Available online at http://scik.org
J. Semigroup Theory Appl. 2019, 2019:6
https://doi.org/10.28919/jsta/3743
ISSN: 2051-2937

COMMON FIXED POINT IN D^* METRIC SPACES

NEVEEN ALZUDE, ABDALLA TALLAFHA*

Department of mathematics, The University of Jordan, Amman-Jordan

Abstract. In this paper we shall obtaine a new results conserning fixed point in D^* Metric Spaces, besides we correct the proves of some results obtaned by, T. Veerapandi and AJI. M Pillai in [35].

Keywords: metric spaces; D^* metric Spaces; semi-linear uniform spaces; contractions; fixed point; common fixed point.

2010 AMS Subject Classification: Primary 54E35, Secondary 41A65.

1. Introduction

The most important principle as a source of existence and uniqueness theorem in different branches of Sciences is Banach contraction principle, since it gives the existence, uniqueness and the sequence of the successive approximation converges to solution.

The study of metric fixed point theory plays an important role in many important areas as differential equation, operation research, mathematical economies and other branches, see $[1],[4],[5],[7],[9]-[11],[13]-[17],[19]-[22],[25]-[28],[36]$ and $[37]$.

*Corresponding author

E-mail address: a.tallafha@ju.edu.jo

Received May 9, 2018

Several generalization of metric spaces were proposed by several mathematicians such as 2-metric spaces, Gahler[8]. D-metric spaces, B. C. Dhage [6], G- metric space [18], L. -G. Huaug at X. Zhang, b- metric spaces, [12] .

Recently A.Tallafha and R. Khalil [29], defined a space which is a mixture of analysis and topology, namely semi-linear uniform space. Semi-linear uniform space is weaker than metric space and stronger than topological space since. Several authors studied the properties of semi-linear uniform spaces and fixed point in such spaces, see [2], [3], [23], [24], and [30]− [34].

In 2011, T. Veerapandi and AJI. M Pillai in [35], established some common fixed point theorems for contraction and generalized contraction mapping in D^*-metric spaces. In the proves of the main results, T. Veerapandi and AJI. M Pillai found a sequence named d_n^* that satisfies $d_{n+1}^* - d_n^* \leq \alpha^n d_0$, since $\alpha \in [0,1)$, then $\alpha^n d_0 \to 0$. From this they conclude that $d_{n+1}^* \leq d_n^*$. This conclusion not correct by the following example.

Example 1. *Let* $x_n = 1 - \frac{1}{2^n}$, *then* $x_{n+1} = 1 - \frac{1}{2^{n+1}}$, *therefor* $x_{n+1} - x_n = \frac{1}{2^{n+1}} - \frac{1}{2^n} = \frac{1}{2^{n+1}} = \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)$, *so* $\alpha = d_0 = \frac{1}{2}$, *clearly* $x_{n+1} \geq x_n$.

In this article, we define an orbit of two and three mapping on a D^*-metric space X, also we correct the proves given in [35] ,under additional condition. Finally we obtained anew results of common fixed point for contraction mapping in D^*-metric spaces.

2. D^* Metric Spaces

Definition 1. *Let* X *be a non-empty set. A generalized metric space or (*D^**-metric space) on* X *is a function* $D^* : X^3 \to [0,\infty)$ *that satisfies the following conditions, for each* $x,y,z,a \in X$.

i) $D^*(x,y,z) \geq 0$.

ii) $D^*(x,y,z) = 0$ *if and only if* $x = y = z$.

iii) $D^*(x,y,z) = D^*(P\{x,y,z\})$*.where* P *is permutation.*

iv) $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$.

The pair (X,D^*) *is called generalized metric space or(* D^**-metric space).*

Definition 2. *An open ball in* D^**-metric space* X *with center* x *and radius* r *is denoted by* $B_{D^*}(x,r)$*, and is defined by* $B_{D^*}(x,r) = \{y \in X : D^*(x,y,y) < r\}$

Definition 3. *Let (X,D^*) be a D^*-metric space and $A \subset X$,*

i) A subset A of X is said to be D^*-bounded if there exists $r > 0$, such that $D^*(x,y,y) < r$, for all $x,y \in A$.

ii) A sequence $\{x_n\}$ in X converges to x if and only if

$D^*(x_n, x_n, x) = D^*(x,x,x_n) \to 0$ as $n \to \infty$.That is, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$

such that $D^*(x,x,x_n) < \varepsilon$ for all $n \geq n_0$.

iii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$

such that $D^*(x_n, x_n, x_m) < \varepsilon$ for all $n,m \geq n_0$.

iv) the D^*-metric space (X,D^*) is said to be complete if every Cauchy sequence is convergent.

