Available online at http://scik.org
J. Math. Comput. Sci. 5 (2015), No. 4, 499-506
ISSN: 1927-5307

ON SYMMETRIES OF GENERALIZED KEPLER PROBLEM WITH DRAG IN 2-D

ARUNAYE FESTUS IRIMISOSE *

Department of Mathematics and Computer Science, Delta State University, Abraka, Nigeria

Abstract: In this paper we considered generalized Kepler problem with drag in two dimensions in the analysis of Lie Symmetry of dynamical systems using reduction method. And we obtain its symmetries via reduction method, many of which are nonlocal type. We obtain the Laplace-Runge-Lenz vectors as well as the corresponding Ermanno-Bernoulli constants of this dynamical system. We also obtain the exact symmetry transformations of the dynamical system.

Keywords: dynamical systems; Kepler problem; Laplace-Runge-Lenz vectors; nonlocal symmetry; Lie groups.

2010 AMS Subject Classification: 34C14; 37C80; 37J15; 70S10.

1. Introduction

Since the inception of the concept of point symmetry for solving for the solutions of differential equations by Siphus Lie, the literature had received volumes in term of Lie point symmetry and group algebra of one differential equation or another. This formidable tool called Lie groups transient most fields of applied Mathematics and theoretical Physics to mention but a few. Kepler problem posed a challenge while determining the complete symmetry groups which specify its equation of motion completely. The manifestation of only five Lie point symmetry groups of the

*Corresponding author

Received August 15, 2014

Kepler equation of motion was not sufficient to specify the equation of motion of the Kepler problem completely (Nucci [1], Katzin and Levine, [2], [3]; Leach [4]); until Krause [5] introduced his concept of complete symmetry groups which combined both Lie point and nonlocal symmetries in the treatment of the Kepler problem. Essentially, the complete symmetry groups of differential equations is the group of symmetry transformations required to specify completely the differential equation (Andriopoulos, Leach and Flessas [6]; Andriopoulos and Flessas [7]. Krause [5] established that the Kepler problem could be completely specified by combining the five Lie point symmetries and some nonlocal symmetries of the Kepler problem (obtained by some ad hoc technique) although this is not the case for every differential equation. Krause [5] also opined that these nonlocal symmetries could not be obtained using the Lie algorithm; but this was not true by the work of Nucci [8]. It is hereby noted that the dimensions of the symmetry groups which specify a differential equation completely is not unique (Arunaye [9]; Andriopoulos, Leach and Flessas [6]).

Fundamentally, since the work of Krause [5], the concern for nonlocal symmetry became prominent in the literature. A significant physical application is the work of J. Krause on the Kepler problem (White [10]) it was observed there that there exist dynamical systems other than the Kepler problem for which its Lie symmetry is a symmetry group (White [11], [12]). More also the Nucci-reduce algorithm for computing symmetry groups of nontrivial dynamical problem with pencil and papers could be very complicated and complex. Suffices to say that one must develop ingenuity in the choice of change of variables and computational skills to reduce the nonlinear system to system of two linear equations with one second order and (n-1) first order (for an n-dimensional argument) equations in the reduced system which admits Lie algorithm to obtain the Lie symmetries. The importance of nonlocal symmetries in the integration of differential equations was treated by Abraham-Shrauner, Govinder and Leach [13] while considering a class of integrable second order equation with no Lie point symmetry (Leach and Andriopoulos [14]). The usefulness of nonlocal symmetries broadens the class of dynamical systems which can be reduced to algebraic form (Géronimi, Felix and Leach [15]) as well as establishing the integrability of differential equations on the existence of infinite nonlocal symmetries of differential equation (Leach and Andriopoulos [14]; Arunaye [9]). We note that the dimensions of the symmetry groups which specify a dynamical system completely are not unique (Andriopoulos, Leach and Flessas [6]; Arunaye [9]). Also the Kepler and related dynamical systems could be reduced to system of oscillator $u'' + u = 0$ and conservation law $L' = 0$; and it is well known that $u'' + u = 0$ has eight dimensional transformation groups. And such groups exist for any dynamical system admitting a Laplace-Runge-lenz (LRL) vector and translation symmetry for first integral arising out of the equation of motion for the angular momentum. These groups were reported to completely specify the dynamical system completely in the content of complete symmetry groups of Krause [5]. However it was later found that there are groups of smaller dimensions which have this property. The oscillator $u'' + u = 0$ was found to possess three complete symmetry algebras that is, algebras which generate complete symmetry groups (Andriopoulos and Leach [7] and Leach, Andriopoulos and Nucci [16]). In section 2, we introduce the generalized Kepler problem with drag force in its two components motion and reduced them to system of oscillator and conservation law. Section 3 presents the Lie symmetry group of the reduced systems as well as the corresponding symmetries of the original dynamical system most of which are nonlocal types.

