Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 2, 152-171
ISSN: 1927-5307

INTERSECTION CURVES OF TWO IMPLICIT SURFACES IN $\mathbb{R}^3$

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Abstract. In this paper, we present algorithms for computing the differential geometry properties of Frenet apparatus $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau)$ of intersection curves of two implicit surfaces in $\mathbb{R}^3$, for transversal and tangential intersection using the implicit function theorem. We obtain a classification of the singularities on the intersection curve. Some examples are given and plotted.

Keywords: Geometric properties; Frenet frame; Frenet apparatus; Surface-surface intersection; Transversal intersection; Tangential intersection; Dupin indicatrices.

2000 AMS Subject Classification: 53A04; 53A05

1. Introduction

The intersection problem is a fundamental process needed in modeling complex shapes in CAD/CAM system. It is useful in the representation of the design of complex objects, in computer animation and in NC machining for trimming off the region bounded by

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Received December 05, 2011

the self-intersection curves of offset surfaces. It is also essential to Boolean operations necessary in the creation of boundary representation in solid modeling [1]. The numerical marching method is the most widely used method for computing intersection curves in $\mathbb{R}^3$. The Marching method involves generation of sequences of points of an intersection curve in the direction prescribed by the local differential geometry [2,3]. Willmore [4] described how to obtain the unit tangent, the unit principal normal, the unit binormal, the curvature and the torsion of the transversal intersection curve of two implicit surfaces [5]. Kruppa [6] explained that the tangential direction of the intersection curve at a tangential intersection point corresponds to the direction from the intersection point towards the intersection of the Dupin indicatrices of the two surfaces. Hartmann [7] provided formulas for computing the curvature of the transversal intersection curves for all types of intersection problems in Euclidean 2-space. Kriezis et al. [8] determined the marching direction for tangential intersection curves based on the fact that the determinant of the Hessian matrix of the oriented distance function is zero. Luo et al. [9] presented a method to trace such tangential intersection curves for parametric-parametric surfaces employing the marching method. The marching direction is obtained by solving an undetermined system based on the equilibrium of the differentiation of the two normal vectors and the projection of the Taylor expansion of the two surfaces onto the normal vector at the intersection point. Ye and Maekawa [1] presented algorithms for computing all the differential geometry properties of both transversal and tangentially intersection curves of two parametric surfaces. They described how to obtain these properties for two implicit surfaces or parametric-implicit surfaces. They also gave algorithms to evaluate the higher-order derivative of the intersection curves. Aléssio [10] studied the differential geometry properties of intersection curves of three implicit surfaces in $\mathbb{R}^4$ for transversal intersection, using the implicit function theorem. Our previous work Soliman et al. [11] presented algorithms for computing differential geometry properties of both transversal and tangentially intersection curves of implicit and parametric surfaces in $\mathbb{R}^3$. Our previous work Abdel-All et al. [12] presented algorithms for computing differential geometry properties of transversal intersection curves of implicit–parametric–parametric and implicit–implicit–parametric hypersurfaces in $\mathbb{R}^4$.

In this paper we present algorithms for computing deferential geometry properties of both transversal and tangentially intersection curves of two implicit surfaces in $\mathbb{R}^3$. The necessary and sufficient conditions for the intersection curve to be a straight line or a plane curve or helix or circular helix or circle are given. Finally some examples of transversal and tangentially intersection are given and plotted.

2. Geometric preliminaries [1,13–15]

Let us first introduce some notation and definitions. Bold letters such as $\boldsymbol{a}$, $\boldsymbol{\alpha}$ will be used for vectors and vector functions. The scalar product and cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ are expressed as $\langle \boldsymbol{a}, \boldsymbol{c} \rangle$ and $\boldsymbol{a} \times \boldsymbol{c}$ respectively. The triple scalar product of three vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are expressed as $\det(\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{d})$ and the length of the vector $\boldsymbol{a}$ is $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$. The transpose and the determinant of a square matrix A are expressed as A^T and $\det A$ respectively. The notation for the differentiation of a curve $\boldsymbol{\alpha}$ in relation to the arc length s is $\boldsymbol{\alpha}'(s) = \frac{d\boldsymbol{\alpha}}{ds}$, $\boldsymbol{\alpha}''(s) = \frac{d^2\boldsymbol{\alpha}}{ds^2}$, $\boldsymbol{\alpha}'''(s) = \frac{d^3\boldsymbol{\alpha}}{ds^3}$ and for a curve $\boldsymbol{\beta}$ with an arbitrary parameter u, it is $\dot{\boldsymbol{\beta}}(u) = \frac{d\boldsymbol{\beta}}{du}$, $\ddot{\boldsymbol{\beta}}(u) = \frac{d^2\boldsymbol{\beta}}{du^2}$, $\dddot{\boldsymbol{\beta}}(u) = \frac{d^3\boldsymbol{\beta}}{du^3}$, $\boldsymbol{\beta}^{(4)} = \frac{d^4\boldsymbol{\beta}}{du^4}$.

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ with arc-length parametrization,

$$\boldsymbol{\alpha}(s) = (x_1(s), x_2(s), x_3(s)). \tag{2.1}$$

Then from elementary differential geometry, we have

$$\boldsymbol{\alpha}'(s) = \boldsymbol{t}, \tag{2.2}$$

$$\boldsymbol{\alpha}''(s) = \kappa \boldsymbol{n}, \tag{2.3}$$

$$\kappa^2(s) = \langle \boldsymbol{\alpha}'', \boldsymbol{\alpha}'' \rangle, \tag{2.4}$$

where $\boldsymbol{t}$ is the unit tangent vector field and $\boldsymbol{\alpha}''$ is the curvature vector. The factor κ is the curvature and $\boldsymbol{n}$ is the unit principal normal vector. The unit binormal vector $\boldsymbol{b}$ is defined as

$$\boldsymbol{b}(s) = \boldsymbol{t} \times \boldsymbol{n}. \tag{2.5}$$

The Frenet–Serret formulas along $\boldsymbol{\alpha}$ are given by

$$\boldsymbol{t}'(s) = \kappa \boldsymbol{n}, \quad \boldsymbol{n}'(s) = -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \quad \boldsymbol{b}'(s) = -\tau \boldsymbol{n}, \tag{2.6}$$

where τ is the torsion which is given by

$$\tau = \frac{\langle \boldsymbol{b}, \boldsymbol{\alpha}''' \rangle}{\kappa}. \tag{2.7}$$

provided that the curvature does not vanish.