Definition 4. *(i) Let $x \in X$ and f is a self mapping then the set*

$O(f,x) = \{f^n(x) : n = 0,1,2,3,..\}$ is called the orbit of x and f.

(ii) Let $x \in X$ and f_1, f_2 are self mapping then the set $O(f_1, f_2, x) = \{x, f_1^i\left(f_2^j(x)\right) : j,i = 0,1,2,3,..\}$ is called the orbit of x, f_1 and f_2

(iii) Let $x \in X$ and f_1, f_2, f_3 are self mapping then the set $O(f_1, f_2, f_3, x) = \{x, f_1^j\left(f_2^i\left(f_3^k(x)\right)\right) : j,i,k = 0,1,2,3,..\}$ is called the orbit of x, f_1, f_2 and f_3

Now we want to present a new definition.

Definition 5. *A three self mapping f_1, f_2 and f_3 on a D^*-metric space (X,D^*), are said to be orbitally bounded if there exists $x_0 \in X$, and a three positive real numbers $\lambda_1, \lambda_2$and $\lambda_3$such that $D^*(a,b,f_i(x)) \leq \lambda_i D^*(a,b,x)$, for all $x \in O(f_1, f_2, f_3, x_0)$ and $\{a,b\} \neq \{x_0\}$.*

If $\max\{\lambda_1, \lambda_2, \lambda_3\} < 1$, then f_1, f_2 and f_3 are called orbitally contractive.

In the above definition we assume a,b not booth equal x_0, since if $a = b = x_0$, the above condition implies that x_0 is the common fixed point.

One of the interesting properties of D^*-metric space, are the following lemmas.

Lemma 1. *In D^*-metric space $D^*(x,y,y) = D^*(x,x,y)$.*

Lemma 2. *Let (X,D^*) be a D^*-metric space then $\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x,y,z)$ whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point (x,y,z) in X^3.That is $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$.*

Proof. Let $\{(x_n, y_n, z_n)\} \in X^3$, be such that $\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$,

$\lim_{n\to\infty} z_n = z$. Then for each $\varepsilon > 0$, there exists n_1, n_2 and $n_3 \in \mathbb{N}$, such that

$D^*(x,x,x_n) < \frac{\varepsilon}{2}$, for all $n \geq n_1, D^*(y,y,y_n) < \frac{\varepsilon}{2}$, for all $n \geq n_2$ and $D^*(z,z,z_n) < \frac{\varepsilon}{2}$, for all $n \geq n_3$. Let $n_0 = \max\{n_1, n_2, n_3\}$, then for every $n \geq n_0$ by triangle inequality we obtain,

$$\begin{aligned} D^*(x_n, y_n, z_n) &\leq D^*(x_n, y_n, z) + D^*(z, z_n, z_n) \\ &\leq D^*(x,y,z) + D^*(x, x_n, x_n) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &< D^*(x,y,z) + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = D^*(x,y,z) + \varepsilon. \end{aligned}$$

Then $D^*(x_n, y_n, z_n) - D^*(x,y,z) < \varepsilon$. Moreover,

$$\begin{aligned} D^*(x,y,z) &\leq D^*(x,y,z_n) + D^*(z_n, z, z) \\ &\leq D^*(z_n, y_n, z_n) + D^*(x_n, x, x) + D^*(y_n, y, y) + D^*(z_n, z, z). \\ &< D^*(x_n, y_n, z_n) + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = D^*(x_n, y_n, z_n) + \varepsilon. \end{aligned}$$

therefore, $D^*(x,y,z) - D^*(x_n, y_n, z_n) < \varepsilon$, which implies,

$|D^*(x_n, y_n, z_n) - D^*(x,y,z)| < \varepsilon$, therefore

$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x,y,z)$ □

One can easly obtained the following Lemmas.

Lemma 3. *Let (X, D^*) be a D^*-metric space, then a convergent sequence has a unique limit.*

Lemma 4. *Let (X, D^*) be a D^*-metric space. Then any convergent sequence in (X, D^*) is a Cauchy sequence .*

3. Main results

In this section, we will present several fixed point results on a complete D^*- metric space.

Theorem 1. *Let X be a complete D^*-metric space and $f_1, f_2 : X \to X$ be any two maps such that,*

$D^*(f_1(x), f_2(y), z) \leq \alpha D^*(x,y,z)$ for all $x, y \in O(f_1, f_2, x_0), z \in X$ and some $x_0 \in X$, and $0 \leq \alpha < \frac{1}{2}$, then f_1, f_2 have a unique common fixed point.