2. Preliminaries

2.1 Reduction of the generalized Kepler problem with drag on the cone of motion

The equation governing the motion of the generalized Kepler problem with drag force is given by

$$\ddot{\mathbf{x}} - \left(\frac{\dot{g}}{2g} + \frac{3\dot{r}}{2r} \right) \dot{\mathbf{x}} + \mu g \mathbf{x} = 0 , \tag{1}$$

where $g = g(r)$, $|\mathbf{x}| = r$, and μ is arbitrary constant; while over-dot represent derivation with respect to time t (Arunaye, [9]). This dynamical system has its radial and transverse components of motion in the plane of motion respectively given by

$$\ddot{r} - r\dot{\theta}^2 - \left(\frac{\dot{g}}{2g} + \frac{3\dot{r}}{2r}\right)\dot{r} + \mu g r = 0\,, \tag{2}$$

and

$$r^{-1}(r^2\dot{\theta})^{\cdot} = \left(\frac{\dot{g}}{2g} + \frac{3\dot{r}}{2r}\right) r\dot{\theta} \quad . \tag{3}$$

Equation (2) implies

$$\dot{L} = \left(\frac{\dot{g}}{2g} + \frac{3\dot{r}}{2r}\right) = \frac{1}{2}\left(\ln g r^2\right)^{\cdot} L\,, \tag{4}$$

where $L = r^2\dot{\theta}$ is the magnitude of the angular momentum of the motion. On integrating (4) we obtain $L = A(gr^3)^{\frac{1}{2}}$ where A, is the constant of integration. On applying the isomorphic transformations formulae of Arunaye [9] for reducing dynamical systems to systems of oscillator(s) and conservation law(s), equations (2) and (3) respectively reduced to

$$u_{1,\theta\theta} + u_1 = 0\,, \tag{5}$$

$$u_{2,\theta} = 0\,. \tag{6}$$

Where $u_1 = u - \mu A^{-2} = u - \mu u_2^{-2}$, $u_2 = L(gr^3)^{-\frac{1}{2}}$; and $u_{2,\theta} = \dfrac{\partial u_2}{\partial \theta}$, $u_{1,\theta\theta} = \dfrac{\partial^2 u_1}{\partial \theta^2}$.

3. Main results

3.1 Symmetry groups of the dynamical system

The Lie symmetry groups of the reduced system is well known in the literature (Leach [17], [18]; Nucci [8], [19]; Prince and Eliezer [20], [21]; Bluman and Kumei [22]; Arunaye [23]), they are nine – eight from the reduced radial component of motion plus one translation symmetry from the reduced transverse component. The Lie symmetries of the reduced systems are

$$w_1 = u_2\partial_{u_2} \;\; ; w_2 = \partial_\theta \;\; ; \;\; w_3 = u_1\partial_{u_1}\,;$$

$$w_{4\pm} = e^{\pm i\theta}\partial_{u_1}\,; w_{6\pm} = e^{\pm i\theta}[\partial_\theta \pm i u_1 \partial_{u_1}]\,; w_{8\pm} = e^{\pm i\theta}[u_1\partial_\theta \pm i u_1^2 \partial_{u_1}]\,. \tag{7}$$

