error(x(t_0 + 5\Delta t)) = 15O(\Delta t^4)$$

Which can be generalized to (it can be shown by induction, but it is given here without proof):

$$error(x(t_0 + n\Delta t)) = \frac{n(n+1)}{2} O(\Delta t^4)$$

If we consider the global error in position between $x(t)$ and $x(t+T)$, where $T = n\Delta t$, it is clear that:

$$error(x(t_0 + T)) = \left(\frac{T^2}{2\Delta t^2} + \frac{T}{2\Delta t} \right) O(\Delta t^4)$$

And therefore, the global (cumulative) error over a constant interval of time is given by:

$$error(x(t_0 + T)) = O(\Delta t^2)$$

Because the velocity is determined in a non-cumulative way from the positions in this integrator, the global error in velocity is also $O(\Delta t^2)$. In molecular dynamics simulations, the global error is typically far more important than the local error, and this integrator is therefore known as a second-order integrator.

4. Second order linear system with singular – A

When the second order linear system with singular-*A* and *B=C=0* is considered, it becomes

$$\ddot{x}(t) = A\dot{x}(t) \tag{1}$$

with initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

By taking $A = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix}$ along with the initial conditions $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\dot{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ equation (1) becomes

$$\begin{aligned} \ddot{x}_1(t) &= 2\dot{x}_2(t) \\ \ddot{x}_2(t) &= -6\dot{x}_2(t) \end{aligned}$$

Therefore the exact solution is

$$x_1(t) = \frac{1}{18} e^{-6t} + \left(\frac{4}{3} \right) t + \frac{17}{18},$$

$$x_2(t) = \frac{7}{6} - \frac{1}{6} e^{-6t},$$

$$\dot{x}_1(t) = -\frac{1}{3}e^{-6t} + \frac{4}{3} \text{ and } \dot{x}_2(t) = e^{-6t}$$

The approximate and exact solutions are calculated for the above problem mention in this section using Leapfrog method and STHW method for x_1 and x_2 for different time intervals and the error between them are shown in the Tables 1 and 2 along with exact solutions. Error graphs are presented in Figures 1and 2 to highlight the effectiveness of the Leapfrog method.

Table 1. Solutions for the problem in section 3 at various values of x_1.

S. No	Time t	Approximate solution of x_1				
		Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
1	0	1.000000	1.000000	0	1.000000	0
2	0.25	1.290174	1.290174	1E-06	1.290174	1E-08
3	0.5	1.613877	1.613877	1E-06	1.613877	1E-08
4	0.75	1.945062	1.945062	2E-06	1.945062	2E-08
5	1	2.277916	2.277916	2E-06	2.277916	2E-08
6	1.25	2.611142	2.611142	2E-06	2.611142	2E-08
7	1.5	2.944451	2.944451	3E-06	2.944451	3E-08
8	1.75	3.277779	3.277779	3E-06	3.277779	3E-08
9	2	3.611111	3.611111	3E-06	3.611111	3E-08

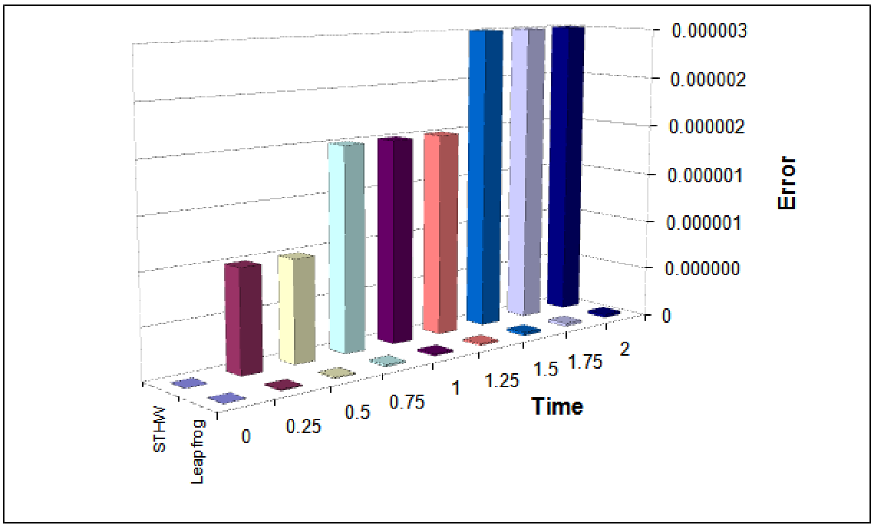

Figure 1. Error graph for problem in section 3 at x_1

Table 2. Solutions for the problem in section 3 at various values of x_2 .