Proof. Let $x_0 \in X$ be any aribitrary element defined a sequence $\{x_n\}$ in X by,

$x_n = \begin{Bmatrix} f_1(x_{n-1}), & \text{if } n \text{ is odd} \\ f_2(x_{n-1}), & \text{if } n \text{ is even} \end{Bmatrix}$. Now we want to prove that $\{x_n\}$ Cauchy sequence, then we have the following cases.

Case 1. If n is odd, set $d_n = D^*(x_n, x_{n+1}, x_{n+1})$, so

$d_n = D^*(x_n, x_{n+1}, x_{n+1}) \ = D^*(f_1(x_{n-1}), f_2(x_n), x_{n+1})$

$\quad \leq \alpha D^*(x_{n-1}, x_n\ , x_{n+1}) \ \ \leq \alpha D^*(x_{n-1}, x_n\ , x_n\ ) + \alpha D^*(x_{n+1}, x_{n+1}, x_n\ ).$

Hence, $d_n \leq \alpha\ d_{n-1} + \alpha\ d_n$, which implies $d_n \leq \frac{\alpha}{1-\alpha} d_{n-1}$.

let $R = \frac{\alpha}{1-\alpha}$ so $R < 1$ since $\alpha < \frac{1}{2}$, and $d_n \leq R\ d_{n-1}$. Repeating this process, we obtained,

$d_n \leq R^n\ d_0$, i.e $d_n \to 0$ as $n \to \infty$.

To prove that $\{x_n\}$ Cauchy sequence, let $n, m \in \mathbb{N}$ be such that $n \leq m$, then

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x_{n+1}) + D^*(x_m, x_m, x_{n+1})$

$\qquad \leq \sum_{k=n}^{m-1} D^*(x_k, x_{k+1}, x_{k+1}) \leq \sum_{k=n}^{m-1} d_k$

$\qquad \leq \sum_{k=n}^{m-1} R^k\ d_0 \ \leq R^n\ d_0\ \left(\frac{1}{1-k}\right) \to 0$ as $n \to \infty$.

Case 2: If n is even, set $d_n = D^*(x_n, x_{n+1}, x_{n+1})$, so

$d_n = D^*(x_{n+1}, x_n, x_{n+1}) \ = D^*(f_1(x_n), f_2(x_{n-1}), x_{n+1})$

$\quad \leq \alpha\ D^*(x_n, x_{n-1}, x_{n+1}) \ \leq \alpha D^*(x_{n-1}, x_n, x_n) + \alpha D^*(x_{n+1}, x_{n+1}, x_n),$

which implies $d_n \leq \alpha\ d_{n-1} + \alpha\ d_n$.As in case 1, we obtain,

$D^*(x_n, x_n, x_m) \to 0$ as $n \to \infty$. So $\{x_n\}$ is Cauchy sequence but since X is complete D^*-metric space then x_n is converges to some element say x.

Now we want to prove that x is fixed point of f_1,suppose that $x \neq f_1(x)$, then $D^*(f_1(x), x, x) = \lim_{n\to\infty} D^*(f_1(x), f_1(x_{2n-1}), x) \leq \lim_{n\to\infty} \alpha D^*(x, x_{2n-1}, x) = 0,$

therefore$D^*(f_1(x), x, x) = 0$, so x is fixed point for f_1. Similarly x is fixed point for f_2.

For uniqueness, suppose there exists $y \in X$ such that, $f_1(y) = f_2(y) = y$.

This implies $D^*(x, y, y) = D^*(f_1(x), f_2(y), y) \leq \alpha D^*(x, y, y) < \frac{1}{2} D^*(x, y, y)$, which implies $x = y$. □

The next theorems give the same result for orbitally contractive maps under some certain conditions.

Theorem 2. *Let X be a complete D^*-metric space and $f_1, f_2, f_3 : X \to X$ be any three orbitally contractive maps that satisfies*

$D^*(f_1(x), f_1(x), f_2(y)) \leq D^*(f_1(x), f_2(y), f_3(z))$ and $D^*(f_1(x), f_2(y), f_3(z)) \leq \alpha D^*(x, y, z)$ for all $x, y, z \in X, \alpha \in (0, 1)$, then $f_1, f_2\ , f_3$ have a unique common fixed point.

Proof. Let X be a complete D^*-metric space and $f_{1,}f_2, f_3 : X \to X$ be any three orbitally contractive maps that satisfies, $D^*(f_1(x), f_1(x), f_2(y)) \leq D^*(f_1(x), f_2(y), f_3(z))$. Since $f_{1,}f_2, f_3$ are orbitally contractive maps, then there exists $x_0 \in X$, and a three positive numbers $\lambda_1, \lambda_2$and $\lambda_3$such that $D^*(a,b, f_i(x)) \leq \lambda_i D^*(a,b,x)$, for all $x \in O(f_1, f_2, f_3, x_0)$ and $\{a,b\} \neq \{x_0\}$, and $\max\{\lambda_1, \lambda_2, \lambda_3\} = \alpha < 1$.