Nucci and Leach [1] presented the backward translation from the reduction variables to the original variables of the reduced symmetry generators in which most symmetry are non-local symmetries. Thus the corresponding symmetry groups in the original variables are

$$W_1 = 3t\partial_t + 2r\partial_r; \quad W_2 = \partial_\theta;$$

$$W_3 = 2[\mu\int r\,dt - A^{-2}t]\partial_t + r[A^{-2}\mu r - 1]\partial_r;$$

$$W_{4\pm} = 2[\int re^{\pm i\theta}dt]\partial_t + r^2 e^{\pm i\theta}\partial_r;$$

$$W_{6\pm} = 2[\int(\mu r + 3A^{-2})e^{\pm 2i\theta}dt]\partial_t + r(\mu r + 3A^{-2})e^{\pm 2i\theta}\partial_r + A^{-2}e^{\pm 2i\theta}\partial_\theta;$$

$$W_{8\pm} = 2[\int\{2\dot{r}A^{-3} \pm ir(\mu - r^3\dot{\theta}^2)(\mu + r^3\dot{\theta}^2)\}e^{\pm i\theta}dt]\partial_t +$$

$$r[2\dot{r}A^{-3} \pm ir(\mu - r^3\dot{\theta}^2)]\partial_r + A^{-2}(\mu - r^2\dot{\theta}^2)e^{\pm i\theta}\partial_\theta. \tag{8}$$

The above presentations of symmetry groups are done in such a way to simplify their complexity.

3.2 **Laplace-Runge-Lenz vector**

Dynamical system (1) possesses the Laplace-Runge-Lenz (LRL) vector (Arunaye [9])

$$J = L^{-2}(\dot{\mathbf{x}} \wedge \mathbf{L}) - \mu A^{-2}r^{-1}\mathbf{x} \tag{9}$$

where $A = \left(\frac{r}{g}\right)^{\frac{1}{2}}\dot{\theta}$; and $\wedge$ denotes vector product symbol. The corresponding Ermanno-Bernoulli constants of the dynamical system (1) are obtain as

$$J_\pm = (v_1 \pm iv_1')e^{\pm i\theta}, \tag{10}$$

where $v_1 = \frac{1}{r} - \mu A^{-2}$ depicts its natural reduction variable for reducing the dynamical systems (1) to system of oscillator(s) and conservations law(s) [see Arunaye [9] and References in it]. However one may take the constant multiple of v_1 as $u_1 = A^2 r^{-1} - \mu$ in the computation of exact symmetry transformations; for details see Arunaye [9] pg 61. We have shown there that exact symmetry transformations obtained using the constant multiples of natural reduction variables of dynamical systems are indifference.

3.3 **Exact symmetries**

We use the symmetries of the reduced Kepler with drag system to compute the exact symmetry transformations as follows. From the variable $u_1 = u - \mu A^{-2} = u - \mu u_2^{-2}$; and the fact $A = L\left(\frac{1}{r^3 g}\right)^{\frac{1}{2}}$, implies $A^{-2} = L^{-2} r^3 g$. The vector field $\alpha u_1 \partial_{u_1}$ generates the flow $f(u_1, u_2, \theta) = f(\bar{u}_1, \bar{u}_2, \bar{\theta})$ from which one gets the exact symmetry transformations

$$\bar{r} = H^{-1} r\,; \quad H = (1-C)\mu A^{-2} r^{-2} g^{-1} + C\,; C \geq 0 \tag{11}$$

and $$\frac{d\bar{t}}{dt} = H^{-2}, \tag{12}$$

where C is a constant.

While the vector field $\alpha u_2 \partial_{u_2}$ generates the flow $f(u_1, u_2, \theta) = f(\bar{u}_1, \bar{u}_2, \bar{\theta})$ that produce exact symmetry transformations

$$\bar{r} = H^{-1} r\,; H = 1 + (C^{-2} - 1)\mu A^{-2} r^{-2} g^{-1}; \quad C > 0 \tag{13}$$

and $$\frac{d\bar{t}}{dt} = C^{-1} H^{-2}, \tag{14}$$

where C is a constant.

3.4 Concluding remarks

It is established here that dynamical systems which possessed LRL vectors are linearizable through the reduction process. The manifestations of their LRL vectors and corresponding Ermanno-Bernoulli constants determine the nature of their symmetries. We note that reduction process enables easy means of obtaining the exact symmetry transformations of dynamical systems. The Lie symmetries of the generalized Kepler problem with drag force are not different from the classical Kepler problem in 2-D plane of motion.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The author is grateful to the Delta State University, Abraka; for the Study Leave with pay granted him for his PhD studies at The University of the West Indies, Mona and St. Augustine's Campuses.

