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FUZZY OSTROWSKI TYPE INEQUALITIES VIA $h-$CONVEX

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Abstract. We would like to state well-known Ostrowski inequality via $h-$convex by using the Fuzzy Reimann integrals. In addition, we establish some Fuzzy Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $h-$convex by Hölder's and power mean inequalities. This class of $h-$convex function, which is the generalization of many important classes including class of Godunova-Levin $s-$convex, $s-$convex in the 2^{nd} kind and hence contains convex functions. It also contains class of $P-$convex and class of Godunova-Levin. In this way we also capture the results with respect to convexity of functions.

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1. Introduction

In recent years, the generalization of classical convex function have emerged resulting in applications in the field of Mathematics. From literature, we recall some definitions for different types of convex functions.

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Definition 1.1. [3] The $\eta : B \subset (0,\infty) \to \mathbb{R}$ is said to be convex, if

$$\eta\left(tx+(1-t)y\right) \leq t\eta(x)+(1-t)\eta(y),$$

$\forall x,y \in B, t \in [0,1]$.

Definition 1.2. [3] The $\eta : B \subset (0,\infty) \to \mathbb{R}$ is $MT-$convex, if $\eta(x) \geq 0$ and

$$\eta\left(tx+(1-t)y\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x)+\frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y),$$

$\forall t \in [0,1], x,y \in B$.

Definition 1.3. [17] The $\eta : B \subset (0,\infty) \to \mathbb{R}$ is a $P-$convex, if $\eta(x) \geq 0$ and $\forall x,y \in B$ and $t \in [0,1]$ we have

$$\eta\left(tx+(1-t)y\right) \leq \eta(x)+\eta(y).$$

Definition 1.4. [20] The $\eta : B \subset (0,\infty) \to \mathbb{R}$ is a GL convex, if $\eta(x) \geq 0$ and $\forall x,y \in B$ and $t \in (0,1)$ we have

$$\eta\left(tx+(1-t)y\right) \leq \frac{1}{t}\eta(x)+\frac{1}{1-t}\eta(y).$$

Definition 1.5. [4] Let $s \in (0,1]$, the $\eta : B \subset (0,\infty) \to \mathbb{R}$ is $s-$convex in the 2^{nd} kind, if

$$\eta\left(tx+(1-t)y\right) \leq t^{s}\eta(x)+(1-t)^{s}\eta(y),$$

$\forall t \in [0,1], x,y \in B$.

Definition 1.6. [9] The $\eta : B \subset (0,\infty) \to \mathbb{R}$ is of GL $s-$convex, with $s \in [0,1)$, if

$$\eta\left(tx+(1-t)y\right) \leq \frac{1}{t^{s}}\eta(x)+\frac{1}{(1-t)^{s}}\eta(y),$$

$\forall t \in (0,1), x,y \in B$.

Now we present the class of $h-$convex, this class contains many classes of convex from literature of convex analysis.

Definition 1.7. [30] Let $h : A \subseteq (0,\infty) \to \mathbb{R}$ with $h \neq 0$. The $\eta : B \subseteq (0,\infty) \to [0,\infty)$ is an $h-$convex if $\forall x, y \in B$, we have

$$\eta\,(tx + (1-t)y) \leq h(t)\eta(x) + h(1-t)\eta(y), \tag{1.1}$$

$\forall t \in [0,1]$.

Remark 1.8. In Definition 1.7, one can see the following.

1. If $h(t) = \frac{1}{t^s}, s \in [0,1]$ in (1.1), then the class of $GL\ s-$convex.
2. If $h(t) = \frac{1}{t}$ in (1.1), then we get the concept of GL convex.
3. If $h(t) = t^s$ with $s \in [0,1]$ in (1.1), then we get the concept of $s-$convex in 2^{nd} kind.
4. If $h(t) = 1$ in (1.1), then we get the concept of $P-$convex.
5. If $h(t) = t$ in (1.1), then we get the concept of ordinary convex.
6. If $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ in (1.1), then the concept of $MT-$convex.

Next we present the clasical ostrowski inequality.

Theorem 1.9. [29] *Let* $\varphi : [a,b] \to \mathbb{R}$ *be differentiable function on* (a,b), $|\varphi'(t)| \leq M$, $\forall t \in (a,b)$. *Then*

$$\left|\varphi(x) - \frac{1}{b-a}\int_a^b \varphi(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right], \tag{1.2}$$

$\forall x \in (a,b)$.

