Available online at http://scik.org

J. Semigroup Theory Appl. 2017, 2017:2

ISSN 2051-2937

ON LIU ALGEBRAS: A NEW COMPOSITE STRUCTURE OF THE *BCL*$^+$ ALGEBRAS AND THE SEMIGROUPS

YONGHONG LIU

School of Automation, Wuhan University of Technology, Wuhan 430070, China

Abstract: This paper offers a new algebra, which is called the Liu algebra (which is named after author), because of its origin in *BCL*$^+$ algebras, and connections between *BCL*$^+$ algebras and semigroups, have more complex structures, or, saying a composite structure. While Liu algebras are dividing into two distinct parts that are structurally independent, we think there are good reasons to mash them up, can be enforced by algebraic operations on distributive laws. Here we introduce several new notions (i.e., ideal, funnel and deductive systems in Liu algebras). We show that if G and H be two algebras, if G ≅ H, then (L; *, •, 1) is an order isomorphism, and discuss some properties for Liu algebras.

Keywords: semigroups; *BCL*$^+$ algebras; Liu algebras; ideal; isomorphism; deductive systems; funnel.

2010 AMS Subject Classification: 03G25, 20M99.

1. Introduction

BCL-algebras and *BCL*$^+$ *algebras* are introduced by author in [1,2], which is a relative new-comer in the history of algebra. Al-Kadi and Hosny in [3] introduced the concept of deformation of such algebra in *BCL-algebras* and illustrate the connection between divisible algebra and deformation function. *Soft BCL-algebras* are studied by Al-Kadi in [4]. The *BCL*$^+$ *algebras* developed recently by author in [5-12], it involved partial order, lattice, topology,

E-mail address: hylinin@163.com

Received Dec. 22, 2016

pseudocomplement, filtrations, ideals, deductive systems and funnels in *BCL*$^+$ *algebras*.

We know that the algebraic theory of *semigroups* occurs naturally in many areas of mathematics, such as combinatorics, automata theory, operator algebras and probability theory. We do also believe that *semigroups*, historically, to large extent, has based its prosperity by Clifford in [13-15], Hall in [16-18] and Nambooripad in [19] with their good works.

Module theory is a composite structure of ring theory and abelian group theory. Similarly, this paper offers a new composite structure, which is based on the well-known *semigroups* and *BCL*$^+$ *algebras* (also known as *Liu algebras*, which is named after author). The aim is that to involve a fresh approach for *Liu algebras*, and research that the properties and the results are interesting. We will also illustrate it with some examples.

2. Preliminaries (1)

In this section 2, let's first review some relevant concepts and state some results, as follows.

Definition 2.1 [2]. A *BCL*$^+$ algebra is a triple (Y; $*$, 1), where Y is a nonempty set, "$*$" is a binary operation on Y, and $1 \in Y$ is an element such that the following three axioms hold for any $x, y, z \in Y$.

(BCL^+1) $x * x = 1$,

(BCL^+2) $x * y = 1$ and $y * x = 1$ imply $x = y$, and

(BCL^+3) $((x * y) * z) * ((x * z) * y) = (z * y) * x$.

Theorem 2.1 [2]. *Assume that* (Y; $*$, 1) *is a BCL*$^+$ *algebra. Then the following hold for any* $x, y, z \in Y$.

(i) $(x * (x * y)) * y = 1$.

(ii) $x * 1 = x$ *implies* $x = 1$.

(iii) $((x * y) * (x * z)) * (z * y) = 1$.

(iv) ($BCL^{+}2$) $x * y = 1$ *and* $y * x = 1$ *imply* $x = y$.

Theorem 2.2 [8]. *Let Y be a* BCL^{+} *algebra. Le*t $x \leq y$ *iff* $x * y = 1$. *Then* $\leq$ *is a partially ordering* on *Y*; *that is* $(Y; \leq)$ *is a partially ordered set with greatest element* 1 (*and unit element*) *of Y*.