Let $\boldsymbol{\beta}\colon J \longrightarrow \mathbb{R}^3$ be a regular curve parametrized by a parameter u, with the same trace as the curve (2.1), i.e.

$$\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(s(u)). \tag{2.8}$$

Then from elementary differential geometry, we have

$$s = \int_{u_0}^{u} \|\dot{\boldsymbol{\beta}}(u)\| du, \quad \boldsymbol{t} = \frac{\dot{\boldsymbol{\beta}}(u)}{\|\dot{\boldsymbol{\beta}}(u)\|}, \tag{2.9}$$

$$\kappa \boldsymbol{n} = \frac{\left\langle \dot{\boldsymbol{\beta}}(u), \dot{\boldsymbol{\beta}}(u) \right\rangle \ddot{\boldsymbol{\beta}}(u) - \left\langle \dot{\boldsymbol{\beta}}(u), \ddot{\boldsymbol{\beta}}(u) \right\rangle \dot{\boldsymbol{\beta}}(u)}{\|\dot{\boldsymbol{\beta}}(u)\|^4}, \quad \kappa = \frac{\|\dot{\boldsymbol{\beta}}(u) \times \ddot{\boldsymbol{\beta}}(u)\|}{\|\dot{\boldsymbol{\beta}}(u)\|^3}, \tag{2.10}$$

$$\tau = \frac{\det(\dot{\boldsymbol{\beta}}(u), \ddot{\boldsymbol{\beta}}(u), \dddot{\boldsymbol{\beta}}(u))}{\|\dot{\boldsymbol{\beta}}(u) \times \ddot{\boldsymbol{\beta}}(u)\|^2} \mid \kappa \neq 0 \tag{2.11}$$

3. Transversal intersection curve

Consider two regular implicit surfaces $f(x_1, x_2, x_3) = 0$ and $h(x_1, x_2, x_3) = 0$. In other words $\boldsymbol{\nabla} f \neq 0$, $\boldsymbol{\nabla} h \neq 0$, where $\boldsymbol{\nabla} f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})$ is the gradient vector of f. We denote to partial derivatives of both surfaces by

$$f_i = \frac{\partial f}{\partial x_i}, \ f_{ii} = \frac{\partial^2 f}{\partial x_i^2}, \ f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \ldots,$$
$$h_i = \frac{\partial h}{\partial x_i}, \ h_{ii} = \frac{\partial^2 h}{\partial x_i^2}, \ h_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j}, \ldots$$

The unit surface normal vector field of the surface f is given by

$$\boldsymbol{N} = \frac{\boldsymbol{\nabla} f}{\|\boldsymbol{\nabla} f\|}. \tag{3.1}$$

Suppose that the curve (2.8) is an intersection curve of both surfaces. Then it can be expressed as

$$\boldsymbol{\beta}(u) = \alpha(s(u)) = (x_1(u), x_2(u), x_3(u)) \mid f(\boldsymbol{\beta}(u)) = 0,\ h(\boldsymbol{\beta}(u)) = 0.$$

Since $\boldsymbol{\beta}$ can be viewed as a curve on the surface f. Then, we have

$$\left\langle \boldsymbol{\nabla} f(\beta(u)), \dot{\boldsymbol{\beta}}(u) \right\rangle = 0,$$

which can be written in a matrix form as the follows

$$\nabla f \dot{\beta} = 0, \tag{3.2}$$

where $\nabla f = \begin{bmatrix} f_1(\beta(u)) & f_2(\beta(u)) & f_3(\beta(u)) \end{bmatrix}$, $\dot{\beta} = \begin{bmatrix} \dot{x}_1(u) & \dot{x}_2(u) & \dot{x}_3(u) \end{bmatrix}^T$. Differentiating Eq. (3.2) with respect to u yields

$$\nabla f \ddot{\beta} = -(\dot{x}_1 \frac{\partial}{\partial x_1}(\nabla f) + \dot{x}_2 \frac{\partial}{\partial x_2}(\nabla f) + \dot{x}_3 \frac{\partial}{\partial x_3}(\nabla f))\dot{\beta}.$$

which can be written as the follows

$$\nabla f \ddot{\beta} = -\dot{\beta}^T H_1 \dot{\beta}, \tag{3.3}$$

where $H_1 = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$ is the Hessian matrix of the function f. Differentiating Eq. (3.3) with respect to u and using the fact $\ddot{\beta}^T H_1 \dot{\beta} = (\dot{\beta}^T H_1 \ddot{\beta})^T = \dot{\beta}^T H_1 \ddot{\beta}$, we have

$$\nabla f \dddot{\beta} = -3\dot{\beta}^T H_1 \ddot{\beta} - \dot{\beta}^T (\nabla H_1 \dot{\beta})\dot{\beta}. \tag{3.4}$$

Differentiating Eq. (3.4) with respect to u yields

$$\nabla f \beta^{(4)} = -5\dot{\beta}^T (\nabla H_1 \dot{\beta})\ddot{\beta} - 4\dot{\beta}^T H_1 \dddot{\beta} - 3\ddot{\beta}^T H_1 \ddot{\beta} - \dot{\beta}^T (\nabla(\nabla H_1 \dot{\beta})\dot{\beta})\dot{\beta}. \tag{3.5}$$

3.1. Tangential direction

Assume that f and h have continuous first derivatives and if, at least one of the Jacobian determinants $\frac{\partial(f,h)}{\partial(x_1,x_2)}$, $\frac{\partial(f,h)}{\partial(x_2,x_3)}$ and $\frac{\partial(f,h)}{\partial(x_1,x_3)}$ is not zero at a point P_0 on the curve $\boldsymbol{\beta}$, then from the implicit function theorem, the surfaces f and h can be solved for two of the variables in terms of the third. Assume that $\frac{\partial(f,h)}{\partial(x_2,x_3)} \neq 0$, then we can write

$$\boldsymbol{\beta}(x_1) = (x_1, x_2(x_1), x_3(x_1)) \mid f(\boldsymbol{\beta}(x_1)) = 0,\ h(\boldsymbol{\beta}(x_1)) = 0. \tag{3.6}$$

Differentiating Eq. (3.6) with respect to x_1 yields

$$\dot{\boldsymbol{\beta}}(x_1) = (1, \dot{x}_2(x_1), \dot{x}_3(x_1)). \tag{3.7}$$

Using Eqs. (3.2), (3.6) and (3.7) yields

$$\nabla f \dot{\beta} = 0, \quad \nabla h \dot{\beta} = 0$$

which can be written in a matrix form as the follows

$$\begin{bmatrix} f_1 & f_2 & f_3 \\ h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} 1 & \dot{x}_2 & \dot{x}_3 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.8}$$