Now we want to show

$D^*(f_1(x), f_2(y), f_3(z)) \leq \lambda_1\lambda_2\lambda_3 D^*(x,y,z)$, for all $x,y,z \in O(f_1, f_2, f_3, x_0)$.

Let $x,y,z \in O(f_1, f_2, f_3, x_0)$, if $x = y = z = x_0$, then $x_0 = f_1(x_0) = f_2(x_0) = f_3(x_0)$, then the required inequality holds.

Defined a sequence $\{x_n\}$ as follows,

$$x_n = \left\{ \begin{array}{l} f_1(x_{n-1}), n = 1,4,7,....3k+1 \\ f_2(x_{n-1}), n = 2,5,8,....3k+2 \\ f_1(x_{n-1}), n = 3,6,9,....3k+3 \end{array} \right\}.$$

Now we want to prove that x_n is Cauchy sequence, that is $D^*(x_n, x_n, x_m) \to 0$ as $n,m \to \infty$, with no loss of generality assume $m \geq n$.

since $D^*(x_n, x_n, x_m) = D^*(x_m, x_m, x_n)$, then we have the following cases.

Case1: $x_n = f_1(x_{n-1}), x_m = f_2(x_{m-1})$,

$$\begin{aligned} D^*(x_n, x_n, x_m) &= D^*(f_1(x_{n-1}), f_1(x_{n-1}), f_2(x_{m-1}) \\ &\leq D^*(f_1(x_{n-1}), f_2(x_{m-1}), f_3(x_{n-1})) \leq \alpha^n D^*(x_0, x_0, f_2(x_{m-n})) \\ &\leq \alpha^n \lambda_2 D^*(x_0, x_0, x_{m-n}) \to 0 \text{ as } m,n \to \infty. \end{aligned}$$

Case 2: $x_n = f_1(x_{n-1}), x_m = f_3(x_{m-1})$,

$$\begin{aligned} D^*(x_n, x_n, x_m) &= D^*(f_1(x_{n-1}), f_1(x_{n-1}), f_2(x_{m-1}) \\ &\leq D^*(f_1(x_{n-1}), f_2(x_{m-1}), f_3(x_{n-1})) \quad \leq \alpha^n D^*(x_0, x_0, f_2(x_{m-n})) \\ &\leq \alpha^n \lambda_2 D^*(x_0, x_0, x_{m-n}) \to 0 \text{ as } m,n \to \infty. \end{aligned}$$

Case 3: $x_n = f_2(x_{n-1}), x_m = f_3(x_{m-1})$,

$$\begin{aligned} D^*(x_n, x_n, x_m) &= D^*(f_2(x_{n-1}), f_2(x_{n-1}), f_3(x_{m-1}) \\ &\leq D^*(f_1(x_{n-1}), f_2(x_{n-1}), f_3(x_{m-1})) \leq \alpha^n D^*(x_0, x_0, f_3(x_{m-n})) \\ &\leq \alpha^n \lambda_3 D^*(x_0, x_0, x_{m-n}) \to 0 \text{ as } m,n \to \infty. \end{aligned}$$

So $\{x_n\}$ is Cauchy sequence in the complete D^*-metric space X, then there exist $x \in X$, such that x_n is converges to x.

Now we want to prove that x is fixed point of f_1 and by a similar idea we can show that x is fixed point of f_2 and f_3.

x is a fixed point for f_1.

$$\begin{aligned}D^*(f_1(x),x,x) &= \lim_{n\to\infty} D^*(f_1(x),x_{3n+2},x_{3n+3})\\ &= \lim_{n\to\infty} D^*(f_1(x),f_2(x_{3n+1}),f_3(x_{3n+2}))\\ &\le \alpha \lim_{n\to\infty} D^*(x,x_{3n+1},x_{3n+2})\\ &= \alpha D^*(x,x,x).\text{so } f_1(x)=x.\end{aligned}$$

Now we want to prove uniqueness of the fixed point x.Assume there exist another common fixed point y, that is $f_1(y)=f_2(y)=f_3(y)=y$. Now $D^*(x,y,y)=D^*(f_1(x),f_2(y),f_3(y)) \le \alpha D^*(x,y,y)$

Since $\alpha \in (0,1)$, then $D^*(x,y,y)=0$, hence $y=x$. □

Theorem 3. *Let X be a complete D^*-metric space and $g,f : X \to X$ be any two maps such that*

