Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 3, 451-461
ISSN: 1927-5307

SPACELIKE CONSTANT ANGLE SURFACES IN MINKOWSKI 3-SPACE

GÜLNUR ŞAFFAK ATALAY*, FATMA GÜLER AND EMİN KASAP

Department of Mathematics, Arts and Science Faculty, Ondokuz Mayis University, Samsun 55139,Turkey

Abstract. In [1], the classifications are given for the spacelike surfaces whose the normals make a constant angle with a timelike constant direction. In this paper, by choosing the constant direction spacelike , we observed that there are different kinds of surfaces in the classification. As a result, a thorough classification is given for constant angle spacelike surfaces. It is shown that the minimal spacelike constant angle surfaces are planes. Finally , examples are given to show the spacelike constant angle surfaces.

Keywords: Constant angle, Minkowski 3–Space,Spacelike Surface.

2000 AMS Subject Classification: 14J26; 53B30; 37K25

1. Introduction

A constant angle surfaces in Euclidean 3-space is a surface whose the unit normal makes a constant angle with a fixed direction. These surfaces generalize the concept of helix, that is, curves whose tangent lines make a constant angle with a fixed vector of E^3.

Helical features characterise all screws and bolts as well as some gears. These components therefore play important roles in mechanical construction and are studied in undergraduate mechanical engineering.

*Corresponding author

E-mail addresses: gulnur.saffak@omu.edu.tr(Gülnur Şaffak Atalay), f.guler@omu.edu.tr (Fatma Güler), kasape@omu.edu.tr (Emin Kasap)

Received December 21, 2011

Their representations in an assembly would be treated in courses on mechanics and machines, and their production, to some extent, in workshop practice. Also, DNA (deoxyribonucleic acid) is a double-stranded molecule that is twisted into a helix like spiral staircase.

Recently, constant angle surfaces have been the subject of some studies: Munteanu and Nistor [6] studied constant angle surfaces in Euclidean 3-space. They obtained classifications for all constant angle surfaces in E^3. Cermelli and Scala [2] proved that constant angle surfaces have some important applications to physics, it was shown how constant angle surfaces can be used to describe interfaces occurring in special equilibrium configurations of liquid crystals and layered fluids. Lopez and Munteanu [1] studied constant angle surfaces in Minkowski 3-space. But , they investigated only spacelike surfaces with the constant timelike direction. They gave a parametrization for such surfaces as

$$r(u,v) = (u\cosh\alpha\cos(v) + \gamma_1(v), u\cosh\alpha\sin(v) + \gamma_2(v), -u\sinh\alpha)$$

where

$$\gamma(v) = (\gamma_1(v), \gamma_2(v)) = \sinh\alpha\left(\int_0^v \sin(\tau)\Gamma(\tau)d\tau, -\int_0^v \cos(\tau)\Gamma(\tau)d\tau\right)$$

By choosing the constant direction spacelike, we obtain different parametrazion for the spacelike surfaces. Morever, we show that minimal spacelike constant angle surfaces are planes.The objective of the study in this paper is to classify spacelike constant angle surfaces in Minkowski 3-space IR_1^3. By choosing the constant direction spacelike , we obtain different parametrizations for the spacelike surfaces. Moreover, we show that minimal spacelike constant angle surfaces are planes.

2. Preliminaries

Let us consider Minkowski 3-space $IR_1^3 = [IR^3, (+,+,-)]$. The *norm* of $X \in IR_1^3$ is denoted by $\|X\|$ and defined as $\|X\| = \sqrt{|\langle X, X\rangle|}$.A vector $X = (x_1, x_2, x_3) \in IR_1^3$ is called a *spacelike, timelike* and *null (lightlike)* vector if $\langle X, X\rangle > 0$ or $X = 0$, $\langle X, X\rangle <$

0 and $\langle X,X\rangle = 0$ for $X \neq 0$, respectively. A timelike vector is said to be *positive (resp.negative)* if and only if $x_3 > 0$ (*resp.*$x_3 < 0$), [4]. A smooth regular curve $\alpha : I \subset IR \to IR^3_1$ is said to be timelike, spacelike or lightlike curve if the velocity vector α' is a timelike, spacelike or lightlike vector, respectively [4]. In fact, a timelike curve corresponds to the path of on observer moving at less than the speed of light. Null curves correspond to moving at the speed of light and spacelike curves to moving faster than light.