S. No	Time t	Approximate solution of x_2				
		Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
1	0	1.000000	1.000000	0	1.000000	0
2	0.25	1.129478	1.129478	1E-06	1.129478	1E-08
3	0.5	1.158368	1.158368	1E-06	1.158368	1E-08
4	0.75	1.164815	1.164815	0	1.164815	0
5	1	1.166254	1.166254	3E-06	1.166254	2E-08
6	1.25	1.166574	1.166574	3E-06	1.166574	2E-08
7	1.5	1.166646	1.166646	4E-06	1.166646	3E-08
8	1.75	1.166662	1.166662	4E-06	1.166662	3E-08
9	2	1.166666	1.166666	4E-06	1.166666	3E-08

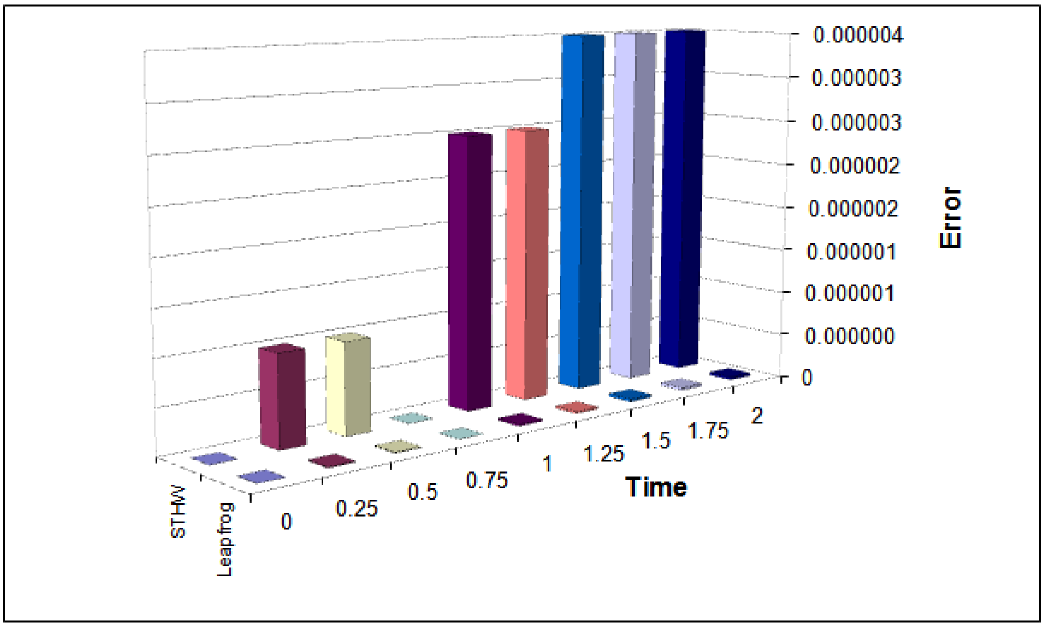

Figure 2. Error graph for problem in section 3 at x_2

5. Second order multivariable linear system with singular-A involving three variables

When a second order linear multivariable system with singular-*A* of the form (1) is considered.

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \text{ and } \quad u = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

with $x(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and $\dot{x}(0) = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}^T$. Hence, the equation (1) becomes

$$\ddot{x}_1 = \dot{x}_3 - 3x_1 + 1$$

$$\ddot{x}_2 = -2x_2$$

$$\ddot{x}_3 = -3x_3$$

The exact solution is

$$x_1 = -\left(\frac{\sqrt{3}}{2}\right)t \sin\sqrt{3}t - \frac{\cos\sqrt{3}t}{3} + \frac{1}{3}$$

$$x_2 = \sqrt{2}\sin\sqrt{2}t$$

$$x_3 = \sqrt{3}\sin\sqrt{3}t$$

Table 3. Solutions for the problem in section 4 at various values of x_1.

S. No	Time t	Approximate solution of x_1				
		Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
1	0	0.000000	0.000000	0	0.000000	0
2	0.25	-0.060082	-0.060082	5E-06	-0.060082	0
3	0.5	-0.212472	-0.212472	8E-06	-0.212472	1E-07
4	0.75	-0.381824	-0.381824	2E-06	-0.381824	1E-07
5	1	-0.467938	-0.467938	1E-06	-0.467938	2E-08
6	1.25	-0.376974	-0.376974	9E-06	-0.376974	2E-08
7	1.5	-0.053164	-0.053164	2E-06	-0.053164	2E-09
8	1.75	0.497501	0.497501	6E-06	0.497501	3E-09
9	2	1.198450	1.198450	9E-06	1.198450	3E-08