Theorem 2.3 [8]. *Let H be a nonempty subset of a* BCL^{+} *algebra* (Y; $*$, 1). *Then H is a filtration if and only if it is a deductive system.*

3. Preliminaries (2)

In this section 3, we recall some basic facts about semigroups which will be needed for this paper.

Definition 3.1. A set *G*, which has generating set $S \subseteq G$ with respect to a given binary operation $\bullet$, forms a semigroup if the following postulates hold:

(i) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, for all $a, b, c \in G$,

(ii) for any $a \in S$ and any $b \in G$, there is at most solution, $x \in G$, we have $a \bullet x = b$, and

(iii) similarly for $x \bullet a = b$.

Definition 3.2. Let *I* be a nonempty subset of semigroups *S*, and assume that for each $s \in S$ and $a \in I$, there is defined an ideal (or a right or left ideal) of *S* denoted $s \bullet a \in I$ and $a \bullet s \in I$. Suppose that for each $s \in S$ and $a \in I$, there is defined a bilateral ideal of *S* denoted $a \bullet s \bullet a \in I$.

Theorem 3.1. *Let S be a semigroup. Then S is a group for* $a, b \in S$, *if and only if*

$$a \bullet x = b \quad \textit{and} \quad y \bullet a = b.$$

for $x, y \in S$, *there exist only solutions in S for these equations*.

Definition 3.3. Let *S* be a semigroup and let $\leq$ be an order relation on *S*. Assume that for all

$a, b, c \in S$, if $a \le b$, we have

$$a \bullet c \le b \bullet c \quad \text{and} \quad c \bullet a \le c \bullet b.$$

Then we say that S is an ordered semigroup $(S; \bullet, \le)$.

Theorem 3.2. *Let S be a semigroup and let J be an index set, where* $\{I_j \mid j \in J\}$ *is an ideal variety. Then*:

(i) $\bigcup_{j \in J} I_j$ *is an ideal of S and*

(ii) *If* $\bigcap_{j \in J} I_j \ne \phi$, *then* $\bigcap_{j \in J} I_j$ *is an ideal of S.*

4. Main results

Definition 4.1. A *Liu algebra* is a tetrad $\mathcal{L} = (L; *, \bullet, 1)$, where L is a nonempty set, two binary operations $*$ and $\bullet$ defined on L. 1 is fixed element of L such that the following axioms hold for each of the elements $x, y, z \in L$.

(LA1) $(L; *, 1)$ is a BCL^+ algebra.

(LA2) $(L; \bullet)$ is a semigroup.

(LA3) $x \bullet (y * z) = (x \bullet y) * (x \bullet z)$.

(LA4) $(y * x) \bullet z = (y \bullet z) * (x \bullet z)$.

Definition 4.2. Let L be a monoid. If $1 \in L$, then for all $x \in L$, and we have

$$1 \bullet x = x \bullet 1 = x.$$

Definition 4.3. Let L be a monoid and let $x \in L$. If $y \in L$, then we have

$$x \bullet y = y \bullet x = 1,$$

and thus x^{-1} is the inverse of x in L, and it follows that $y = x^{-1}$.

Remark 4.1. Hint that Definition 4.3, any nonempty subset of H closed under multiplication and taking inverses in H is a subgroup.

Definition 4.4. By Definition 4.3, it suffices to show that a group is made of multiplication to the nonvanishing element on L, then $\mathcal{L} = (L; *, \bullet, 1)$ is called Liu group. If $H \subseteq L$ is nonempty subset, that two binary operations $*$ and $\bullet$ on L are closed, then H is called Liu subgroup.