For X and Y be spacelike vectors in IR^3_1.

If the inequality $|\langle X,Y\rangle| > \|X\| \|Y\|$ is satisfied, there is a unique real number α such that, $\langle X,Y\rangle = \|X\| \|Y\| \cosh\alpha$.

If the inequality $|\langle X,Y\rangle| \leq \|X\| \|Y\|$ is satisfied, there is a unique real number α such that, $\langle X,Y\rangle = \|X\| \|Y\| \cos\alpha$.

Let X be α spacelike vector and Y be a positive timelike vector in IR^3_1. Then there is a unique nonnegative real number α such that $\langle X,Y\rangle = \|X\| \|Y\| \sinh\alpha$.For X and Y be timelike vectors in IR^3_1.Then there is a unique real number α such that, $\langle X,Y\rangle = \|X\| \|Y\| \cosh\alpha$, [5] A surface M in IR^3_1 is called a *spacelike surface* if the induced metric g on the surface is a Euclid metric. The normal vector on the timelike surface is a spacelike vector, [6]. Let M is a spacelike surface in IR^3_1 and N is unit normal of M. For all $X,Y \in \chi(M)$, we get $\overline{D}_X Y = D_X Y - g(S(X),Y)N$ where is induced metric on the surface M , $\overline{D}$ and D are Levi-Civita connections on IR^3_1 and M, respectively. $S : \chi(M) \to \chi(M)$, $S(X) = -\overline{D}_X N$ is the *shape operator* of M , [4] . The *mean curvature* H of the spacelike surface $r = r(u,v)$ is given by

$$H = \frac{1}{2}(2FM - GL - En)\left(F^2 - EG\right)^{-1} \tag{2.1}$$

where $E = g(r_u,r_u)$, $F = g(r_u,r_v)$, $G = g(r_v,r_v)$, $L = g(r_{uu},N)$, $M = g(r_{uv},N)$, $n = g(r_{vv},N)$ are coefficients of the Minkowski first and second fundamental forms,[7] . A spacelike surface with vanishing mean curvature is called a minimal surface, [8] .

3.Spacelike Constant Angle with Constant Spacelike Direction

Let M be a spacelike surface and α be constant angle between the unit normal $N = (n_1, n_2, n_3)$ and the fixed spacelike direction k. Without loss of generality, the fixed direction is taken to be the first real axis.

There is one case for angle α :

$g(N, k) = \sinh \alpha.$

Now we will examine this case :

Since k is a spacelike unit vector, for an unitary spacelike vector field e_1 on M, we get

$$k = \cosh \alpha e_1 + \sinh \alpha N \tag{3.1}$$

Lemma 1. *Let e_2 be an unitary vector field on M and orthogonal to e_1. For the orthonormal basis $\{e_1, e_2\}$ of $\chi(M)$, we get*

$$\overline{D}_{e_2} N = \lambda e_2, \quad \overline{D}_{e_2} e_1 = -\lambda \tanh \alpha e_2, \quad \lambda = \lambda(u, v)$$

$$\overline{D}_{e_1} N = \overline{D}_{e_1} e_1 = \overline{D}_{e_1} e_2 = 0.$$

Proof. By applying $\overline{D}_{e_2}$ to the equality (3.1), we have

$$\overline{D}_{e_2} k = \cosh \alpha \overline{D}_{e_2} e_1 + \sinh \alpha \overline{D}_{e_2} N \tag{3.2}$$

Since $e_2 \left[g \left(N, e_1 \right) \right] = 0$, we get

$$g\left(\overline{D}_{e_2} N, e_1\right) + g\left(N, \overline{D}_{e_2} e_1\right) = 0 \tag{3.3}$$

As $e_2\left[g\left(N,N\right)\right]=0$. This clearly implies that $\overline{D}_{e_2}N\in\chi(M)$. Therefore

$$\overline{D}_{e_2}N=\lambda_1 e_1+\lambda e_2 \tag{3.4}$$

From (3.2) and (3.4) , we get

$$\overline{D}_{e_2}e_1=-\tanh\alpha\left(\lambda_1 e_1+\lambda e_2\right) \tag{3.5}$$

we will investigate the case $\alpha=0$ later. It is easy to see, from (3.3), that $\overline{D}_{e_2}N=\lambda e_2$ and $\overline{D}_{e_2}e_1=-\lambda\tanh\alpha e_2$.