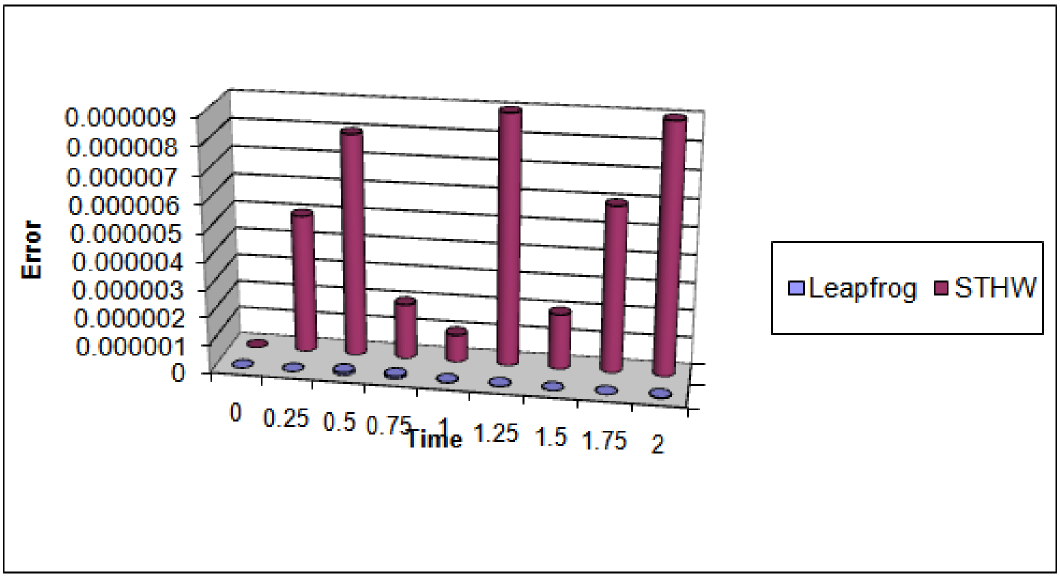

Figure 3. Error graph for problem in section 4 at x_1

Table 4. Solutions for the problem in section 4 at various values of x_2.

S. No	Time t	Approximate solution of x_2				
		Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
1	0	0.000000	0.000000	53E-06	0.000000	1E-06
2	0.25	0.489648	0.489648	53E-06	0.489648	1E-07
3	0.5	0.918725	0.918725	78E-06	0.918725	1E-07
4	0.75	1.234153	1.234153	92E-06	1.234153	1E-07
5	1	1.396911	1.396911	91E-06	1.396911	2E-08
6	1.25	1.386868	1.386868	19E-06	1.386868	2E-08
7	1.5	1.205264	1.205264	62E-06	1.205264	2E-09
8	1.75	0.874565	0.874565	26E-06	0.874565	3E-09
9	2	0.435679	0.435679	99E-06	0.435679	3E-08

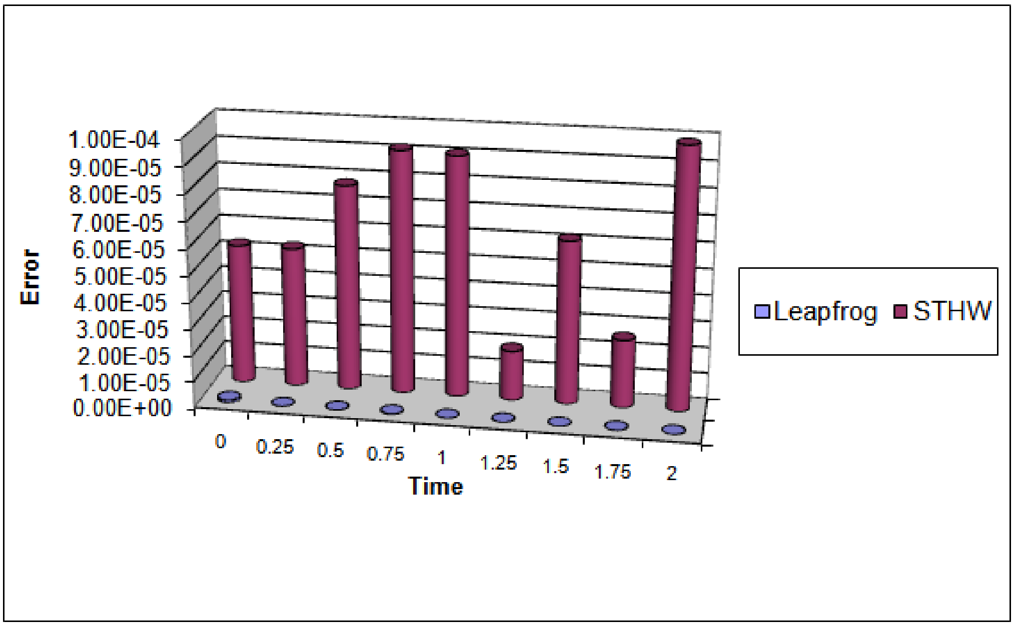

Figure 4. Error graph for problem in section 4 at x_2

Table 5. Solutions for the problem in section 4 at various values of x_3.