Definition 1.10. [6] A fuzzy number is $\phi : \mathbb{R} \to [0,1]$ can be defined as

1. $[\phi]^0 = \text{Closure}(\{r \in \mathbb{R} : \phi(r) > 0\})$ is compact.
2. ϕ is Normal.(i.e, $\exists\ r_0 \in \mathbb{R}$ such that $\phi(r_0) = 1$).
3. ϕ is fuzzy convex, i.e, $\phi(\eta r_1 + (1-\eta)r_2) \geq \min\{\phi(r_1), \phi(r_2)\}$, $\forall r_1, r_2 \in \mathbb{R}$, $\eta \in [0,1]$.
4. $\forall r_0 \in R$ and $\varepsilon > 0$, $\exists$ Neighborhood $V(r_0)$, such that $\phi(r) \leq \phi(r_0) + \varepsilon$, $\forall r \in \mathbb{R}$.

Definition 1.11. [7] For any $\zeta \in [0,1]$, and ϕ be any fuzzy number, then $\zeta-$level set $[\phi]^\zeta = \{r \in \mathbb{R} : \phi(r) \geq \zeta\}$. Moreover $[\phi]^\zeta = \left[\phi_-^{(\zeta)}, \phi_+^{(\zeta)}\right], \forall \zeta \in [0,1]$.

Proposition 1.12. [26] *Let* $\phi, \varphi \in F_{\mathbb{R}}$*(Set of all Fuzzy numbers) and* $\eta \in \mathbb{R}$, *then the following properties holds:*

(1) $[\phi]^{\zeta_1} \subseteq [\varphi]^{\zeta_2}$ *whenever* $0 \leq \zeta_2 \leq \zeta_1 \leq 1$.

(2) $[\phi + \varphi]^{\zeta} = [\phi]^{\zeta} + [\varphi]^{\zeta}$.

(3) $[\eta \odot \phi]^{\zeta} = \eta\, [\phi]^{\zeta}$.

(4) $\phi \oplus \varphi = \varphi \oplus \phi$.

(5) $\eta \odot \phi = \phi \odot \eta$.

(6) $\widetilde{1} \odot \phi = \phi$.

$\forall \zeta \in [0,1]$, *where* $\widetilde{1} \in F_{\mathbb{R}}$, *defined by* $\forall r \in \mathbb{R}, \widetilde{1}(r) = 1$.

Definition 1.13. [6] Let $D : F_{\mathbb{R}} \times F_{\mathbb{R}} \to \mathbb{R}_+ \cup \{0\}$, defined as

$$D(\phi, \varphi) = \sup_{\zeta \in [0,1]} \max \left\{ \left| \phi_-^{(\zeta)}, \phi_+^{(\zeta)} \right|, \left| \varphi_-^{(\zeta)}, \varphi_+^{(\zeta)} \right| \right\}$$

$\forall \phi, \varphi \in F_{\mathbb{R}}$. Then D is metric on $F_{\mathbb{R}}$.

Proposition 1.14. [6] *Let* $\phi_1, \phi_2, \phi_3, \phi_4 \in F_{\mathbb{R}}$ *and* $\eta \in F_{\mathbb{R}}$, *we have*

(1) $(F_{\mathbb{R}}, D)$ *is complete.*

(2) $D(\phi_1 \oplus \phi_3, \phi_2 \oplus \phi_3) = D(\phi_1, \phi_2)$.

(3) $D(\eta \odot \phi_1, \eta \odot \phi_2) = |\eta| D(\phi_1, \phi_2)$.

(4) $D(\phi_1 \oplus \phi_2, \phi_3 \oplus \phi_4) = D(\phi_1, \phi_3) + D(\phi_2, \phi_4)$.

(5) $D(\phi_1 \oplus \phi_2, \widetilde{0}) = D(\phi_1, \widetilde{0}) + D(\phi_2, \widetilde{0})$.

(6) $D(\phi_1 \oplus \phi_2, \phi_3) = D(\phi_1, \phi_3) + D(\phi_2, \widetilde{0})$,

where $\widetilde{0} \in F_{\mathbb{R}}$, *defined by* $\forall r \in \mathbb{R}, \widetilde{0}(r) = 0$.

