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ON PAIRWISE L-CLOSED SPACES IN BITOPOLOGICAL SPACES

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Abstract. The purpose of this paper is to introduce some further properties of the concept of pairwise L-closed spaces as a continuation of a previous study of this notion.

Keywords: pairwise L-closed spaces; bitopological spaces.

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1. Introduction

Kelly [1] introduced and studied the notion of bitopological spaces in 1963. He defined pairwise Hausdorff, pairwise regular and pairwise normal spaces.

Several mathematicians studied various concepts in bitopological spaces which are turned to be an important field in general topology. In this paper, we study the notion of pairwise L-closed spaces in bitopological spaces and their relations with other bitopological concepts.

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We use $\mathbb{R}$ and $\mathbb{N}$ to denote the set of all real and natural numbers respectively, p- to denote pairwise, *cl* to denote the closure of a set, and τ_s, τ_l to denote Sorgenfrey and left ray topologies on $\mathbb{R}$ or $\mathbb{N}$.

2. Pairwise L-Closed spaces

Definition 2.1: A bitopological space (X,τ_1,τ_2) is said to be *pairwise L-closed space* if each τ_1-Lindl$\ddot{o}$f subset of X is τ_2-closed and each τ_2-Lindl$\ddot{o}$f subset of X is τ_1-closed.

Definition 2.2: [5] A bitopological space (X,τ_1,τ_2) is called *pairwise T_1space* if for each pair of distinct points *x*,*y* in X, there exists a τ_1-neighbourhood U of x and a τ_2-neighbourhood V of y such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Definition 2.3: [4] A bitopological space (X,τ_1,τ_2) is called *p-Hausdorff space* if $\forall x \neq y$ in X, there exists a τ_1-neighbourhood U of x and a τ_2-neighbourhood V of y such that $x \in U$, $y \in V$, and $U \cap V = \phi$

Definition 2.4: [3] A bitopological space (X,τ_1,τ_2) is called *Lindl$\ddot{o}$f* (resp. *compact*) if it is τ_1-Lindl$\ddot{o}$f (resp. τ_1-compact) and τ_2-Lindl$\ddot{o}$f (resp. τ_2-compact).

Example 2.5: Consider $\mathbb{R}$ equipped with Sorgenfreyline topology τ_s and the left ray topology τ_l,then the bitopological space is not pairwise L-closed space because if A is a τ_s-open subset, then A is τ_s-Lindl$\ddot{o}$f. However A is not τ_l-closed.

Proposition 2.6: If every countable subset of (X,τ_1,τ_2) is closed, then every countable subset of X is discrete.

Proof: Let A be a τ_i-subset of X such that $A = \cup_{k=1}^{\infty}\{x_k\}$ where $x_k \in X$, $k \in \mathbb{N}$,

then A is a countable subset of X and hence closed by the assumption,see [6] .

Each point of A is an isolated point. Thus A is discrete.

Definition 2.7: [3] A bitopological space (X,τ_1,τ_2) is called *first* (*second*) *countable* if (X,τ_1) is first (second) countable and (X,τ_2) is first (second) countable.

Definition 2.8: A topological space $(X,\tau)$is called *Fre$'$chet Urysohn* if and only if for every $A \subseteq X$, and every $x \in clA$, there exists a sequence (x_n) in A converges to x $\forall n \in \mathbb{N}$. A bitopological space (X,τ_1,τ_2) is said to be *pairwise Fre$'$chet Urysohn* if it is τ_1-Fre$'$chet Urysohn and τ_2-Fre$'$chet Urysohn.

Definition 2.9: A topological space (X,τ) is called *sequential* if every non closed subset F of X contains a sequence converging to a point in X-F. A bitopological space (X,τ_1,τ_2) is said to be *pairwise sequential* if it is τ_1-sequential and τ_2-sequential.

Definition 2.10: A topological space (X,τ) is said to have a *countable tightness property* if whenever $A \subseteq X$ and $x \in clA$, there exists a countable subset B of A such that $x \in clB$. A bitopological space (X,τ_1,τ_2) is said to have a *pairwise countable tightness property* if it has τ_1-countable tightness and τ_2-countable tightness property.