Lemma 4.1. *Suppose that the Liu group H and that* $D \subseteq H$ *be a nonempty subset. Let* $x * y^{-1} \in D$ and $x \bullet y^{-1} \in D$ *for all* $x, y \in D$. *Then D is a Liu subgroup of H.*

Proof. Choose $x \in D$. It suffices to show that

$$1 = x * x^{-1} \in D \quad \text{and} \quad 1 = x \bullet x^{-1} \in D.$$

Now for $y \in D$, we have

$$y^{-1} = 1 * y^{-1} \in D \quad \text{and} \quad y^{-1} = 1 \bullet y^{-1} \in D,$$

let $x \in D$, we get

$$x * y = x * \left(y^{-1}\right)^{-1} \in D \quad \text{and} \quad x \bullet y = x \bullet \left(y^{-1}\right)^{-1} \in D.$$

Since $1 \in D, y^{-1} \in D$ for all $y \in D$, we conclude that D has an identity and inverses and so is a Liu subgroup.

Lemma 4.2. *Let* $(L; *, \bullet, 1)$ *be a Liu algebra. For all* $a, b \in L$, *we have* $a^{-1} = b^{-1}$ *if and only if* $a = b$.

Proof. Let $a = b$. Then $a^{-1} = b^{-1}$. Conversely, suppose $a^{-1} = b^{-1}$, for all $x \in L$, we have

$$x * a = x * b \quad \text{and} \quad x \bullet a = x \bullet b.$$

Now we can take $x = a$, or b. Then

$$a * b = b * a = 1 \quad \text{and} \quad a \bullet b = b \bullet a = 1,$$

so that $a = b$.

Definition 4.5. Let $(L; *, \bullet, 1)$ be a Liu algebra. A nonempty subset $S \subseteq L$ is a subalgebra if S is closed under tow binary operations $*$ and $\bullet$ in L. Assume that for all $x, y \in S$, there is defined a unique element $x * y \in S$ and $x \bullet y \in S$. Then we say that S is a subalgebra of L. (Of course, the (S; $*$, $\bullet$, 1) itself is a Liu algebra.)

We now give an example of Liu algebras.

Example 4.1. Let $L = \{a, b, c, 1\}$ and two binary operations $*$ and $\bullet$ on L can be represented by

Table 4.1. BCL^+ operation

$*$	a	b	c	1
a	1	1	c	1
b	a	1	c	1
c	1	c	1	1
1	1	1	1	1

Table 4.2. $L(X)$ operation

$\bullet$	a	b	c	1
a	1	1	1	1
b	a	1	1	1
c	1	c	1	1
1	1	1	1	1

By Definition 4.1, Definition 4.3 and Definition 4.4, if we let $x = a, y = b$ and $z = c$, then (L; $*$, $\bullet$, 1) is a Liu algebra, but is not a Liu group. If $a, b, c \in S$, the elements, $c * b \in S$, $b * a \in S$ and $c \bullet b \in S$, $b \bullet a \in S$, then $\{c, b\}$ and $\{b, a\}$ are subalgebras of L.

Theorem 4.1. *Let* (L; $*$, $\bullet$, 1) *be a Liu algebra. Then the following hold for each* $x \in L$:

(i) $1 \bullet x = x \bullet 1 = 1$ *and*

(ii) *Suppose that b is the maximal element in L. Then $x \bullet b$ and $b \bullet x$ are also the maximal* *elements*.

Proof. For part (i), by Definition 4.1 (LA1), (LA4), we see that

$$1 = (1 \bullet x) * (1 \bullet x) = (1 * 1) \bullet x = 1 \bullet x$$

and

$$1 = (x \bullet 1) * (x \bullet 1) = x \bullet (1 * 1) = x \bullet 1.$$

Thus $1 \bullet x = x \bullet 1 = 1$.

For part (ii), let $b = b * (1 * b)$ be the maximal element, by Definition 4.1 (L3), we see that

$$\begin{aligned}(x \bullet b) * ((x \bullet 1) * (x \bullet b)) &= (x \bullet b) * (x \bullet (1 * b)) \\ &= x \bullet (b * (1 * b)) \\ &= x \bullet b.\end{aligned}$$

We also use the same method to prove $b \bullet x$. □

Theorem 4.2. *Let* $(L; *, \bullet, 1)$ *be a Liu algebra and let* $x, y, z \in L$. *Then* $x \le y$ *implies* $z \bullet x \le z \bullet y$ *and* $x \bullet z \le y \bullet z$.