Solving the coefficients $\dot{x}_2$ and $\dot{x}_3$ from linear system (3.8) and substituting into (3.7) yields

$$\dot{\boldsymbol{\beta}}(x_1) = \frac{(A_{23}, -A_{13}, A_{12})}{A_{23}}, \tag{3.9}$$

where

$$A_{ij} = \det \begin{bmatrix} f_i & f_j \\ h_i & h_j \end{bmatrix}. \tag{3.10}$$

Then the speed and the arc length of $\boldsymbol{\beta}$ can be obtained respectively, by

$$\|\dot{\boldsymbol{\beta}}(x_1)\| = \frac{\sqrt{A_{12}^2 + A_{13}^2 + A_{23}^2}}{|A_{23}|}. \tag{3.11}$$

$$S = \int_{x_1^0}^{x_1} \frac{\sqrt{(A_{12})^2 + (A_{13})^2 + (A_{23})^2}}{|A_{23}|} dx_1, \tag{3.12}$$

where x_1^0 is the value of x_1 at P_0. The unit tangent vector field of the intersection curve is given by

$$\boldsymbol{t} = \frac{\sigma(A_{23}, -A_{13}, A_{12})}{\sqrt{A_{12}^2 + A_{13}^2 + A_{23}^2}}, \ \sigma = \begin{cases} 1 & \text{if} \quad A_{23} > 0 \\ -1 & \text{if} \quad A_{23} < 0 \end{cases}. \tag{3.13}$$

3.2. Curvature and curvature vector

Differentiating Eq. (3.7) with respect to x_1 and using Eqs. (3.3) and (3.6), we obtain

$$\ddot{\boldsymbol{\beta}}(x_1) = (0, \ddot{x}_2(x_1), \ddot{x}_3(x_1)), \tag{3.14}$$

$$\nabla f \ddot{\beta} = -\dot{\beta}^T H_1 \dot{\beta}, \quad \nabla h \ddot{\beta} = -\dot{\beta}^T H_2 \dot{\beta}, \tag{3.15}$$

where $H_2 = (h_{ij})$. The linear system (3.15) can be written in a matrix form as the follows

$$\begin{bmatrix} f_2 & f_3 \\ h_2 & h_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} -\dot{\beta}^T H_1 \dot{\beta} \\ -\dot{\beta}^T H_2 \dot{\beta} \end{bmatrix}.$$

Solving the coefficients $\ddot{x}_2$ and $\ddot{x}_3$ from above linear system yields

$$\begin{aligned} \ddot{x}_2 &= \frac{1}{A_{23}^3}\begin{pmatrix} B_{311}A_{23}^2 + B_{322}A_{13}^2 + B_{333}A_{12}^2 \\ +2(B_{313}A_{12}A_{23} - B_{312}A_{13}A_{23} - B_{323}A_{13}A_{12}) \end{pmatrix}, \\ \ddot{x}_3 &= \frac{-1}{A_{23}^3}\begin{pmatrix} B_{211}A_{23}^2 + B_{222}A_{13}^2 + B_{233}A_{12}^2 \\ +2(B_{213}A_{12}A_{23} - B_{212}A_{13}A_{23} - B_{223}A_{13}A_{12}) \end{pmatrix}, \end{aligned} \tag{3.16}$$

where

$$B_{ijk} = \det \begin{bmatrix} f_i & h_i \\ f_{jk} & h_{jk} \end{bmatrix}. \tag{3.17}$$

The curvature vector and the curvature can be calculated using Eqs. (2.10), (3.9) and (3.14) respectively, as

$$\kappa \boldsymbol{n} = \left(\frac{A_{23}}{A_{12}^2 + A_{13}^2 + A_{23}^2}\right)^2 \begin{pmatrix} A_{23}A_{13}\ddot{x}_2 - A_{23}A_{12}\ddot{x}_3, (A_{12}^2 + A_{23}^2)\ddot{x}_2 \\ +A_{13}A_{12}\ddot{x}_3, (A_{13}^2 + A_{23}^2)\ddot{x}_3 + A_{13}A_{12}\ddot{x}_2 \end{pmatrix}, \tag{3.18}$$

$$\kappa = A_{23}^2\sqrt{\frac{(A_{12}^2+A_{23}^2)\ddot{x}_2^2+(A_{13}^2+A_{23}^2)\ddot{x}_3^2+2A_{12}A_{13}\ddot{x}_2\ddot{x}_3}{(A_{12}^2+A_{13}^2+A_{23}^2)^3}}. \tag{3.19}$$

3.3. Torsion and third-order derivative vector

Differentiating Eq. (3.14) with respect to x_1 and using Eqs. (3.4) and (3.6) yields

$$\dddot{\boldsymbol{\beta}}(x_1) = (0, \dddot{x}_2(x_1), \dddot{x}_3(x_1)), \tag{3.20}$$

$$\nabla f\dddot{\beta} = -3\dot{\beta}^T H_1\ddot{\beta} - \dot{\beta}^T(\nabla H_1\dot{\beta})\dot{\beta}, \quad \nabla h\dddot{\beta} = -3\dot{\beta}^T H_2\ddot{\beta} - \dot{\beta}^T(\nabla H_2\dot{\beta})\dot{\beta}.$$

Solving the coefficients $\dddot{x}_2$ and $\dddot{x}_3$ from above linear system yields

$$\begin{bmatrix}\dddot{x}_2\\ \dddot{x}_3\end{bmatrix} = \frac{-1}{A_{23}^4}\begin{bmatrix}h_3 & -f_3\\ -h_2 & f_2\end{bmatrix}\begin{bmatrix}c_{11}\ddot{x}_2+c_{12}\ddot{x}_3+c_{13}\\ c_{21}\ddot{x}_2+c_{22}\ddot{x}_3+c_{23}\end{bmatrix}.$$