$D^*(gf(x),f(x),y) \le \alpha D^*(f(x),x,y)$, *for all* $x,y \in O(f,g,x_0)$, $x_0 \in X$ *and* $0 \le \alpha < \frac{1}{2}$, *then g,f have a unique common fixed point.*

Proof. Defined a sequence $\{x_n\}$ in X as follow,

$$x_n = \begin{Bmatrix} f(x_{n-1}), & \text{if } n \text{ is dd} \\ g(x_{n-1}), & \text{if } n \text{ is even}\end{Bmatrix},$$

Set $d_n = D^*(x_n, x_{n+1}, x_{n+1})$ for all $n = 0,1,2,...,$

$$\begin{aligned}\text{now } d_1 = D^*(x_1,x_2,x_2) &= D^*(f(x_0),gf(x_0),x_2)\\ &\le \alpha D^*(x_0,f(x_0),x_2) = \alpha D^*(x_0,x_1,x_2)\\ &\le \alpha D^*(x_0,x_1,x_1)+\alpha D^*(x_1,x_2,x_2) = \alpha d_0 + \alpha d_1, \text{ therefor}\end{aligned}$$

$d_1 \le \beta d_0$, where $\beta = \frac{\alpha}{1-\alpha} < 1$. Also,

$$\begin{aligned}d_2 &= D^*(x_2,x_3,x_3) = D^*(gf(x_0),f(x_2),x_3)\\ &\le \alpha D^*(f(x_0),x_2,x_3) = \alpha D^*(x_1,x_2,x_3)\\ &\le \alpha D^*(x_1,x_2,x_2)+\alpha D^*(x_2,x_3,x_3)\\ &= \alpha d_1 + \alpha d_2,\end{aligned}$$

therefor $d_2 \le \beta d_1$. By a similar arguments we have,

$d_n \le \beta d_{n-1} \le \beta^n d_0 \to 0$ as $n \to \infty$.

Now we shall prove that $\{x_n\}$ is Cauchy sequence in X.

Let $n,m > n_0$, for some $n_0 \in \mathbb{N}$, then

$D^*(x_n,x_n,x_m) \le \sum_{k=n}^{m-1} D^*(x_k,x_{k+1},x_{k+1})$

$= \sum_{k=n}^{m-1} \beta^n \frac{1}{1-\beta} d_0 \to 0$ as $n, m \to \infty$, therefore $\{x_n\}$ is a D^*-Cauchy sequence in a complete D^*-metric space X, hence converges to some $x \in X$. Since,

$$D^*(f(x),x,x) = \lim_{n\to\infty} D^*(f(x),x_{2n},x) = \lim_{n\to\infty} D^*(f(x),g(x_{2n-1}),x)$$
$$= \lim_{n\to\infty} D^*(f(x),gf(x_{2n-2}),x) \le \alpha \lim_{n\to\infty} D^*(f(x),x_{2n-2},x)$$
$$= \alpha \lim_{n\to\infty} D^*(f(x),x_{2n-2},x) = \alpha \lim_{n\to\infty} D^*(f(x),x,x)$$

therefore $D^*(gf(x), f(x), x) = D^*(f(x),x,x) = 0$, which imlies $g(x) = f(x) = x$.

To prove uniqueness, if possible suppose ther exist $x \neq y$, such that $g(x) = f(x) = x$ and $g(y) = f(y) = y$,

then $D^*(x,x,y) = D^*(gf(x), f(x), y) \le \alpha D^*(f(x),x,y) = \alpha D^*(x,x,y)$.

This implies $\alpha \ge 1$. $x = y$. □

Theorem 4. *Let X be a complete D^*-metric space and $f,g,h : X \to X$ be any three maps satisfy the following conditions*

1) There exists, $x_0 \in X$, and $0 < \lambda < 1$,such that $D^*(a,b,f(x)) \le \lambda D^*(a,b,x)$, for all $a,b,x \in O(f_1, f_2, f_3, x_0)$ and $\{a,b,x\} \neq \{x_0\}$

2) $D^*(hgf(x), gf(x), f(x)) \le \alpha D^*(gf(x), f(x), x)$ for all $x \in X$ and $0 < \alpha < 1, \lambda < 1$.

3) For all $x,y,z \in O(f,g,h,x_0)$, we have

$D^*(f(x), f(x), g(y)) \le D^*(f(x), g(y), h(z))$,

$D^*(f(x), f(x), h(z)) \le D^*(f(x), g(y), h(z))$

and $D^*(g(y), g(y), h(z)) \le D^*(f(x), g(y), h(z))$.

Then f,g,h have a unique common fixed point.