Proof. Since $x \le y$, and we can write $x * y = 1$. We now have

$$(z \bullet x) * (z \bullet y) = z \bullet (x * y) = z \bullet 1 = 1,$$

so that $z \bullet x \le z \bullet y$, as desired. In the case where $x \bullet z \le y \bullet z$, is proved similarly. □

Theorem 4.3. *Let* $(L; *, \bullet, 1)$ *be a Liu algebra and let* $x \le y$. *For all* $x, y, z \in L$, *if* $x * y = 1$. *Then* $\le$ *is a partial ordering on L.* (*The partial ordering Liu algebras*, *denoted by* $(L; *, \bullet, \le, 1)$ *or po-Liu algebras*.)

Proof. The proof is immediate by Theorem 2.2 and Definition 3.2. □

Definition 4.6. Let E be a nonempty subset of BCL^+ algebras $(Y; *, 1)$, we say that E is called an ideal of Y if

(BE1) $1 \in E$,

(BE2) For all $x, y \in Y$, $y * x \in E$ and $x \in E$ imply $y \in E$, and

(BE3) For all $x, y, z \in Y$, $(z * (y * x)) * x \in E$.

The following is the extension of Theorem 3.2 to BCL^+ algebras.

Theorem 4.4. *Let* $\{I_k \mid k \in K\}$ *is an ideal variety of* BCL^+ *algebras* $(Y; *, 1)$. *Then*

(i) *If* $\{I_k \mid k \in K\}$ *is an ideal chain* (*i.e. between any two elements can compare in* I_k). *Then* $\bigcup_{k \in K} I_k$ *is an ideal of* Y.

(ii) $\bigcap_{k \in K} I_k$ *is an ideal of* Y.

Proof. For part (i), of course, we have $1 \in \bigcup_{k \in K} I_k$. Now suppose that $y * x \in \bigcup_{k \in K} I_k$ and $x \in \bigcup_{k \in K} I_k$, we have $k_1, k_2 \in K$. Then $y * x \in I_{k_1}$ and $x \in I_{k_2}$. We may assume that $I_{k_1} \subseteq I_{k_2}$, then $y * x \in I_{k_2}$ and $x \in I_{k_2}$, where I_{k_2} is an ideal of Y. Since $y \in I_{k_2} \subseteq \bigcup_{k \in k} I_k$. We prove Definition 4.6 (BE2). We know that we can write

$$\begin{aligned} z * ((z * (y * x)) * x) &= (z * (y * x)) * (z * x) \\ &\geq (y * x) * x \\ &\geq y, \end{aligned}$$

and

$$y, z \in \bigcup_{k \in K} I_k, \text{ imply } (z * (y * x)) * x \in \bigcup_{k \in K} I_k.$$

In fact, since $y * x \in \bigcup_{k \in K} I_k$, we have $k_1,\ k_2 \in K$, and so $z * (y * x) \in I_{k_1},\ I_{k_2}$. Since $x \in I_{k_1},\ I_{k_2}$. Then

$$(z * (y * x)) * x \in I_{k_1},\ I_{k_2} \subseteq \bigcup_{k \in K} I_k,$$

and the proof that Definition 4.6 (BE3) is satisfied. Thus $\bigcup_{k \in K} I_k$ is an ideal of Y.