Explicitly

$$\begin{aligned}\dddot{x}_2 &= A_{23}^{-4}((f_3c_{21}-h_3c_{11})\,\ddot{x}_2+(f_3c_{22}-h_3c_{12})\,\ddot{x}_3+f_3c_{23}-h_3c_{13}),\\ \dddot{x}_3 &= -A_{23}^{-4}((f_2c_{21}-h_2c_{11})\,\ddot{x}_2+(f_2c_{22}-h_2c_{12})\,\ddot{x}_3+f_2c_{23}-h_2c_{13}),\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}c_{11} &= 3(A_{23}^3f_{12}-A_{13}A_{23}^2f_{22}+A_{12}A_{23}^2f_{23}),\\ c_{21} &= 3(A_{23}^3h_{12}-A_{13}A_{23}^2h_{22}+A_{12}A_{23}^2h_{23}),\\ c_{12} &= 3(A_{23}^3f_{13}-A_{13}A_{23}^2f_{23}+A_{12}A_{23}^2f_{33}),\\ c_{22} &= 3(A_{23}^3h_{13}-A_{13}A_{23}^2h_{23}+A_{12}A_{23}^2h_{33}),\\ c_{13} &= \left(\begin{array}{c}A_{23}^3f_{111}-6A_{12}A_{13}A_{23}f_{123}-A_{13}^3f_{222}+A_{12}^3f_{333}\\ +3\left(\begin{array}{c}A_{12}A_{23}^2f_{113}-A_{13}A_{23}^2f_{112}+A_{13}^2A_{23}f_{122}\\ +A_{12}^2A_{23}f_{133}-A_{13}A_{12}^2f_{233}+A_{12}A_{13}^2f_{223}\end{array}\right)\end{array}\right),\end{aligned} \tag{3.21}$$

$$c_{23} = \left(\begin{array}{c}A_{23}^3h_{111}-6A_{12}A_{13}A_{23}h_{123}-A_{13}^3h_{222}+A_{12}^3h_{333}\\ +3\left(\begin{array}{c}A_{12}A_{23}^2h_{113}-A_{13}A_{23}^2h_{112}+A_{13}^2A_{23}h_{122}\\ +A_{12}^2A_{23}h_{133}-A_{13}A_{12}^2h_{233}+A_{12}A_{13}^2h_{223}\end{array}\right)\end{array}\right).$$

The torsion can be calculated by using Eqs. (2.10), (2.11) and (3.19) as the follows

$$\tau = \frac{(\ddot{x}_2 \dddot{x}_3 - \ddot{x}_3 \dddot{x}_2)}{\|\dot{\boldsymbol{\beta}}\|^6 \kappa^2}.$$

Explicitly

$$\tau = \frac{A_{23}^2(\ddot{x}_2 \dddot{x}_3 - \ddot{x}_3 \dddot{x}_2)}{(A_{12}^2 + A_{23}^2)\ddot{x}_2^2 + (A_{13}^2 + A_{23}^2)\ddot{x}_3^2 + 2A_{12}A_{13}\ddot{x}_2\ddot{x}_3}. \tag{3.23}$$

From the foregoing results, we have the following corollaries:

Corollary 3.1. The necessary and sufficient condition for the curve to be a straight line ($\kappa = 0$) is given by

$$\begin{aligned} B_{311}A_{23}^2 + B_{322}A_{13}^2 + B_{333}A_{12}^2 &= 2(B_{312}A_{13}A_{23} + B_{323}A_{13}A_{12} - B_{313}A_{12}A_{23}), \\ B_{211}A_{23}^2 + B_{222}A_{13}^2 + B_{233}A_{12}^2 &= 2(B_{212}A_{13}A_{23} + B_{223}A_{13}A_{12} - B_{213}A_{12}A_{23}). \end{aligned} \tag{3.24}$$

Corollary 3.2. The necessary and sufficient condition for the curve to be a plane curve ($\tau = 0$) is given by

$$\left(\begin{array}{l} (c_{22}f_3 - c_{12}h_3)\ddot{x}_3^2 + (c_{21}f_2 - c_{11}h_2)\ddot{x}_2^2 + (c_{23}f_3 - c_{13}h_3)\ddot{x}_3 \\ +(c_{23}f_2 - c_{13}h_2)\ddot{x}_2 + (c_{21}f_3 + c_{22}f_2 - c_{11}h_3 - c_{12}h_2)\ddot{x}_2\ddot{x}_3 \end{array} \right) = 0 \tag{3.25}$$

Corollary 3.3. The necessary and sufficient condition for the curve to be a helix ($\frac{\kappa}{\tau} = c_1 =$const.) is given by

$$\begin{aligned} &(A_{12}^2 + A_{13}^2 + A_{23}^2)^{-3}((A_{12}^2 + A_{23}^2)\ddot{x}_2^2 + (A_{13}^2 + A_{23}^2)\ddot{x}_3^2 + 2A_{12}A_{13}\ddot{x}_2\ddot{x}_3)^3 \\ &= c_1 A_{23}^{-8} \left(\begin{array}{l} (c_{22}f_3 - c_{12}h_3)\ddot{x}_3^2 + (c_{21}f_2 - c_{11}h_2)\ddot{x}_2^2 + (c_{23}f_3 - c_{13}h_3)\ddot{x}_3 \\ +(c_{23}f_2 - c_{13}h_2)\ddot{x}_2 + (c_{21}f_3 + c_{22}f_2 - c_{11}h_3 - c_{12}h_2)\ddot{x}_2\ddot{x}_3 \end{array} \right)^2 \end{aligned} \tag{3.26}$$

Corollary 3.4. The necessary and sufficient condition for the curve to be a circular helix ($\kappa = c_2 =$const., $\tau = c_3 =$const.) is given by

$$\begin{aligned} &A_{23}^4((A_{12}^2 + A_{23}^2)\ddot{x}_2^2 + (A_{13}^2 + A_{23}^2)\ddot{x}_3^2 + 2A_{12}A_{13}\ddot{x}_2\ddot{x}_3) = c_2(A_{12}^2 + A_{13}^2 + A_{23}^2)^3, \\ &\left(\begin{array}{l} (c_{22}f_3 - c_{12}h_3)\ddot{x}_3^2 + (c_{21}f_2 - c_{11}h_2)\ddot{x}_2^2 + (c_{23}f_3 - c_{13}h_3)\ddot{x}_3 \\ +(c_{23}f_2 - c_{13}h_2)\ddot{x}_2)^2 + (c_{21}f_3 + c_{22}f_2 - c_{11}h_3 - c_{12}h_2)\ddot{x}_2\ddot{x}_3 \end{array} \right) \\ &= c_3 A_{23}^2((A_{12}^2 + A_{23}^2)\ddot{x}_2^2 + (A_{13}^2 + A_{23}^2)\ddot{x}_3^2 + 2A_{12}A_{13}\ddot{x}_2\ddot{x}_3) \end{aligned} \tag{3.27}$$