Definition 1.15. [7] Let $\phi, \varphi \in F_{\mathbb{R}}$, if $\exists\ \theta \in F_{\mathbb{R}}$, such that $\phi = \varphi \oplus \theta$, then θ is $H-$difference of ϕ and φ, denoted by $\theta = \phi \ominus \varphi$.

Definition 1.16. [7] A function $\phi : [r_0, r_0 + \varepsilon] \to F_{\mathbb{R}}$ is $H-$differentiable at r, if $\exists\ \phi'(r) \in F_{\mathbb{R}}$, i.e both limits

$$\lim_{h \to 0^+} \frac{\phi(r+h) \ominus \phi(r)}{h}, \lim_{h \to 0^+} \frac{\phi(r) \ominus \phi(r-h)}{h}$$

exists and are equal to $\phi'(r)$.

Definition 1.17. [19] Let $\phi : [a,b] \to F_{\mathbb{R}}$, if $\forall \zeta > 0, \exists \eta > 0$, for any partition $P = \{[u,v] : \delta\}$ of $[a,b]$ with norm $\Delta(P) < \eta$, we have

$$D\left(\sum_{P}^{*}(v-u)\phi(\delta), \varphi\right) < \zeta,$$

then we say that ϕ is Fuzzy–Riemann integrable to $\varphi \in F_{\mathbb{R}}$, we write it as

$$\varphi = (FR)\int_a^b \phi(x)dx.$$

2. Fuzzy Ostrowski Type Inequalities via $h-$Convex Functions

In order to prove our main results, we need the following lemma that has been obtained in [5].

Lemma 2.1. *Let $\varphi : [a,b] \to F_{\mathbb{R}}$ be an absolutely continuous mapping on (a,b) with $a < b$. If $\varphi' \in C_F[a,b] \cap L_F[a,b]$, then for $x \in (a,b)$ the following identity holds:*

$$\begin{aligned}&\frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt \oplus \frac{(x-a)^2}{b-a} \odot (FR)\int_0^1 t \odot \varphi'(tx+(1-t)a)dt \\ &= \varphi(x) \oplus \frac{(b-x)^2}{b-a} \odot (FR)\int_0^1 t \odot \varphi'(tx+(1-t)b)dt. \qquad (2.1)\end{aligned}$$

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u}u^{x-1}du$.

Theorem 2.2. *Suppose all the assumptions of Lemma 2.1 hold. Additionally, $\lambda \in (0,1], \phi : (0,1) \to (0,\infty)$ be a measurable function with $h(t) \neq \frac{1}{t}$, $D(\varphi', \widetilde{0})$ be a $h-$convex function on $[a,b]$ and $D(\varphi'(x), \widetilde{0}) \leq M$. Then $\forall x \in (a,b)$ the following inequality holds:*

$$\begin{aligned}&D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \\ &\leq M\left(\int_0^1 (t\,h(t) + t\,h(1-t))\,dt\right) I(x), \qquad (2.2)\end{aligned}$$

where $I(x) = \frac{(x-a)^2+(b-x)^2}{b-a}$.

Proof. From the Lemma 2.1,

$$
\begin{aligned}
& D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \\
& \leq D\left(\frac{(x-a)^2}{b-a} \odot (FR)\int_0^1 t \odot \varphi'(tx+(1-t)a)dt,\right. \\
& \left.\frac{(b-x)^2}{b-a} \odot (FR)\int_0^1 t \odot \varphi'(tx+(1-t)b)dt\right),
\end{aligned}
$$

$$
\begin{aligned}
& \leq D\left(\frac{(x-a)^2}{b-a} \odot (FR)\int_0^1 t \odot \varphi'(tx+(1-t)a)dt, \widetilde{0}\right) \\
& + D\left(\frac{(b-x)^2}{b-a} \odot (FR)\int_0^1 t \odot \varphi'(tx+(1-t)b)dt, \widetilde{0}\right),
\end{aligned}
$$

$$
\begin{aligned}
& = \frac{(x-a)^2}{b-a} D\left((FR)\int_0^1 t \odot \varphi'(tx+(1-t)a)dt, \widetilde{0}\right) \\
& + \frac{(b-x)^2}{b-a} D\left((FR)\int_0^1 t \odot \varphi'(tx+(1-t)b)dt, \widetilde{0}\right),
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{(x-a)^2}{b-a} \int_0^1 tD\left(\varphi'(tx+(1-t)a), \widetilde{0}\right) dt \\
& + \frac{(b-x)^2}{b-a} \int_0^1 tD\left(\varphi'(tx+(1-t)b), \widetilde{0}\right) dt, \qquad (2.3)
\end{aligned}
$$