Proposition 2.11: For a pairwise T_1 pairwise L-closed space, the following are equivalent:

a. X is first countable.

b. X is pairwise Fre$'$chet Urysohn.

c. X is pairwise sequential.

d. X has pairwise countable tightness property.

e. X is discrete.

Proof: a→b) Suppose that (X,τ_1,τ_2) is a first countable space, let A be a subset of X such that $x \in cliA$ $\forall i$=1,2.

Then there exists a τ_i-countable local base $\widetilde{\beta} = \{\beta_n : n \in \mathbb{N}\}$ at x such that $x_1 \in \beta_1$, $x_2 \in \beta_2 - \beta_1$, ... each of which must intersect A. Now choose x_m such that $x_m \in \beta_n \cap A$ $\forall m \geq 1$ $m \in \mathbb{N}$,so x_m is a τ_i-sequence in A.

Given any τ_i-neighborhood U of x, U must contain β_n for some $n \in \mathbb{N}$.So x_n belongs to U $\forall n \geq n_\circ$ and x_n converges to x. Thus X is pairwise Fre$'$chet Urysohn.

b→c) Suppose that X is pairwise Fre$'$chet Urysohn. Let C be a τ_i-sequentially closed subset of X such that C is not $\tau_i - closed$ ∀i=1,2. Let $x \in cl_iC$ such that $x \notin C$.

Since X is pairwise Fre$'$chet Urysohn, there exists a τ_i-sequence (x_n)such that $(x_n) \to x$ $\forall n \in \mathbb{N}$.

C is τ_i-sequentially closed, hence $x \in C$.Thus C is $\tau_i - closed$ which is a contradiction.

c→d) Let X be a pairwise sequential space, let D be a τ_i-subset of X such that $x \in cl_iD$ $\forall i$=1,2.

Suppose that K=$\cup\{$ cl_iE: $E \subseteq D$ $|$ E $|\leq \omega_\circ\}$. Clearly, $K \subset cl_iD$,hence K is τ_i-sequentially closed.

Let x_n be a τ_i-sequence in D such that $x_n \to x$ $\forall n \in \mathbb{N}$, $x \notin D$,there exists a τ_i-countable subset E_n of D such that $x_n \in \cup_{n=1}^{\infty} cl_i E_n$, so $x \in K$. Since X is pairwise sequential, K=$cl_i D$, i.e K is τ_i-closed.

Thus X is pairwise countable tightness.

d$\to$e) Let X be a pairwise countable tightness, let B be a τ_i-open subset of X such that

$x \in cl_i B$ $\forall$i=1,2, $\exists$ a τ_i-countable subset $M \subseteq B$ such that $x \in$cl$_i M$. But X is a pairwise L-closed space,

so M=$cl_i M$ and hence M is τ_i-discrete by proposition2.6. Thus X is discrete.

e$\to$a) Let X be a discrete space, $\forall x \in X$ $\exists$ a τ_i-open set U containing x. But $x \in \{x\}$ because X is discrete, so $\{x\} \subseteq U$.

Hence $\{\{x\}$: $x \in X\}$ is a τ_i-local base for X.

Thus X is a τ_i-first countable $\forall$i=1,2, and hence X is first countable.

Definition 2.12: A topological space (X,τ) is said to be *sequentially compact* if every infinite sequence has a convergent subsequence. A bitopological space (X,τ_1,τ_2) is called *pairwise sequentially compact* if it is τ_1-sequentially compact and τ_1-sequentially compact.

Definition 2.13: A topological space (X,τ) is said to be *strongly countably compact* if the closure of every countable subset of X is compact. A bitopological space (X,τ_1,τ_2) is called *pairwise strongly countably compact* if it is τ_1-strongly countably compact and τ_2-strongly countably compact.

Definition 2.14: A topological space (X,τ) is said to be *countably compact* if every countable open cover of X has a finite subcover. A bitopological space (X,τ_1,τ_2) is called *pairwise countably compact* if it is τ_1-countably compact and τ_2-countably compact.

Proposition 2.15: If (X,τ_1,τ_2) is p-Hausedorff pairwise L-closed, then the following are equivalent:

a. X is compact.

b. X is pairwise sequentially compact.

c. X is pairwise strongly countably compact.

d. X is pairwise countably compact.

e. X is finite.