Corollary 3.5. The necessary and sufficient condition for the curve to be a circle ($\kappa = c_4$ =const., $\tau = 0$) is given by

$$A_{23}^4((A_{12}^2 + A_{23}^2)\ddot{x}_2^2 + (A_{13}^2 + A_{23}^2)\ddot{x}_3^2 + 2A_{12}A_{13}\ddot{x}_2\ddot{x}_3) = c_4(A_{12}^2 + A_{13}^2 + A_{23}^2)^3,$$

(3.28)
$$\left(\begin{array}{c} (c_{22}f_3 - c_{12}h_3)\ddot{x}_3^2 + (c_{21}f_2 - c_{11}h_2)\ddot{x}_2^2 + (c_{23}f_3 - c_{13}h_3)\ddot{x}_3 \\ +(c_{23}f_2 - c_{13}h_2)\ddot{x}_2 + (c_{21}f_3 + c_{22}f_2 - c_{11}h_3 - c_{12}h_2)\ddot{x}_2\ddot{x}_3 \end{array}\right) = 0.$$

4. Tangentially Intersection curve

Two surfaces intersect tangentially when whose normal vectors are parallel to each other. Assume that the surfaces $f(x_1, x_2, x_3) = 0$ and $h(x_1, x_2, x_3) = 0$ intersect tangentially at a point P, so that $\boldsymbol{N}^{(f)} // \boldsymbol{N}^{(h)}$ at P, i.e.

$$\frac{\boldsymbol{\nabla} f(\boldsymbol{\alpha}(s))}{\|\boldsymbol{\nabla} f(\boldsymbol{\alpha}(s))\|} = \pm \frac{\boldsymbol{\nabla} h(\boldsymbol{\alpha}(s))}{\|\boldsymbol{\nabla} h(\boldsymbol{\alpha}(s))\|}.$$

Which can be written as

$$\boldsymbol{\nabla} f(\alpha(s)) = A(\alpha(s))\boldsymbol{\nabla} h(\alpha(s)), \quad A(\alpha(s)) = \pm \frac{\|\boldsymbol{\nabla} f(\alpha(s))\|}{\|\boldsymbol{\nabla} h(\alpha(s))\|}. \tag{4.1}$$

Then

$$f_i(\alpha(s)) = A(\alpha(s))h_i(\alpha(s)). \tag{4.2}$$

Thus, the tangential points or tangential curve are determined by solving the following system of equations

$$f = 0, \quad h = 0, \quad f_1 = Ah_1, \quad f_2 = Ah_2, \quad f_3 = Ah_3. \tag{4.3}$$

4.1. Tangential direction

Assume that both surfaces are intersected tangentially at a curve $\boldsymbol{\beta}$ which is parametrized by the variable x_1, say. Then

$$\boldsymbol{\beta}(x_1) = (x_1, x_2(x_1), x_3(x_1)) \mid f = 0, \; h = 0, \; f_1 = Ah_1, \; f_2 = Ah_2, \; f_3 = Ah_3. \tag{4.4}$$

Differentiating Eq. (4.4) with respect to x_1 and using Eqs. (3.2) and (4.4) yields

$$\dot{\boldsymbol{\beta}}(x_1) = (1, \dot{x}_2(x_1), \dot{x}_3(x_1)), \quad \ddot{\boldsymbol{\beta}}(x_1) = (0, \ddot{x}_2(x_1), \ddot{x}_3(x_1)), \tag{4.5}$$

$$\nabla f \dot{\beta} = 0, \quad \nabla h \dot{\beta} = 0. \tag{4.6}$$

By projecting $\ddot{\boldsymbol{\beta}}$ onto the normals of both surfaces, we obtain

$$\left\langle \ddot{\boldsymbol{\beta}}(x_1), \boldsymbol{\nabla} f(\boldsymbol{\beta}(x_1)) \right\rangle = A \left\langle \ddot{\boldsymbol{\beta}}(x_1), \boldsymbol{\nabla} h(\boldsymbol{\beta}(x_1)) \right\rangle. \tag{4.7}$$

Using the system (3.15) and Eq. (4.7) yields

$$\dot{\beta}^T (H_1 - AH_2)\dot{\beta} = 0.$$

Explicitly

$$\left(\begin{array}{c} (f_{22} - Ah_{22})(\dot{x}_2)^2 + (f_{33} - Ah_{33})(\dot{x}_3)^2 + 2(f_{23} - Ah_{23})\dot{x}_2\dot{x}_3 \\ +2(f_{12} - Ah_{12})\dot{x}_2 + 2(f_{13} - Ah_{13})\dot{x}_3 + (f_{11} - Ah_{11}) \end{array} \right) = 0. \tag{4.8}$$

Using (4.6), we obtain

$$\dot{x}_2 = -\frac{f_1 + f_3\dot{x}_3}{f_2}, \ f_2 \neq 0 \quad or \quad \dot{x}_3 = -\frac{f_1 + f_2\dot{x}_2}{f_3}, \ f_3 \neq 0.$$

Assume that $f_2 \neq 0$, and substituting in Eq. (4.8), we get

$$a_{11}(\dot{x_3})^2 + 2a_{12}\dot{x_3} + a_{13} = 0, \tag{4.9}$$

where

(4.10)

$$a_{11} = f_2^2(f_{33} - Ah_{33}) - 2f_2f_3(f_{23} - Ah_{23}) + f_3^2(f_{22} - Ah_{22}),$$
$$a_{12} = f_2^2(f_{13} - Ah_{13}) - f_1f_2(f_{23} - Ah_{23}) - f_2f_3(f_{12} - Ah_{12}) + f_1f_3(f_{22} - Ah_{22}),$$
$$a_{13} = f_1^2(f_{22} - Ah_{22}) - 2f_1f_2(f_{12} - Ah_{12}) + f_2^2(f_{11} - Ah_{11}).$$

Solving Eq. (4.9) yields

(4.11)

$$\dot{x}_1 = 1, \quad \dot{x}_2 = -\frac{(a_{11}f_1 - a_{12}f_3) \pm f_3\sqrt{a_{12}^2 - a_{11}c}}{a_{11}f_2}, \quad \dot{x_3} = \frac{-a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{13}}}{a_{11}}.$$

Thus

$$\left\| \dot{\boldsymbol{\beta}}(x_1) \right\| = \sqrt{1 + (\dot{x}_2)^2 + (\dot{x_3})^2}, \tag{4.12}$$

$$S=\int_{x_0}^{x}\left\|\dot{\boldsymbol{\beta}}(x_1)\right\|dx_1,\quad \boldsymbol{t}=\frac{\dot{\boldsymbol{\beta}}(x_1)}{\left\|\dot{\boldsymbol{\beta}}(x_1)\right\|}. \tag{4.13}$$

From the previous formulas, it is easy to see that there are four distinct cases for the solution of Eq. (4.9) depending upon the discriminant $\Delta = a_{12}^2 - a_{11}a_{13}$, these cases are as follows [1]:

Lemma 4.1. The point P is a branch point of the intersection curve (4.4), if $\Delta > 0$ and there is another intersection branch crossing the curve (4.4) at that point.