For part (ii), assume $k \in K$ and let $1 \in I_k$. Then we can choose $1 \in \bigcap_{k \in K} I_k$. Let

$y*x \in \bigcap_{k\in K} I_k$ and $x \in \bigcap_{k\in K} I_k$. Since for each $k \in K$. Then $y*x \in I_k$ and $x \in I_k$. If I_k is an ideal of *Y*. Than $y \in I_k$. This yields $y \in \bigcap_{k\in K} I_k$, and thus $\bigcap_{k\in K} I_k$ is an ideal of *Y*. We prove Definition 4.6 (BE2), is also. We also use the part (i) method to prove $(z*(y*x))*x \in I_{k_1}$, $I_{k_2} \subseteq \bigcap_{k\in K} I_k$. Thus $\bigcap_{k\in K} I_k$ is an ideal of *Y*. □

Definition 4.7. Let *K* be a nonempty subset of Liu algebras (L; $*$, $\bullet$, 1), let $a \in K$, $x \in L$. We say that *K* is called an ideal of *L* if

(LK1) $x*a \in K$ implies $x \in K$,

(LK2) $a \bullet x \in K$ and $x \bullet a \in K$ imply $x \in K$.

Remark 4.2. Hint that Definition 4.7, we discuss a few examples of ideals. For any Liu algebras, of course, the *L* itself and $\{1\}$ are ideals of *L*. Beyond that is called a principal ideal Liu algebra, or a *PKL*.

Example 4.2. Example 4.1 shows that $\{c, b\}$ and $\{b, a\}$ are subalgebras of *L*, but $\{1, a\}$ is not a *PKL*, since $b*a = a$, but we conclude that $b \notin \{1, a\}$.

Theorem 4.5. *Let* (L; $*$, $\bullet$, 1) *be a Liu algebra and suppose* $x \in L$. *Then the following are ideal of semigroups* (L; $\bullet$):

(i) $M(L) = \{x \in L \mid 1*x = 1\}$ *and*

(ii) $V(L) = \{x \in L \mid 1*x = x\}$.

Proof. For part (i), let $a \in M(L)$ (viewed as a set) and $x \in L$. To see that this hold, write $1*x = 1$. Then

$$\begin{aligned}(1\bullet x)*(a\bullet x) &= (a\bullet 1)*(a\bullet x)\\ &= a\bullet(1*x)\\ &= a\bullet 1\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}(1 \bullet x) * (x \bullet a) &= (1 \bullet a) * (x \bullet a)\\ &= (1 * x) \bullet a\\ &= 1 \bullet a\\ &= 1,\end{aligned}$$

and so $a \bullet x \in M(L), x \bullet a \in M(L)$, we now know that $M(L)$ is an ideal of semigroups (L; •). Part (ii) is proved similarly. □

That is as far as an application of Theorem 4.4 and Theorem 3.1, a similarly result is an ideal variety, there also exists in Liu algebras, and we give the following corollary.

Corollary 4.1. *If* $\{I_k \mid k \in K\}$ *is an ideal variety of Liu algebras* (L; ∗, •, 1), *then*

(i) *If* $\{I_k \mid k \in K\}$ *is an ideal chain* (*i.e. between any two elements can compare in* I_k). *Then* $\bigcup_{k \in K} I_k$ *is an ideal of L.*

(ii) $\bigcap_{k \in K} I_k$ *is an ideal of L.*

Corollary 4.2. *Let Q be a nonempty subset of Liu algebras* (L; ∗, •, 1). *Then Q is an ideal if and only if*

(i) *Q is an ideal of* BCL^+ algebras (L; ∗, 1), *and*

(ii) *Q is an ideal of* semigroups (L; •).

Definition 4.8. If D be a nonempty subset of a Liu algebra (L; ∗, •, 1). Then we say that D be a deductive system of L if the following two axioms are satisfied:

(DS1) $1 \in D$ and

(DS2) $x \in D$, $x * y \in D$ and $x \bullet y \in D$ imply $y \in D$.