Next by applying $\overline{D}_{e_1}$ for the equality (3.1), we obtain

$$\overline{D}_{e_1}k=\cosh\alpha\overline{D}_{e_1}e_1+\sinh\alpha\overline{D}_{e_1}N \qquad (3.6).$$

Since $e_1\left[g\left(N,N\right)\right]=0$, we get

$$\overline{D}_{e_1}N=\mu_1 e_1+\mu_2 e_2 \quad , \quad \mu_1,\mu_2\in IR \tag{3.7}$$

From (3.6) , (3.7) and $e_1\left[g\left(e_1,e_1\right)\right]=0$, we have

$$\overline{D}_{e_1}e_1=-\mu_2\tanh\alpha e_2 \tag{3.8}$$

Because the shape operator S of M is symmetric, we see that, $g(S(e_1),e_2)=g(e_1,S(e_2))$, $\mu_2=\lambda_1=0$. Hence, we get $\overline{D}_{e_1}N=\overline{D}_{e_1}e_1=0$.

With respect to the basis $\{e_1,e_2,N\}$ of IR^3_1, we can write $\overline{D}_{e_1}e_2=a_1e_1+a_2e_2+a_3N$, $a_1,a_2,a_3\in IR$. Simply calculations give
$\overline{D}_{e_1}e_2=0$.

Now, we can give the following results:

Corollary 3.1

$$D_{e_1}e_1 = D_{e_1}e_2 = 0,\ D_{e_2}e_1 = -\lambda \tanh \alpha e_2,\ D_{e_2}e_2 = \lambda \tanh \alpha e_1.$$

The spacelike surface M can be expressed as $r = r(u,v) = (x(u,v), y(u,v), z(u,v))$.

We may assume that $r_u = e_1$, $r_v = \beta(u,v)e_2$ where $\beta : M \to IR$ is a smooth function. By corollary 3.1, we get

$$r_{uu} = 0, \quad r_{vu} = \beta_u \frac{1}{\beta} r_v, \quad r_{uv} = -\lambda \tanh \alpha r_v$$

and

$$r_{vv} = \lambda \beta^2 \tanh \alpha r_u + \frac{1}{\beta}\beta_v r_v + \lambda \beta^2 N$$

Since $r_{uv} = r_{vu}$ and $N_{uv} = N_{vu}$, we obtain the following diferential equations:

$$\beta_u + \lambda \beta \tanh \alpha = 0,\ \lambda_u - \lambda^2 \tanh \alpha = 0 \tag{3.9}$$

solving the equations (3.9), we have

$$\lambda(u,v) = -\frac{\coth \alpha}{u + \Gamma(v)}, \quad \beta(u,v) = \varphi(v)(u + \Gamma(v)) \tag{3.10}$$

or

$$\lambda(u,v) = 0,\ \beta(u,v) = \beta(v) \tag{3.11}$$

Now we can give the classifications for M. Let (3.10) be a solution for equations (3.9). Since $g(r_u, k) = \cosh \alpha$ and $g(r_v, k) = 0$, we get

$$r(u,v) = (u\cosh\alpha, h(u,v))$$

where $h(u,v) \in IR_1^2$. As $g(r_u, r_u) = 1$, we have that

$$g(h_u, h_u) = y_u^2 - z_u^2 = -\sinh^2\alpha.$$

Therefore, we obtain

$$h_u = (\sinh\alpha f_1(v), \sinh\alpha f_2(v))$$

where $f(v) = (f_1(v), f_2(v)) \in IR_1^2$ and $\|f(v)\| = 1$. Thus, we get

$$r(u,v) = (u\cosh\alpha, u\sinh\alpha f_1(v) + \gamma_1(v), u\sinh\alpha f_2(v) + \gamma_2(v))$$