Lemma 4.2. The surfaces f and h intersect at the point P and at its neighborhood, if $\Delta = 0$ and $a_{11}^2 + a_{12}^2 + a_{13}^2 \neq 0$. (Tangential intersection curve)

Lemma 4.3. The point P is an isolated contact point of the surfaces f and h, if $\Delta < 0$.

Lemma 4.4 The surfaces f and h have contact of at least second order at the point P, if $a_{11} = a_{12} = a_{13} = 0$. (Higher-order contact point)

4.2. Curvature and curvature vector

By projecting $\dddot{\boldsymbol{\beta}}(x_1)$ onto unit normal vectors of both surfaces, we obtain

$$\nabla f\dddot{\beta} = A(\nabla h\dddot{\beta}). \tag{4.14}$$

Using Eqs. (3.3), (3.4) and (4.14) yields

$$\nabla h\ddot{\beta} = -\dot{\beta}^T H_2\dot{\beta}, \tag{4.15}$$

$$3\dot{\beta}^T(H_1 - AH_2)\ddot{\beta} = \dot{\beta}^T((\nabla H_2 - A\ \nabla H_1)\dot{\beta})\dot{\beta}. \tag{4.16}$$

Eqs (4.15) and (4.16) can be written as the follows

$$\begin{bmatrix} 3((f_{12}-A\ h_{12})+\dot{x}_2(f_{22}-A\ h_{22})+\dot{x}_3(f_{23}-A\ h_{23})) & f_2 \\ 3((f_{13}-A\ h_{13})+\dot{x}_2(f_{23}-A\ h_{23})+\dot{x}_3(f_{33}-A\ h_{33})) & f_3 \end{bmatrix}^T \begin{bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}$$
$$= \begin{bmatrix} \dot{\beta}^T((\nabla H_2 - A\ \nabla H_1)\dot{\beta})\dot{\beta} \\ -\dot{\beta}^T H_2\dot{\beta} \end{bmatrix}.$$

Thus

$$\begin{bmatrix}\ddot{x}_2\\ \ddot{x}_3\end{bmatrix} = \frac{1}{\Omega}\begin{bmatrix} f_3 & -3((f_{13}-Ah_{13})+\dot{x}_2(f_{23}-Ah_{23})+\dot{x}_3(f_{33}-Ah_{33}))\\ -f_2 & 3((f_{12}-Ah_{12})+\dot{x}_2(f_{22}-Ah_{22})+\dot{x}_3(f_{23}-Ah_{23}))\end{bmatrix} \times \begin{bmatrix}\dot{\beta}^T((\nabla H_2 - A\nabla H_1)\dot{\beta})\dot{\beta}\\ -\dot{\beta}^T H_2\dot{\beta}\end{bmatrix}, \quad \Omega \neq 0, \tag{4.17}$$

where

$$\Omega = 3\det\begin{bmatrix}(f_{12}-Ah_{12})+\dot{x}_2(f_{22}-Ah_{22})+\dot{x}_3(f_{23}-Ah_{23}) & \frac{1}{3}h_2\\ (f_{13}-Ah_{13})+\dot{x}_2(f_{23}-Ah_{23})+\dot{x}_3(f_{33}-Ah_{33}) & \frac{1}{3}h_3\end{bmatrix}^T. \tag{4.18}$$

Then the curvature and curvature vector are given by

$$\begin{aligned}\kappa\boldsymbol{n} &= \frac{(\dot{x}_2\ddot{x}_2+\dot{x}_3\ddot{x}_3,(1+\dot{x}_3^2)\ddot{x}_2-\dot{x}_2\dot{x}_3\ddot{x}_3,(1+\dot{x}_2^2)\ddot{x}_3-\dot{x}_2\dot{x}_3\ddot{x}_2)}{(1+\dot{x}_2^2+\dot{x}_3^2)^2},\\ \kappa &= \sqrt{\frac{(1+\dot{x}_3^2)\ddot{x}_2^2+(1+\dot{x}_2^2)\ddot{x}_3^2-2\dot{x}_2\dot{x}_3\ddot{x}_2\ddot{x}_3}{(1+\dot{x}_2^2+\dot{x}_3^2)^3}}.\end{aligned} \tag{4.19}$$

4.3. Torsion and third-order derivative vector

By Projecting $\boldsymbol{\beta}^{(4)}(x_1)$ onto unit normal vectors of both surfaces, we have

$$\nabla f\beta^{(4)} = A\nabla h\beta^{(4)}. \tag{4.20}$$

Using Eqs. (3.4), (3.5) and (4.20) yields

$$\nabla h\dddot{\beta} = -3\dot{\beta}^T H_2\ddot{\beta} - \dot{\beta}^T(\nabla H_2\dot{\beta})\dot{\beta},$$

$$\begin{bmatrix}4\dot{\beta}^T(H_1 - AH_2)\\ \nabla h\end{bmatrix}\begin{bmatrix}0\\ \dddot{x}_2\\ \dddot{x}_3\end{bmatrix} = \begin{bmatrix}5\dot{\beta}^T(\nabla(H_2-AH_1)\dot{\beta})\ddot{\beta}+3\ddot{\beta}^T(H_2-AH_1)\ddot{\beta}\\ -3\dot{\beta}^T H_2\ddot{\beta}\end{bmatrix},$$

$$+\begin{bmatrix}\dot{\beta}^T(\nabla(\nabla(H_2-AH_1))\dot{\beta})\dot{\beta})\dot{\beta}\\ -\dot{\beta}^T(\nabla H_2\dot{\beta})\dot{\beta}\end{bmatrix}$$