Since $D(\varphi', \widetilde{0})$ be $h-$convex function and $D(\varphi'(x), \widetilde{0}) \leq M$, we have

$$
\begin{aligned}
D\left(\varphi'(tx+(1-t)a), \widetilde{0}\right) & \leq h(t)D\left(\varphi'(x), \widetilde{0}\right) + h(1-t)D\left(\varphi'(a), \widetilde{0}\right) \\
& \leq M\left[h(t)+h(1-t)\right] \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
D\left(\varphi'(tx+(1-t)b), \widetilde{0}\right) & \leq h(t)D\left(\varphi'(x), \widetilde{0}\right) + h(1-t)D\left(\varphi'(b), \widetilde{0}\right) \\
& \leq M\left[h(t)+h(1-t)\right]. \qquad (2.5)
\end{aligned}
$$

Now using (2.4) and (2.5) in (2.3) we get (2.2). □

Corollary 2.3. *In Theorem* 2.2, *one can see the following.*

(1) *If one takes $h(t) = t^{-s}$ in (2.2), then one has the Fuzzy Ostrowski inequality for Godunova-Levin $s-$convex functions:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq M\left(\frac{1}{1-s}\right) I(x).$$

(2) *If one takes $h(t) = t^s$ where $s \in (0,1]$ in (2.2), then one has the Fuzzy Ostrowski inequality for $s-$convex functions in 2^{nd} kind:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq M\left(\frac{1}{1+s}\right) I(x).$$

(3) *If one takes $h(t) = 1$ in (2.2), then one has the Fuzzy Ostrowski inequality for $P-$convex function:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq MI(x).$$

(4) *If one takes $h(t) = t$ in (2.2), then one has the Fuzzy Ostrowski inequality for convex function:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq \frac{M}{2} I(x).$$

(5) *If one takes $h(t) = \frac{t}{2\sqrt{t(1-t)}}$ in in (2.2), then one has the Fuzzy Ostrowski inequality for $MT-$convex function:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq \frac{M\pi}{4} I(x).$$

Theorem 2.4. *Suppose all the assumptions of Lemma 2.1 hold. Additionally, $h(t) \neq \frac{1}{t}$, $[D(\varphi', \widetilde{0})]^q$ for $q \geq 1$ be $h-$convex function on $[a,b]$ and $D(\varphi'(x), \widetilde{0}) \leq M$. Then $\forall x \in (a,b)$ the following inequality holds:*

$$\begin{aligned} &D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \\ &\leq \frac{M}{2^{1-\frac{1}{q}}}\left(\int_0^1 (t\,h(t) + t\,h(1-t))\,dt\right)^{\frac{1}{q}} I(x). \end{aligned} \tag{2.6}$$

Proof. From the inequality (2.3) and power mean inequality [31]

$$
\begin{aligned}
&D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right)\\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)a),\widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(\varphi'(tx+(1-t)b),\widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \qquad (2.7)
\end{aligned}
$$

Since $[D(\varphi',\widetilde{0})]^q$ be $h-$convex function and $D(\varphi'(x),\widetilde{0}) \leq M$, we have

$$
\begin{aligned}
&\left[D\left(\varphi'(tx+(1-t)a),\widetilde{0}\right)\right]^q \leq h(t)\left[D\left(\varphi'(x),\widetilde{0}\right)\right]^q\\
&+h(1-t)\left[D\left(\varphi'(a),\widetilde{0}\right)\right]^q \leq M^q\left[h(t)+h(1-t)\right], \qquad (2.8)
\end{aligned}
$$

$$
\begin{aligned}
&\left[D\left(\varphi'(tx+(1-t)b),\widetilde{0}\right)\right]^q \leq h(t)\left[D\left(\varphi'(x),\widetilde{0}\right)\right]^q\\
&+h(1-t)\left[D\left(\varphi'(b),\widetilde{0}\right)\right]^q \leq M^q\left[h(t)+h(1-t)\right], \qquad (2.9)
\end{aligned}
$$

Now using (2.8) and (2.9) in (2.7) we get (2.6).