Lemma 4.3. *Let* (L; ∗, •, 1) *be a Liu algebra and* $x, y, z \in L$. *Then*

$$x \bullet (y \bullet z) = y \bullet (x \bullet z).$$

Proof. In fact

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z = (y \bullet x) \bullet z = y \bullet (x \bullet z).$$

This completes the proof. □

Theorem 4.6. *A nonempty subset K of a Liu algebra* $(L; *, \bullet, 1)$ *is an ideal if and only if it is a deductive system.*

Proof. Let K be an ideal, we show now that if $1 \in K$. Then Definition 4.8 (DS1) is satisfied. To prove Definition 4.8 (DS2) suppose $a \in K$ and $x * a = a_1 \in K$ for some $x \in L$. Then by Definition 4.7 (LK1), since $1 \in K$, we have

$$a_2 = (x * a) * x \in K,$$

and so

$$\begin{aligned} x &= 1 * x \\ &= (((x * a) * x) * ((x * a) * x)) * x \\ &= (a_2 * (a_1 * x)) * x \in K. \end{aligned}$$

Thus $a \in K$ and $a * x \in K$. Using the same methods we get

$$x = 1 \bullet x \in K \quad \text{and} \quad x = x \bullet 1 \in K.$$

Taken together, it imply $x \in K$. We prove Definition 4.8 (DS2). Then K is a deductive system.

Conversely, if K is a deductive system, then $1 \in K$, we have

$$\begin{aligned} a * (a * x) &\leq (a * a) * (a * x) \\ &= 1 * (a * x) \\ &= 1 * (a * a) \qquad , \\ &= 1 * 1 \\ &= 1 \in K, \end{aligned}$$

by Lemma 4.3, we have

$$a \bullet (x \bullet a) = x \bullet (a \bullet a) = a \bullet 1 = 1 \in K,$$

and

$$a \bullet (a \bullet x) = (a \bullet a) \bullet x = 1 \bullet a = 1 \in K.$$

This for $a \in K$ imply $x \bullet a \in K$ and $a \bullet x \in K$, for every $x \in L$. Hence Definition 4.7 (LK1) and (LK2) are satisfied. $\square$

Definition 4.9. Let G and H be two Liu algebras and assume that the corresponding mapping $\theta : G \to H$ is a bijection. We say that θ is an isomorphism for all $x, y \in G$, then

(i) $\theta(x * y) = \theta(x) \tilde{*}\, \theta(y)$ and

(ii) $\theta(x \bullet y) = \theta(x) \tilde{\bullet}\, \theta(y)$,

where, $\tilde{*}, \tilde{\bullet}$ are two binary operations of H. We say that G and H are isomorphic, and we write $G \cong H$ if an isomorphism between them exists.

Remark 4.3. Note that if $\theta : G \to H$ is an isomorphism, then $\theta^{-1} : H \to G$ is an isomorphism also. In addition, if $\mu : H \to P$ is another isomorphism, it is routine to check that $\pi : G \to P$ is an isomorphism, and $\mu \circ \theta = \pi$.

$$\begin{tikzcd}
G \arrow[rr, "\pi"] \arrow[dr, "\theta"'] & & P \\
& H \arrow[ur, dashed, "\mu"'] &
\end{tikzcd}$$

Theorem 4.7. *Let* $G \cong H$. *Then* (L; $*$, $\bullet$, 1) *is an order isomorphism.*

Proof. Choose θ such that $\theta : G \to H$, let $x \le y$, we can write $x * y = 1$.

Then we have

$$\theta(x) \tilde{*}\, \theta(y) = \theta(x * y) = \theta(1) = 1,$$

and let $x \bullet y = 1$. Then

$$\theta(x) \tilde{\bullet}\, \theta(y) = \theta(x \bullet y) = \theta(1) = 1.$$

As above, this order $\theta(x) \le \theta(y)$. This completes the proof. $\square$

Theorem 4.8. *Let G and H be two Liu algebras and assume that the corresponding mapping*

$\theta : G \to H$ is a bijection. Let Q be an ideal for the Liu algebras G. Then $\theta(Q)$ is an ideal of H.