Proof. Let X be a complete D^*-metric space and $f,g,h : X \to X$ be three maps satisfy the above conditions. By (2), there exists, $x_0 \in X$, and a positive real number λ, such that $D^*(a,b,f_1(x)) \le \lambda D^*(a,b,x)$, for all $x \in O(f,g,h,x_0)$ and $\{a,b,x\} \neq \{x_0\}$. Define a sequence $\{x_n\}$ in X as follow

$$x_n = \left\{ \begin{array}{ll} f(x_{n-1}), & n = 1,4,7,\ldots..3k+1 \\ g(x_{n-1}), & n = 2,5,8,,\ldots 3k+2 \\ h(x_{n-1}), & n = 3,6,9\ldots\ldots 3k+3 \end{array} \right\}$$

To prove $\{x_n\}$ is Cauchy sequence, with no loss of generality assume $m \ge n$

Case 1: $x_n = f(x_{n-1}), x_m = g(x_{m-1})$

$$
\begin{aligned}
D^* (x_n,x_n,x_m) &= D^* ((f(x_{n-1}),f(x_{n-1}),g(x_{m-1}) \\
&\leq D^*(h(x_{n-1}),g(x_{m-1}),f(x_{n-1})) \\
&= D^* (hgf(x_{n-3}),gf(x_{m-2}),f(x_{n-1})) \\
&\leq \alpha D^* (gf(x_{n-3}),f(x_{m-2}),(x_{n-1})) \\
&= \alpha D^* (x_{n-1},x_{m-1},x_{n-1}) \leq ... \\
&\leq \alpha^n D^*(x_0,x_0,f(x_{m-n-1})) \\
&\leq \alpha^n (\lambda)^{m-n} D^*(x_0,x_0,f(x_0)) \to 0 \text{ as } n,m \to \infty
\end{aligned}
$$

Case 2: $x_n = g(x_{n-1}), x_m = h(x_{m-1})$,

$$
\begin{aligned}
D^* (x_n,x_n,x_m) &= D^* (g(x_{n-1}),g(x_{n-1}),h(x_{m-1}) \\
&\leq D^*(h(x_{m-1}),g(x_{n-1}),f(x_{n-1})) \\
&= D^* (hgf(x_{m-3}),gf(x_{n-2}),f(x_{n-1})) \\
&\leq \alpha D^* (gf(x_{m-3}),f(x_{n-2}),(x_{n-1})) \\
&= \alpha D^* (x_{n-1},x_{m-1},x_{n-1}) \leq ... \\
&\leq \alpha^n D^*(x_0,f(x_{m-n-1}),x_0) \\
&\leq \alpha^n (\lambda)^{m-n} D^*(x_0,x_0,f(x_0)) \to 0 \text{ as } n,m \to \infty
\end{aligned}
$$

Cas 3: $x_n = f(x_{n-1}), x_m = h(x_{m-1})$,

$$
\begin{aligned}
D^* (x_n,x_n,x_m) &= D^* ((f(x_{n-1}),f(x_{n-1}),h(x_{m-1}) \\
&\leq D^*(h(x_{m-1}),g(x_{n-1}),f(x_{n-1})) \\
&= D^* (hgf(x_{m-3}),gf(x_{n-2}),f(x_{n-1})) \\
&\leq \alpha D^* (gf(x_{m-3}),f(x_{n-2}),(x_{n-1})) \\
&= \alpha D^*(x_{n-1},x_{m-1},x_{n-1}) \leq ... \\
&\leq \alpha^n D^*(x_0,f(x_{m-n-1}),x_0) \\
&\leq \alpha^n (\lambda)^{m-n} D^*(x_0,x_0,f(x_0)) \to 0 \text{ as } n,m \to \infty.
\end{aligned}
$$

So $\{x_n\}$ converges to some element say $x \in X$.

Now we want to prove that x is fixed point of f,g,h.

$$
\begin{aligned}
(i)\ D^*(f(x),x,x) &= \lim_{n\to\infty} D^*(f(x),x_{3n-3},x_{3n-1}) \\
&= \lim_{n\to\infty} D^*(hgf(x_{3n-5}),gf(x_{3n-2}),f(x)) \\
&\leq \alpha \lim_{n\to\infty} D^*(x_{3n-3},x_{3n-1},x) = \alpha D^*(x,x,x)
\end{aligned}
$$

$(ii)\ D^*(x,g(x),x) = lim_{n\to\infty} D^* (f(x),g(x),x_{3n-3})$

$= lim_{n\to\infty} D^*(hgf(x_{3n-5}), gf(x), f(x))$

$\leq \alpha lim_{n\to\infty} D^*(gf(x_{3n-5}), f(x), x)$

$= \alpha lim_{n\to\infty} D^*(x_{3n-5}, x, x))D^*(x, x, x))$

(iii) $D^*(x, x, h(x)) = D^*(f(x), gf(x), hgf(x))$

$\leq \alpha D^*(x, f(x), fg(x))$

$= \alpha D^*(x, x, x)$

If possible suppose there exist $x, y \in X$, $x \neq y$ and $f(x) = g(x) = h(x) = x$ and $f(y) = g(y) = h(y) = y$.