Explicitly

(4.21)
$$\begin{bmatrix} \dddot{x}_2 \\ \dddot{x}_3 \end{bmatrix} = \frac{1}{\rho} \begin{bmatrix} h_3 & 4((f_{13}-Ah_{13})+(f_{23}-Ah_{23})+(f_{22}-Ah_{22}))\dot{x}_3 \\ -h_2 & 4((f_{12}-Ah_{12})+((f_{33}-Ah_{33})+(f_{23}-Ah_{23}))\dot{x}_2 \end{bmatrix}$$
$$\times \begin{bmatrix} 5\dot{\beta}^T(\nabla(H_2-AH_1)\dot{\beta})\ddot{\beta}+3\ddot{\beta}^T(H_2-AH_1)\ddot{\beta}+\dot{\beta}^T(\nabla(\nabla(H_2-AH_1))\dot{\beta})\dot{\beta})\dot{\beta} \\ -3\dot{\beta}^T H_2\ddot{\beta}-\dot{\beta}^T(\nabla H_2\dot{\beta})\dot{\beta} \end{bmatrix},$$

where

$$\rho = 4\begin{pmatrix} ((f_{12}-Ah_{12})+(f_{22}-Ah_{22})h_2+(f_{23}-Ah_{23}))h_3\dot{x}_3 \\ -((f_{13}-Ah_{13})+(f_{23}-Ah_{23})+(f_{33}-Ah_{33}))h_2\dot{x}_2 \end{pmatrix}. \tag{4.22}$$

The torsion can be calculated by substituting in

$$\tau = \frac{\ddot{x}_2\dddot{x}_3-\ddot{x}_3\dddot{x}_2}{(1+\dot{x}_3^2)\ddot{x}_2^2+(1+\dot{x}_2^2)\ddot{x}_3^2-2\dot{x}_2\dot{x}_3\ddot{x}_2\ddot{x}_3}. \tag{4.23}$$

5. Examples

Example 5.1. Consider the intersection of the two implicit surfaces (ellipsoids) [1]

$$\begin{aligned} f(x_1,x_2,x_3) &= \frac{x_1^2}{0.6^2}+\frac{x_2^2}{0.8^2}+\frac{x_3^2}{1^2}-1=0, \\ h(x_1,x_2,x_3) &= \frac{x_1^2}{0.45^2}+\frac{x_2^2}{0.8^2}+\frac{x_3^2}{1.25^2}-1=0. \end{aligned} \tag{5.1}$$

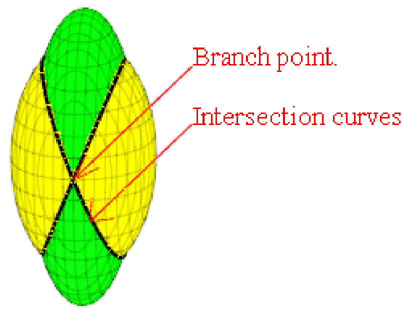

Fig. 1. Transversal and tangential intersection.

Transversal intersection: Since $\frac{\partial(f,h)}{\partial(x_2,x_3)} \neq 0$, $x_3 \neq 0$, then we can solve (5.1) for the variables x_2, x_3 in terms of x_1. Using Eqs (3.9) and (5.1), we obtain

$$\dot{\boldsymbol{\beta}} = (1, -\frac{4096x_1}{729x_2}, \frac{4375x_1}{729x_3}). \tag{5.2}$$

The speed and unit tangent vector of the intersection curve can be calculated using Eqs (3.12) and (3.13) as the follows

$$\dot{s} = \frac{\sqrt{182.33x_1^2x_2^2 + 5.06x_2^2x_3^2 + 159.82x_1^2x_3^2}}{|2.25x_2x_3|}, \tag{5.3}$$

$$\boldsymbol{t} = \begin{cases} \dfrac{(-2.25x_2x_3, 12.64x_1x_3, -13.50x_1x_2)}{\sqrt{182.33x_1^2x_2^2 + 5.06x_2^2x_3^2 + 159.82x_1^2x_3^2}} & \text{if} \quad x_2x_3 < 0 \\ \dfrac{(2.25x_2x_3, -12.64x_1x_3, 13.50x_1x_2)}{\sqrt{182.33x_1^2x_2^2 + 5.06x_2^2x_3^2 + 159.82x_1^2x_3^2}} & \text{if} \quad x_2x_3 > 0 \end{cases}. \tag{5.4}$$

The curvature vector and the curvature of the intersection curve can be calculated by using Eqs(3.16), (3.18) and (3.19) as the follows

$$\begin{aligned} \kappa\boldsymbol{n} &= (\frac{A_{23}}{A_{12}^2 + A_{13}^2 + A_{23}^2})^2(A_{23}A_{13}\ddot{x}_2, (A_{12}^2 + A_{23}^2)\ddot{x}_2, A_{13}A_{12}\ddot{x}_2), \\ \kappa &= \sqrt{\frac{A_{23}^4(A_{12}^2 + A_{23}^2)\ddot{x}_2^2}{(A_{12}^2 + A_{13}^2 + A_{23}^2)^3}}; \end{aligned} \tag{5.5}$$

$$\ddot{\boldsymbol{\beta}} = (0, -\frac{64.x_2^2 + 359.594x_1^2}{11.391x_2^3}, 0). \tag{5.6}$$

Using Eqs. (3.22) and (3.23) yields

$$\dddot{x}_2 = -\frac{27A_{13}\ddot{x}_2}{4(A_{23})^2}, \quad \dddot{x}_3 = 0. \tag{5.7}$$

Thus

$$\tau = 0. \tag{5.8}$$

Thus, the intersection curve β is a plane curve as shown in Fig 1.

Tangentially intersection: Solving the system (4.3) for the surfaces (5.1), we find that the surfaces are intersecting tangentially at the points $P(0, \pm 0.8, 0)$. Using (4.3), (4.10) and (5.1) at $P_1(0, 0.8, 0)$ yields

$$A = 1, \quad a_{11} = \frac{9}{2}, \quad a_{12} = 0, \quad a_{13} = -\frac{4375}{162}.$$

Then $\Delta > 0$ this means that the point P_1 is a branch point. The unit tangent vector of the intersection curve at P_1 can be calculated by using Eqs (4.11) and (4.13) as the follows

$$\boldsymbol{t} = (\frac{27}{4\sqrt{319}}, 0, \pm\frac{25\sqrt{7}}{4\sqrt{319}}). \tag{5.9}$$

The curvature vector, the curvature and the torsion of the intersection curve at P_1 can be calculated by using Eqs. (4.17), (4.19), (4.23) and (5.1) as the follows

$$\kappa\boldsymbol{n} = (0, -\frac{320}{319}, 0), \quad \kappa = \frac{320}{319}, \quad \tau = 0. \tag{5.10}$$

Thus, the intersection point P_1 is a flat point as shown in Fig 1.