□

Corollary 2.5. *In Theorem* 2.4, *one can see the following.*

(1) If one takes $q=1$, *one has the Theorem* 2.2.

(2) If one takes $h(t)=t^{-s}$ *in* (2.6), *then one has Fuzzy Ostrowski inequality for Godunova-Levin* $s-$*convex functions:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}} I(x).$$

(3) If one takes $h(t)=t^s$ *where* $s \in [0,1]$ *in* (2.6), *then one has Fuzzy Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}} I(x).$$

(4) If one takes $h(t)=1$, *in* (2.6), *then one has the Fuzzy Ostrowski inequality for* $P-$*convex function:*

$$D\left(\varphi(x), \frac{1}{b-a} \odot (FR)\int_a^b \varphi(t)dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} I(x).$$

(5) If one takes $h(t)=t$, in (2.6), then one has the Fuzzy Ostrowski inequality for convex function:

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{2}I(x).$$

(6) If one takes $h(t)=\frac{t}{2\sqrt{t(1-t)}}$ in (2.6), then one has the Fuzzy Ostrowski inequality for $MT-$convex function:

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}}I(x).$$

Theorem 2.6. *Suppose all the assumptions of Lemma 2.1 hold. Additionally $h(t)\neq\frac{1}{t}$, $[D(\varphi',\widetilde{0})]^q$ be a $h-$convex function on $[a,b],q>1$ and $D(\varphi'(x),\widetilde{0})\leq M$. Then $\forall x\in(a,b)$, the following inequality holds:*

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)$$

$$\leq\frac{M}{(p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t)+h(1-t))\,dt\right)^{\frac{1}{q}}I(x), \tag{2.10}$$

where $p^{-1}+q^{-1}=1$.

Proof. From the inequality (2.3) and Hölder's inequality [32]

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)$$

$$\leq\frac{(x-a)^2}{b-a}\left(\int_0^1t^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left[D\left(\varphi'(tx+(1-t)a),\widetilde{0}\right)\right]^qdt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^2}{b-a}\left(\int_0^1t^pdt\right)^{\frac{1}{p}}\left(\int_0^1\left[D\left(\varphi'(tx+(1-t)b),\widetilde{0}\right)\right]^qdt\right)^{\frac{1}{q}}. \tag{2.11}$$

Since $[D(\varphi',\widetilde{0})]^q$ be $h-$convex function and $D(\varphi'(x),\widetilde{0})\leq M$, we have

$$\left[D\left(\varphi'(tx+(1-t)a),\widetilde{0}\right)\right]^q\leq h(t)\left[D\left(\varphi'(x),\widetilde{0}\right)\right]^q+$$

$$h(1-t)\left[D\left(\varphi'(a),\widetilde{0}\right)\right]^q\leq M^q\left[h(t)+h(1-t)\right],$$

(2.12)

$$\left[D\left(\varphi'(tx+(1-t)b),\widetilde{0}\right)\right]^q \leq h(t)\left[D\left(\varphi'(x),\widetilde{0}\right)\right]^q$$
$$+h(1-t)\left[D\left(\varphi'(b),\widetilde{0}\right)\right]^q \leq M^q\left[h(t)+h(1-t)\right],$$

(2.13)

Now using (2.12) and (2.13) in (2.11) we get (2.10).

□

Corollary 2.7. *In Theorem* 2.6, *one can see the following.*

(1) If one takes $h(t)=t^{-s}$ *where* $s\in[0,1)$ *in* (2.10), *then one has the Fuzzy Ostrowski inequality for Godunova-Levin* $s-$*convex functions:*

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1-s}\right)^{\frac{1}{q}}I(x).$$

(2) If one takes $h(t)=t^s$, *where* $s\in(0,1]$ *in* (2.10), *then one has the Fuzzy Ostrowski inequality for* $s-$*convex functions in* 2^{nd} *kind:*

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{1+s}\right)^{\frac{1}{q}}I(x).$$

(3) If one takes $h(t)=1$, *in* (2.10), *then one has the Fuzzy Ostrowski inequality for* $P-$*convex function:*

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{2^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}}I(x).$$

(4) If one takes $h(t)=t$, *in* (2.10), *then one has the Fuzzy Ostrowski inequality for convex function:*

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{(p+1)^{\frac{1}{p}}}I(x).$$

(5) If one takes $h(t)=\frac{t}{2\sqrt{t(1-t)}}$ *in* (2.10), *then one has the Fuzzy Ostrowski inequality for* $MT-$*convex function:*

$$D\left(\varphi(x),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M\left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}}I(x).$$

2.1. Fuzzy Ostrowski type midpoint inequalties via $h-$convex functions.

Remark 2.8. In Theorem 2.4, one can see the following.