Proof. (i) To prove that $\theta(Q)$ *is an ideal of BCL*$^+$ algebras (L; $*$, 1). If Q is an ideal of G, since $1 \in Q$, we have $1 = \theta(1) \in \theta(Q)$. Let $x, y \in G$ and $\tilde{x}, \tilde{y} \in H$. If $\tilde{x}, \tilde{y} \,\tilde{*}\, \tilde{x} \in \theta(Q)$. Then there exists $x, z \in Q$, and hence $\tilde{x} = \theta(x)$, $\tilde{y} = \theta(y)$ and $\tilde{y} \,\tilde{*}\, \tilde{x} = \theta(z)$, we have

$$\theta(z) = \theta(y) \,\tilde{*}\, \theta(x) = \theta(y * x),$$

and so

$$\theta((y * x) * z) = \theta(y * x) \,\tilde{*}\, \theta(z) = 1,$$

and

$$\tilde{y} = \theta(y) \,\tilde{*}\, 1 = \theta(y) \,\tilde{*}\, \theta((y * x) * z) = \theta(y * ((y * x) * z)),$$

where

$$((y * x) * ((y * x) * z)) * z = 1 \in Q,$$

we conclude that $\tilde{y} \in \theta(Q)$ and $\theta(Q)$ are ideal of H.

(ii) To prove that $\theta(Q)$ is an ideal of semigroups (L; •). Let $\tilde{x} \in H, \tilde{a} \in \theta(Q)$, since $G \cong H$. Then there exists $x \in G, a \in Q$, and we have $\tilde{x} = \theta(x)$ and $\tilde{a} = \theta(a)$. Since Q is an ideal of semigroups, by Definition 4.7 (LK2), we can write $a \bullet x \in Q$ and $x \bullet a \in Q$, and we have

$$\tilde{a} \,\tilde{\bullet}\, \tilde{x} = \theta(a) \,\tilde{\bullet}\, \theta(x) = \theta(a \bullet x) \in \theta(Q),$$

and

$$\tilde{x} \,\tilde{\bullet}\, \tilde{a} = \theta(x) \,\tilde{\bullet}\, \theta(a) = \theta(x \bullet a) \in \theta(Q).$$

This completes the proof.

Corollary 4.3. *Let G and H be two Liu algebras and map $\theta : G \to H$. Suppose P is an ideal of H. Then $\theta^{-1}(P)$ is an ideal of G.*

The following definition is very useful.

Definition 4.10. Let F be a nonempty subset of Liu algebras (L; $*$, $\bullet$, 1) and assume that

$$L * F = F \quad \text{and} \quad L \bullet F = F .$$

Then F is a funnel of Liu algebras.

Example 4.3. Example 4.1 show that F = {1, c} is a subset of L, it is also a funnel of Liu algebras (L; $*$, $\bullet$, 1).

Theorem 4.9. *If* (L; $*$, $\bullet$, 1) *is a Liu algebra*, *then L is a funnel.*

Proof. The proof is simple. Let 1 = F, we get

$$1 = L * L = F ,$$

and

$$1 = L \bullet L = F .$$

By Definition 4.10, we have $L = F$. □

Theorem 4.10. *Let H be a funnel of Liu algebras* (L; $*$, $\bullet$, 1). *Then there exists* $1 \subseteq H$.

Proof. Let H be a funnel of Liu algebras, in particular, $0 \neq H$. Then $H = L * H$. By Definition 4.10, if $a \subseteq H$, then

$$1 = a * a \subseteq L * H = H ,$$

and

$$1 = a \bullet a \subseteq L \bullet H = H ,$$

and so $1 \subseteq H$. □

Theorem 4.11. *Let H be a funnel of Liu algebras* (L; $*$, $\bullet$, 1). *Then H is a subalgebra of Liu algebras*.