REFERENCES

[1] M.C. Nucci, and P.G..L. Leach, The harmony in the Kepler and related problems. J. Math. Phys. 42 (2) (2001), 746-764.

[2] G.H. Katzin, and Jark Levine, A guage Invariant formulation of time-dependent dynamical symmetry mappings and associated constants of motion for Lagrangian particle mechanic I. J. Math. Phys. 17(7) (1976), 1345-1350.

[3] G.H. Katzin, and Jark Levine, Time-dependent dynamical symmetries and constants of motion III. Time-dependent harmonic oscillator. J. Math. Phys. 18(6) (1977), 1267-1274.

[4] P.G.L. Leach, The first integrals and orbit for the Kepler problem with drag. J. Phys. A: Math. Gen. 20 (1987), 997-2002.

[5] J. Krause, On the complete symmetry group of the classical Kepler system. J. Math. Phys. 35 (11) (1994), 5734-5748.

[6] K. Andriopoulos, P.G.L. Leach and G.P. Flessas, Complete symmetry Groups of ordinary Differential Equations and Their Integrals: Some Basic considerations. J. Math. Anal. and Appl. 262 (2001), 256-273.

[7] K. Andriopoulos and P.G.L. Leach, The Economy of complete symmetry groups for linear higher dimensional systems. J. Nonlinear Math. Phys. 9 (2) (2002), 10-23.

[8] M.C. Nucci, The complete Kepler group can be derived by Lie group analysis. J. Math. Phys. 37 (4) (1996), 1772-1775.

[9] F.I. Arunaye, Symmetries of differential equations: Topics in Nonlocal symmetries of dynamical systems.

GmbH, Berlin: LAP Lambert Academic Publishing, (2011).

[10] H. White, On Linearization of Dynamical systems Admitting a Laplace-Runge-Lenz Vector. JARDCS 1 (2) (2009), 1-17.

[11] H. White, Vector constants of motion for dynamical systems with nonlocal symmetries and applications. J. Advanced Research in Name. Vol. 00 No.0, (2009), 1-16.

[12] H. White, On vector constants of motion for dynamical systems admitting symmetry groups of the inverse cube law equation.IL NUOVO CIMENTO (2009), DOI 10.1393/ncb/i2009-10798-2.

[13] B. Abraham-Shrauner, K.S. Govinder and P.G.L. Leach, Integration of second order ordinary differential equations not possessing Lie point symmetries. Phys. Lett. A, 203 (1995), 169-174.

[14] P.G.L. Leach and K. Andriopoulos, Nonlocal symmetries Past, Present and Future. Appl. Anal. and Discrete Math. 1(2007), 150-171.

[15] C. Géronimi, M.R. Felix and P.G.L. Leach, First Integrals and parametric solutions for equations Integrable Through Lie symmetries. J. Nonlinear Math. Phys. 8(2) (2001), 211-216.

[16] P.G.L. Leach, K. Andriopoulos and M.C.Nucci, The Ermanno-Bernoulli constants and representations of the complete symmetry group of the Kepler problem. J. Math. Phys. 44 (9) (2003), 4090-4106.

[17] P.G.L. Leach, The complete symmetry group of the one-dimension time-dependent harmonic oscillator. J. Math. Phys. 21(2) (1980), 300-304.

[18] P.G.L. Leach, SL(3,R) and the repulsive oscillator. J. Phys. A: Math. Gen. 13 (1980), 1991-2000.

[19] M.C. Nucci, The role of symmetries in solving differential equations. Math. Comput. Modell. 25 (1997), 181-193.

[20] G.E. Prince and C.J. Eliezer, Symmetries of the time-dependent n-dimensional oscillator. J. Phys. A: Math. Gen.13 (1980), 815-823.

[21] G.E. Prince and C.J. Eliezer, On the Lie symmetries of the classical Kepler problem. J.Phys. A: Math. Gen. 14 (1981), 587-596.

[22] G.W. Bluman and S. Kumei Symmetries and Differential Equations. New York: Springer Verlag, (1989).

[23] F.I. Arunaye, Computing exact symmetries of dynamical systems from their reduced system of equations can be interesting II. WSEAS Transactions on Mathematics 7(7) (2008), 451-185.