(1) If one takes $x=\frac{a+b}{2}$ in (2.6), then one has the Fuzzy Ostrowski Midpoint inequality for $h-$convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)$$
$$\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\int_0^1(th(t)+th(1-t))\,dt\right)^{\frac{1}{q}}(b-a).$$

(2) If one takes $x=\frac{a+b}{2}$ and $h(t)=t^{-s}$ where $s\in[0,1)$ in (2.6), then one has Fuzzy Ostrowski Midpoint inequality for Godunova-Levin $s-$convex functions:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}(b-a).$$

(3) If one takes $x=\frac{a+b}{2}$ and $h(t)=t^s$ where $s\in[0,1]$ in (2.6), then one has Fuzzy Ostrowski Midpoint inequality for $s-$convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{2^{2-\frac{1}{q}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}(b-a).$$

(4) If one takes $x=\frac{a+b}{2}$ and $h(t)=1$ in (2.6), then one has the Fuzzy Ostrowski Midpoint inequality for $P-$convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{2^{2-\frac{1}{q}}}(b-a).$$

(5) If one takes $x=\frac{a+b}{2}$ and $h(t)=t$ in (2.6), then one has the Fuzzy Ostrowski Midpoint inequality for convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{4}(b-a).$$

(6) If one takes $h(t)=\frac{t}{2\sqrt{t(1-t)}}$ in (2.6), then one has the Fuzzy Ostrowski inequality for $MT-$convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}}I(x).$$

Remark 2.9. In Theorem 2.6, one can see the following.

(1) If one takes $x=\frac{a+b}{2}$ in (2.10), one has the Fuzzy Ostrowski Midpoint inequality for $h-$convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)$$
$$\leq\frac{M}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1(h(t)+h(1-t))\,dt\right)^{\frac{1}{q}}(b-a).$$

(2) If one takes $x=\frac{a+b}{2}$ and $h(t)=t^{-s}$ where $s\in[0,1)$ in (2.10), then one has the Fuzzy Ostrowski Midpoint inequality for Godunova-Levin $s-$convex functions:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1-s}\right)^{\frac{1}{q}}(b-a).$$

(3) If one takes $x=\frac{a+b}{2}$ and $h(t)=t^{s}$, where $s\in(0,1]$ in (2.10), then one has the Fuzzy Ostrowski Midpoint inequality for $s-$convex functions in 2^{nd} kind:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1+s}\right)^{\frac{1}{q}}(b-a).$$

(4) If one takes $x=\frac{a+b}{2}$ and $h(t)=1$ in (2.10), then one has the Fuzzy Ostrowski Midpoint inequality for $P-$convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{2^{\frac{1}{q}-1}M}{(p+1)^{\frac{1}{p}}}(b-a).$$

(5) If one takes $x=\frac{a+b}{2}$ and $h(t)=t$ in (2.10), then one has the Fuzzy Ostrowski Midpoint inequality for convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M}{2(p+1)^{\frac{1}{p}}}(b-a).$$

(6) If one takes $h(t)=\frac{t}{2\sqrt{t(1-t)}}$ in (2.10), then one has the Fuzzy Ostrowski inequality for $MT-$convex function:

$$D\left(\varphi\left(\frac{a+b}{2}\right),\frac{1}{b-a}\odot(FR)\int_a^b\varphi(t)dt\right)\leq\frac{M\pi^{\frac{1}{q}}}{2^{\frac{1}{q}+1}(1+p)^{\frac{1}{p}}}(b-a).$$

3. CONCLUSION

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $h-$convex function which is the generalization of many important classes including class of Godunova-Levin $s-$convex [9], $s-$convex in the 2^{nd} kind [4] (and hence contains class of convex functions [3]). It also contains class of $P-$convex functions [17] and class of Godunova-Levin functions [20]. We would like to state well-known Fuzzy Ostrowski inequality via $h-$convex function. In addition, we establish some Fuzzy Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $h-$convex functions by using different techniques including Hölder's inequality [32] and power mean inequality [31].

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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