then $D^*(x, y, y) = D^*(f(x), gf(y), hgf(y))$

$\leq \alpha D^*(x, y, y)$

This implies $\alpha \geq 1$, which is contradiction, hence h, g and f have a unique fixed point. □

Now we will prove the next theorem

Theorem 5. *Let X be a complete metric space D^*-metric space $f_1, f_2, f_3 : X \to X$ be any three maps satisfy the following conditions*

1)D^* $(f_1(x), f_2(y), f_3(z)) \leq \alpha\{D^*(x, y, z) + D^*(x, f_1(x), f_2(y)) + D^*(y, f_2(y), f_3(z))$ *For all* $x, y, z \in X$ *and* $0 \leq \alpha < 1/3$

2) f_1, f_2, f_3 *are orbitally contractive.*

3) $D^*(f_1(x), f_1(x), f_2(y)) \leq D^*(f_1(x), f_2(y), f_3(z))$,

$D^*(f_1(x), f_1(x), f_3(z)) \leq D^*$ $(f_1(x), f_2(y), f_3(z))$

and $D^*(f_2(y), f_2(y), f_3(z)) \leq D^*$ $(f_1(x), f_2(y), f_3(z))$ *for all* $x, y, z \in X$

then f_1, f_2 *and* f_3 *have a unique common fixed point*

Proof. Since f_1, f_2, f_3 are orbitally contractive, there exists $x_0 \in X$, and a three positive real numbers $\lambda_1, \lambda_2$and $\lambda_3$such that $D^*(a, b, f_i(x)) \leq \lambda_i D^*(a, b, x)$, for all $x \in O(f_1, f_2, f_3, x_0)$ and $\{a, b, x\} \neq \{x_0\}$. Defined a sequence x_n by,

$$x_n = \left\{ \begin{array}{c} f_1(x_{n-1}), n = 1, 4, \ldots .3k-2 \\ f_2(x_{n-1}), n = 2, 5, \ldots .3k-1 \\ f_3(x_{n-1}), n = 3, 6, \ldots .3k \end{array} \right\}$$

To prove that $\{x_n\}$ is Cauchy sequence, assume $m \geq n$.

Case1: $x_n = f_1(x_{n-1})$,$x_m = f_2(x_{m-1})$.

$$\begin{aligned}
D^*(x_n,x_n,x_m) &= D^*(f_1(x_{n-1}),f_1(x_{n-1}),f_2(x_{m-1}))\\
&\leq D^*(f_1(x_{n-1}),f_2(x_{m-1}),f_3(x_{n-1}))\\
&\leq \alpha\{D^*(x_{n-1},x_{n-1},x_{m-1})+D^*(x_{n-1},f_1(x_{n-1}),f_2(x_{m-1}))\\
&+D^*(x_{m-1},f_2(x_{m-1}),f_3(x_{n-1}))\}\\
&\leq \alpha\{D^*(x_{n-1},x_{n-1},x_{m-1})+\lambda_1 D^*(x_{n-1},x_{n-1},f_2(x_{m-1}))\\
&+\lambda_2 D^*(x_{m-1},x_{m-1},f_3(x_{n-1}))\}\\
&\leq \alpha\{D^*(x_{n-1},x_{n-1},x_{m-1})+\lambda_1\lambda_2 D^*(x_{n-1},x_{n-1},x_{m-1})\\
&+\lambda_2\lambda_3 D^*(x_{m-1},x_{m-1},x_{n-1})\}\\
&= \beta D^*(x_{n-1},x_{n-1},x_{m-1}),\text{ were } \beta=\alpha\{1+\lambda_1\lambda_2+\lambda_2\lambda_3\}<1.
\end{aligned}$$

Continuo this way, we obtain,

$$\begin{aligned}
D^*(x_n,x_n,x_m) &\leq \beta\ D^*(x_{n-1},x_{n-1},x_{m-1}) \leq \beta^2 D^*(x_{n-2},x_{n-2},x_{m-2})\\
&\leq \beta^n D^*(x_0,x_0,x_{m-n}) \leq \beta^n\,(\lambda_3)^{m-n} D^*(x_0,x_0,f(x_0)) \to 0 \text{ as } n,m\to\infty
\end{aligned}$$