S. No	Time t	Approximate solution of x_3				
		Exact Solutions	STHW Solutions	STHW Error	Leapfrog Solutions	Leapfrog Error
1	0	0.000000	0.000000	53E-06	0.000000	1E-09
2	0.25	0.726781	0.726781	53E-06	0.726781	1E-09
3	0.5	1.319407	1.319407	78E-06	1.319407	1E-09
4	0.75	1.668485	1.668485	92E-06	1.668485	1E-09
5	1	1.709580	1.709580	91E-06	1.709580	2E-09
6	1.25	1.435106	1.435106	19E-06	1.435106	2E-09
7	1.5	0.895728	0.895728	62E-06	0.895728	2E-09
8	1.75	0.191008	0.191008	26E-06	0.191008	3E-09
9	2	-0.548969	-0.548969	99E-06	-0.548969	3E-09

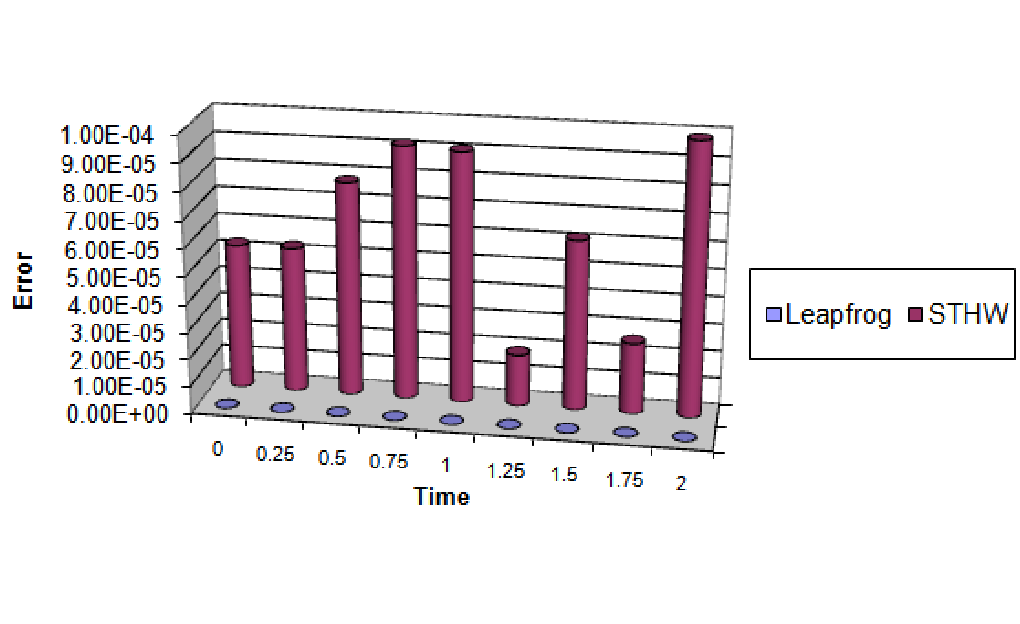

Figure 5. Error graph for problem in section 4 at x_3

Using Leapfrog method and STHW method to solve the above problem mention in this section, the approximate solutions and the exact solutions have been determined and are presented in Tables 3 - 5. The error graph for the three variables x_1, x_2 and x_3 are presented in the Figures 3 -5 respectively.

6. Conclusions

The Leapfrog is a powerful, accurate, and flexible tool for solving many types of singular systems (problems) in scientific computation. The obtained approximate solutions of the second order linear system with singular-*A* is compared with exact solutions and it reveals that the Leapfrog method works well for finding the approximate solutions. From the Tables 1 – 5, one can observe that for most of the time intervals, the absolute error is less (almost no error) in Leapfrog method when compared to the STHW method, which yields a little error, along with the exact solutions. From the Figures 1 – 5, it can be predicted that the error is very less in Leapfrog method when compared to the STHW method. Hence, Leapfrog method is more suitable for studying second order linear system with singular-A.

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