Example 5.2. Consider the intersection of two implicit surfaces

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_1^2 + x_2^2 + x_3^2 + 3)^2 - 16(x_1^2 + x_2^2) = 0, \\ h(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 - 1 = 0. \end{aligned} \tag{5.11}$$

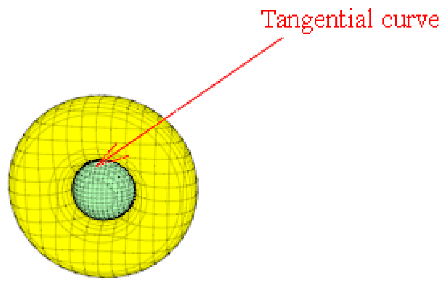

Fig. 2. Tangential intersection

At $x_3 = 0$, we have $\boldsymbol{N}^{(f)} \; // \; \boldsymbol{N}^{(h)}$.Using Eqs. (4.10) and (5.11), we obtain $\Delta = 0$, this means that the surfaces are intersect tangentially at a curve. Then from Eq. (4.11), we obtain

$$\dot{\boldsymbol{\beta}} = (1, -\frac{x_1}{x_2}, 0). \tag{5.12}$$

Using Eqs. (4.12) and (4.13) yields

$$\dot{S} = \frac{\sqrt{x_1^2 + x_2^2}}{x_2}, \quad \boldsymbol{t} = \frac{(x_2, -x_1, 0)}{\sqrt{x_1^2 + x_2^2}}. \tag{5.13}$$

Using Eq. (4.18), we have $\Omega = 0$, then Eq. (4.16) vanish on the intersection curve. From Eq. (4.15), we have

$$x_2 \ddot{x}_2 + x_3 \ddot{x}_3 = -\frac{x_1^2 + x_2^2}{x_2^2}.$$

On the intersection curve we have $x_3 = 0$. Thus

$$\ddot{\boldsymbol{\beta}} = (0, -\frac{1}{x_2^3}, 0). \tag{5.14}$$

The curvature vector and curvature of the intersection curve can be calculated using Eqs (4.19), (5.12) and (5.14) as the follows

$$\kappa \boldsymbol{n} = \frac{-(x_1, x_2, 0)}{(x_1^2 + x_2^2)}, \quad \kappa = \frac{1}{\sqrt{x_1^2 + x_2^2}}, \quad \boldsymbol{n} = \frac{-(x_1, x_2, 0)}{\sqrt{x_1^2 + x_2^2}}. \tag{5.15}$$

Using Eqs. (4.21), (5.12) and (5.14), we get

$$\dddot{\boldsymbol{\beta}} = (0, \frac{-3x_1}{x_2^5}, 0). \tag{5.16}$$

The torsion of the intersection curve can be calculated by using Eqs (4.23), (5.12), (5.14) and (5.16), thus

$$\tau = 0.$$

Then the tangential intersection curve is a plane curve as shown in Fig 2.

Example 5.3. Consider the intersection of the two implicit surfaces

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 - 1 = 0, \\ h(x_1, x_2, x_3) &= x_2 - 1 = 0. \end{aligned} \tag{5.17}$$

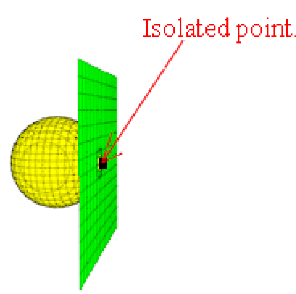

Fig. 3. Tangential intersection.

At the point $P(0,1,0)$, we have $\mathbf{N}^{(f)} \,//\, \mathbf{N}^{(h)}$. By using Eqs. (4.8) and (5.17) yielding $\Delta < 0$. Then the point P is an isolated tangential contact point as shown in Fig. 3.

Example 5.4. Consider the intersection of the two implicit surfaces

$$\begin{aligned} f(x_1, x_2, x_3) &= x_1^2 - x_2^2 + x_3 - 1 = 0, \\ h(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^3 = 0. \end{aligned} \tag{5.18}$$

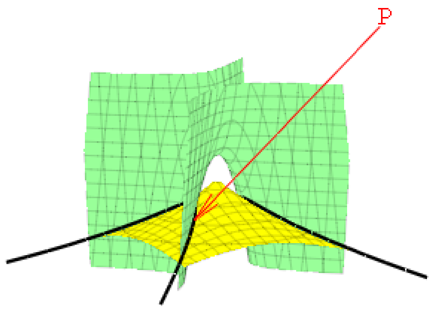

Fig. 4. Transversal intersection.

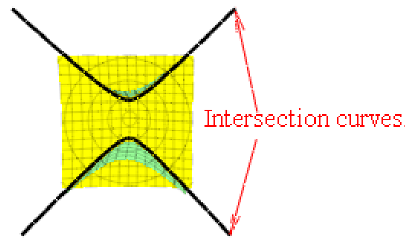

Fig. 5. Transversal intersection.

The point of the intersection curve is $p = (\sqrt{\frac{11}{2}}, \sqrt{\frac{5}{2}}, -2) \in S^f \cap S^h$.In this point, we have

The unit tangent vector can be calculated using Eqs. (3.10) and (3.13), as

$$\boldsymbol{t}(p) = (\frac{13}{2}\sqrt{\frac{5}{654}}, \frac{11}{2}\sqrt{\frac{11}{654}}, -\sqrt{\frac{55}{327}}). \tag{5.19}$$

The curvature vector and curvature can be calculated using Eqs (3.16) and (3.17), as

$$\begin{gathered} \kappa\boldsymbol{n}(p) = (\frac{271}{23762}\sqrt{22}, -\frac{737}{71286}\sqrt{10}, \frac{1231}{71286}), \\ \kappa(p) = 6.5028 \times 10^{-2}. \end{gathered} \tag{5.20}$$

The torsion can be calculated using Eqs. (3.21), (3.22) and (3.23), as

$$\tau(p) = -0.515\,28. \tag{5.21}$$

The intersection curves are shown in Fig. 5.

5. Conclusion

Algorithms for computing the differential geometry properties of intersection curves of two implicit surfaces in $\mathbb{R}^3$ are given for transversal and tangential intersection. This paper also includes the necessary and sufficient conditions for the curve to be a straight line or a planer curve or helix or circular helix or circle. The types of singularities on the intersection curve are characterized. The questions of how to exploit and extend these algorithms to compute the differential geometry properties of self-intersection curves of an implicit surface in $\mathbb{R}^3$ can be topics of future research.

Acknowledgements

The authors would like to thank the reviewers for their valuable comments and suggestions.

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