Proof: a→b) Let (x_n) be a τ_i-sequence that has no τ_i-convergent subsequence ∀i=1,2 ∀n∈ ℕ. $w.l.o.g$ let $x_k \neq x_l$ $\forall k,l \in \mathbb{N}$, $k \neq l$. Each term of the τ_i-sequence (x_n) is a τ_i-isolated point, since otherwise (x_n) would have a τ_i-convergent subsequence. So ∀i=1,2 ∃ a τ_i-open set u_n neighborhood of x_k such that $x_l \notin u_n$ $\forall k \neq l$ because X is p-Hausdorff.

Let $u_n\prime$=X-$\{x_n$:n∈ ℕ$\}$ be a τ_i-open set, then it's complement consists only of τ_i-isolated points.

So it is τ_i-closed. Hence $\{u_n\prime\} \cup \{u_n : n \in \mathbb{N}\}$ is a τ_i-open cover of X that has no τ_i-finite subcover because any finite subcollection of these sets would fail to include infinitely many τ_i-terms from (x_n) in its union.

Thus X is not compact.

b→d) Let X be a pairwise sequentially compact and A be a τ_i- infinite countable subset of X ∀i=1,2.

Let (x_n) be a τ_i-sequence in A. (x_n) has a τ_i-convergent subsequence (x_{n_k}) $\forall n,k \in \mathbb{N}$.

There exists a point $x \in X$ such that $(x_{n_k}) \to x$.

If U is a τ_i-neighborhood of x, then U contains a tail of (x_{n_k}).$(U$-$\{x\}) \cap A \neq \phi$.Thus x is a τ_i-cluster point of A.

Hence X is pairwise countably compact.

Definition 2.16: [8] If (X,τ) is a L-closed space, then X is called a L_4-*space* if whenever $A \subseteq X$ is Lindlöf, there exists a Lindlöf $F_\sigma-$set F such that $A \subseteq F \subseteq clA$.

Definition 2.17: [8] If (X,τ_1,τ_2) is a pairwise L-closed space, then X is called a *pairwise* L_4-*space* if $\forall \tau_i-$Lindlöf subset A of X, there exists a $\tau-$Lindlöf F_σ-subset F such that $A \subseteq F \subseteq cl_i A$ $\forall i$=1,2.

Proposition 2.18: If (X,τ_1,τ_2) is a pairwise L-closed space, then X is pairwise L_4-*space.*

Proposition 2.19: If (X,τ_1,τ_2) is a pairwise L-closed space, then every τ_i-Lindlöf F_σ-set is τ_j-closed ∀i,j=1,2 i≠j.

Proposition 2.20: If every τ_i-Lindlöf subset of (X,τ_1,τ_2) is a τ_i-F_σ-set, then X is pairwise T_1 and every countable closed subset of X is discrete.

Proposition 2.21: In a p-Hausdorff space (X,τ_1,τ_2), the following are equivalent:

a. (X,τ_1,τ_2) is pairwise L-closed space.

b. every τ_i-Lindl$\ddot{o}$f F_σ-set is τ_j-closed ∀i,j=1,2 i≠j and τ_i-closure of a τ_i-Lindl$\ddot{o}$f subset of X is τ_i-Lindl$\ddot{o}$f.

Definition 2.22:[7] A topological space (X,τ) is called *hypercountably compact* if the union of every countable family of compact subsets of X has a compact closure.

Definition 2.23: A bitopolgical space (X,τ_1,τ_2) is said to be *pairwise hypercountably compact* if it is τ_i-hypercountably compact ∀i$= 1,2$.

Proposition 2.25: If a bitopolgical space (X,τ_1,τ_2) is p-Hausdorff pairwise L_4 strongly countably compact, then X is pairwise hypercountably compact.

Proof: Let $\tilde{U}$={ A_k: $A_k \subseteq X$, A_k is τ_i-compact} $\forall k \in \mathbb{N}$, ∀i$= 1,2.X$ is pairwise strongly countably compact, hence cl_iA_k is compact. Since X is pairwise L_4,there exists a Lindl$\ddot{o}$f $F_\sigma-$subset F such that $A_k \subseteq F \subseteq cl_iA_k$.

Now $\cup_{k=1}^{\infty} cl_iA_k$ is compact, thus X is pairwise hypercountably compact.

Conflict of Interests

The authors declare that there is no conflict of interests.

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