As $r_{uv} = r_{vu}$, it is easy to see that

$$\frac{d\gamma_1}{dv} = \sinh\alpha \frac{df_1}{dv}\Gamma(v)$$

$$\frac{d\gamma_2}{dv} = \sinh\alpha \frac{df_2}{dv}\Gamma(v).$$

Without loss of generality, if we get $f(v) = (\cosh v, \sinh v)$, this implies that

$$r(u,v) = (u\cosh\alpha, u\sinh\alpha\cosh v + \gamma_1(v), u\sinh\alpha\sinh v + \gamma_2(v)) \tag{3.12}$$

where

$$\gamma(v)=(\gamma_1(v),\gamma_2(v))=\sinh\alpha\left(\int_0^v \sinh\tau\Gamma(\tau)d\tau,\int_0^v \cosh\tau\Gamma(\tau)d\tau\right)$$

Let (3.11) be a solution for equations (3.9). Since $\beta_u=0$, we have $r_{vu}=0$. As $\mathrm{r}_{uu}=0$ and $r_{uv}=0$, we get $h_{uu}=0$ and $h_{uv}=0$. This implies that h_u is a constant vector in IR^2_1. We may assume that $h_u=(\sinh\alpha\sinh\mu,-\sinh\alpha\cosh\mu)$ such that μ is a constant. Therefore, we can write $h(u,v)=(u\sinh\alpha\sinh\mu+\gamma_1(v),-u\sinh\alpha\cosh\mu+\gamma_2(v))$. Because r_u and r_v are orthogonal, we obtain

$$\gamma(v)=(\gamma_1(v),\gamma_2(v))=(\cos\mu\Gamma(v),-\sinh\mu\Gamma(v)).$$

The last equation implies that

$$r(u,v)=(u\cosh\alpha,u\sinh\alpha\sinh\mu+\cosh\mu\Gamma(v),-u\sinh\alpha\cosh\mu-\sinh\mu\Gamma(v))$$

By applying a Lorentz transformation of the form

$$\begin{bmatrix} 0 & \cosh\mu & \sinh\mu \\ 0 & \sinh\mu & \cosh\mu \\ 1 & 0 & 0 \end{bmatrix}$$

We obtain

$$r(u,v)=(\Gamma(v),-\mathrm{u}\sinh\alpha,\cosh\alpha)$$

This is a parametrization for the Lorentz plane

$$\cosh\alpha y+\sinh\alpha z=0.$$

Special Case: If $\alpha = 0$ then $k = e_1$. Thus , k is a tangent to M . It follows that

$$r(u, v) = (u, \cosh \mu\Gamma(v), \sinh \mu\Gamma(v)).$$

In this case, M is a part of the cylindrical surface.

Finally, we can give the following theorem:

Theorem 3.1. *Every spacelike constant angle surface M with constant spacelike direction is a congruent to the following surfaces:*

(i) $r(u, v) = (u \cosh \alpha, u \sinh \alpha \cosh v + \gamma_1(v), u \sinh \alpha \sinh v + \gamma_2(v))$

where

$$\gamma(v) = (\gamma_1(v), \gamma_2(v)) = \sinh \alpha \left(\int_0^v \sinh \tau\Gamma(\tau) d\tau, \int_0^v \cosh \tau\Gamma(\tau) d\tau \right).$$

(ii) A part of the cylindrical surface:

$$r(u, v) = (u, \gamma_1(v), \gamma_2(v))$$

where

$$\gamma(v) = (\gamma_1(v), \gamma_2(v)) = (\cos \mu\Gamma(v), -\sinh \mu\Gamma(v))$$

(iii) A Lorentz plane which have the equation

$$\cosh \alpha y + \sinh \alpha z = 0$$

From (2.1), The mean curvature of the spacelike constant angle surface M is given by

$$H = -\frac{1}{2}\varepsilon\lambda.$$

where $\varepsilon = g(k, k)$. Thus we can give the following result:

Corollary 3.2. Minimal spacelike constant angle surface are the planes.