Proof. Let $a, b \in H$. Then by Definition 4.10, we have

$$a * b \in H * H \subseteq L * H = H \subseteq L ,$$

and

$$a \bullet b \in H \bullet H \subseteq L \bullet H = H \subseteq L.$$

therefore, this proves the funnel H is a subalgebra of Liu algebras. □

Remark 4.4. Strictly speaking, the converse of Theorem 4.11 is not true in next example.

Example 4.4. $H = \{a, c\}$ is a subset of L, but it is not a funnel of Liu algebras $(L; *, \bullet, 1)$ in Example 4.1.

5. Conclusion

It turns out that BCL^+ algebras are a very interesting area of research in the theory of algebraic systems in mathematics. In the present paper, the BCL^+ algebras plays a central role in Liu algebras, and the semigroup is actually helping one important aspect of Liu algebras, so we put some useful definitions and properties into Section 4, Main Results.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Y. H. Liu, A new branch of the pure algebra: *BCL*-algebras, *Advances in Pure Mathematics*, 1 (2011), 297-299. http://dx.doi.org/10.4236/apm.2011.15054

[2] Y. H. Liu, On BCL^+ -algebras, *Advances in Pure Mathematics*, 2 (2012), 59-61. http://dx.doi.org/10.4236/apm.2012.21012

[3] D. Al-Kadi and R. Hosny, On *BCL*-algebra. *Journal of Advances in Mathematics,* 3 (2013), 184-190.

[4] D. Al-Kadi, Soft *BCL*-algebra, *International Journal of Algebra*, 8 (2014), 57-65. http://dx.doi.org/10.12988/ija.2014.311122

[5] Y. H. Liu, Partial orders in BCL^+-algebra, *Journal of Advances in Mathematics,* 5 (2013), 630-634.

[6] Y. H. Liu, Topological BCL^+-algebras, *Pure and Applied Mathematics Journal*, 3 (2014), 11-13.

[7] Y. H. Liu, Some distributions of BCL^+- algebras, *International Journal of Algebra*, 8 (2014), 495-503. http://dx.doi.org/10.12988/ija.2014.4556

[8] Y. H. Liu, Filtrations and deductive systems in BCL^+ algebras, *British Journal of Mathematics & Computer Science*, 8 (2015), 274-285.

[9] Y. H. Liu, Funnels in BCL^+ algebras, *International Journal of Mathematical Sciences & Engineering Applications*, 9 (2015), 179-185.

[10] Y. H. Liu, Improve results for set identities, *Applied Mathematics*, 5 (2014), 677-684. http://dx.doi.org/10.4236/am.2014.54065

[11] Y. H. Liu, Liu's laws and *p*- BCL^+ algebras, *International Journal of Pure & Engineering Mathematics*, 3 (2015), 51-58.

[12] Y. H. Liu, Standard ideals in BCL^+ algebras, *Journal of Mathematics Research*, 8 (2016), 37-44.

[13] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, *Mathematical Surveys*, *American Mathematical Society*, *Providence*, *RI*. *No*. 7, *Vol*. 1. 1961.

[14] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, *Mathematical Surveys*, *American Mathematical Society*, *Providence*, *RI*. *No*. 7, *Vol*. 2. 1967.

[15] A. H. Clifford, The fundamental representation of a completely regular semigroup, *Semigroup Forum*, 12 (1978), 341-346.

[16] T. E. Hall, On orthodox semigroups and uniform and anti-uniform bands, *Journal of Algebra*, 16 (1970), 204-217.

[17] T. E. Hall, On regular semigroups, *Journal of Algebra*, 24 (1973), l-24.

[18] T. E. Hall, Representation extension and amalgamation for semigroups, *Quart. J. Math. Oxford*, 29 (1978), 309-334.

[19] K. S. S. Nambooripad, Structure of regular semigroups, I. Fundamental regular semigroups, II. The general case, *Semigroup Forum*, 9 (1975), 354-372.