Example 3.1.

a) *If we take* $\Gamma(v) = 1$ *and* $\alpha = 2$ *in* (3.12), *we obtain the following parametrization for*

$$M : r(u, v) = (u\cosh(2), \cosh(v)\,(u\sinh(2) + 1) - 1, \sinh(v)(u\sinh(2) + 1) \;\; (Fig.1)$$

b) *If we take* $\Gamma(v) = v$ *and* $\alpha = 2$ *in* (3.12), *we obtain the following parametrization for*

$$M : r(u, v) = \begin{pmatrix} u\cosh(2), \cosh(v)\,(u\sinh(2) - 1) + 1 + v\sinh(v), \\ \sinh(v)(u\sinh(2) - 1 + v\cosh(v) \end{pmatrix} \;\; (Fig.2)$$

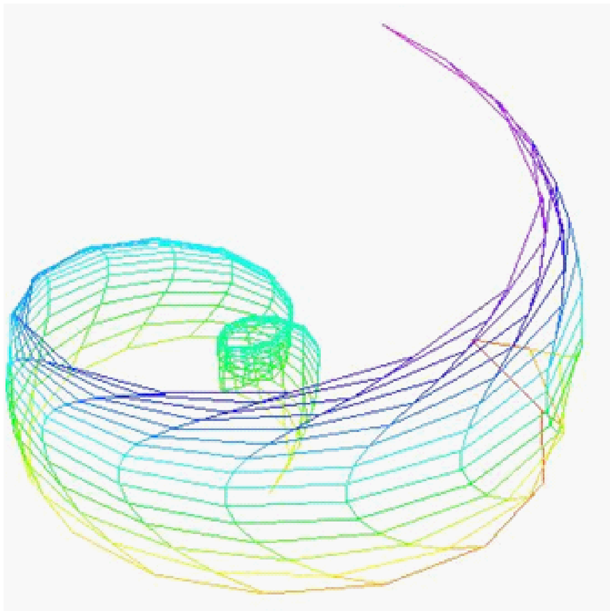

Figure 1: Spacelike constant angle surface with spacelike direction

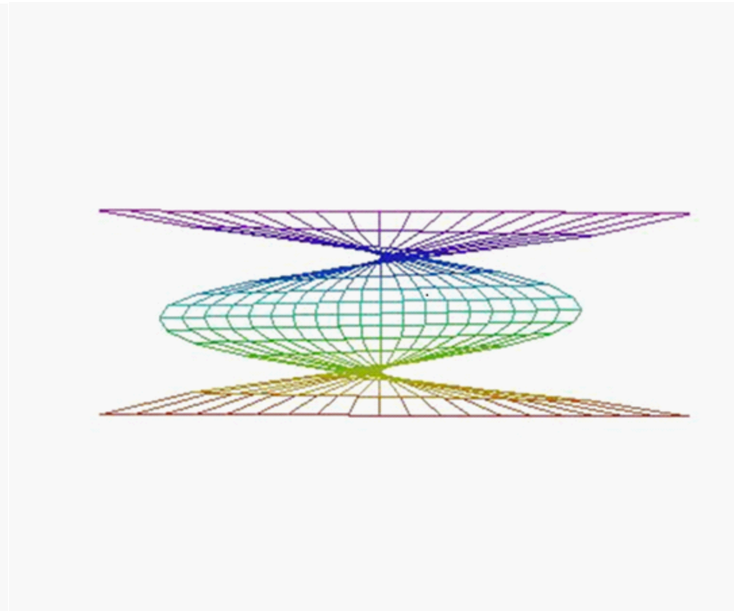

Figure 2: Spacelike constant angle surface with spacelike direction

References

[1] *Lopez and Munteanu Constant-angle surfaces in Minkowski space, Bulletin of the BElgian Mathematical Society, Volume18, Number 2 (2011), 271-286.*

[2] *Cermelli, P. and Di Scala, A. J.: Constant-angle surfaces in liquid crystals, Philosophical Magazine, 87(12),1871-1888 (2007).*

[3] *M.I.Munteanu, A.I.Nistor, A New Approach on Constant Angle Surface in E^3, Turkish J. Math. 33(2009).2, 1-10*

[4] *O'Neill, B. Semi-Riemannian Geometry. Academic Press. New York, 1983.*

[5] *Ratcliffe, J. G. Foundations of Hyperbolic Manifolds. Springer 2005.*

[6] *Beem, J. K. and Ehrlich, P. E. Global Lorentzian Geometry. Marcel Dekker. Inc. New York, 1981.*

[7] *Weinstein, T. Lorentz Surfaces. Walter de Gruyter, New York 1996.*

[8] *Woestijne, V. D. I. Minimal surfaces of the 3-dimensional Minkowski space, World Scientific Publishing Singapore: 344-369, 1990.*