Case2: $x_n = f_1(x_{n-1})$,$x_m = f_3(x_{m-1})$.

$$\begin{aligned}
D^*(x_n,x_n,x_m) &= D^*(f_1(x_{n-1}),f_1(x_{n-1}),f_3(x_{m-1}))\\
&\leq D^*(f_1(x_{n-1}),f_2(x_{n-1}),f_3(x_{m-1}))\\
&\leq \alpha\{D^*(x_{n-1},x_{n-1},x_{m-1})+D^*(x_{n-1},f_1(x_{n-1}),f_2(x_{n-1}))\\
&+D^*(x_{n-1},f_2(x_{n-1}),f_3(x_{m-1}))\}\\
&\leq \alpha\{D^*(x_{n-1},x_{n-1},x_{m-1})+\lambda_1 D^*(x_{n-1},x_{n-1},f_2(x_{n-1}))\\
&+\lambda_2 D^*(x_{m-1},x_{n-1},f_3(x_{m-1}))\}\\
&\leq \alpha\{D^*(x_{n-1},x_{n-1},x_{m-1})+\lambda_1\lambda_2 D^*(x_{n-1},x_{n-1},x_{n-1})\\
&+\lambda_2\lambda_3 D^*(x_{m-1},x_{n-1},x_{m-1})\}\\
&= \beta D^*(x_{n-1},x_{n-1},x_{m-1}),\text{ were } \beta=\alpha\{1+\lambda_1\lambda_2+\lambda_2\lambda_3\}<1.
\end{aligned}$$

Continuo this way, as in case1, we obtain,

$$D^*(x_n,x_n,x_m) \leq \beta^n\,(\lambda_3)^{m-n} D^*(x_0,x_0,f(x_0)) \to 0 \text{ as } n,m\to\infty$$

Case3: $x_n = f_2(x_{n-1}), x_m = f_3(x_{m-1})$ is similar to the other cases.

from the above cases, x converges to some point $x\in X$.

Now we want to prove that x is fixed point for f_1.

$$\begin{aligned}
D^*(f_1(x),x,x) &= \lim_{n\to\infty} D^*(f_1(x),x_{(3n-1)},x_{3n})\\
&= \lim_{n\to\infty} D^*(f_1(x),f_2(x_{3n-2}),f_3(x_{3n-1}))
\end{aligned}$$

$\leq \lim_{n\to\infty} \alpha\{D^*(x, x_{3n-2}, x_{3n-1}) + D^*(x, f_1(x), f_2\ (x_{3n-2}))$

$+D^*\ (x_{(3n-2)}, f_2(x_{3n-2}), f_3(x_{3n-1})\}$

$\leq \lim_{n\to\infty} \alpha\{D^*(x, x_{3n-2}, x_{3n-1}) + D^*(x, f_1(x), x_{3n-1})$

$+D^*\ (x_{3n-2}, x_{3n-1}, x_{3n}\}$

$\leq \alpha\ D^*\ (x, f_1(x), x)$

This implies $D^*\ (x, f_1(x), x) = 0$. So x is fixed point for f_1, similarly for f_2, f_3.

Now we want to prove that x is a unique common fixed point of f_1, f_2, f_3.

Let $y \neq x$ be such that

$f_1(x) = f_2(x) = f_3(x) = x$ and $f_1(y) = f_2(y) = f_3(y) = y$,

then $D^*(x,y,y) = D^*(f_1(x), f_2(y), f_3(y)) \leq \alpha\,(D^*(x,y,y) + D^*(x,x,y) + D^*(y,y,y))$
$= 2\alpha\{D^*(x,y,y)\}$, which is a contradiction, unless $D^*(x,y,y) = 0$, which implies that f_1, f_2, f_3have a unique common fixed point. □

Using the same procedures we can prove the following.

Theorem 6. *Let X be a complete D^*-metric space and $f_1, f_2, f_3 : X \to X$ be any three maps satisfy the following conditions*

1)$D^*(f_1(x), f_2(y), f_3(z)) \leq \alpha_1 D^*(x,y,z) + \alpha_2\{D^*(x, f_1(x), f_2(y)) + D^*(y, f_2(y), f_3(z))\}$
$+\alpha_3\{D^*(x,y,f_2(y)) + D^*(y,z,f_3(z))\}$

for all $x,y,z \in X$ and $0 \leq \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$

2) f_1, f_2, f_3 are orbitally contractive .

3) $D^*(f_1(x), f_1(x), f_2(y)) \leq D^*(f_1(x), f_2(y), f_3(z))$,

$D^*(f_1(x), f_1(x), f_3(z)) \leq D^*\ (f_1(x), f_2(y), f_3(z))$

and $D^*(f_2(y), f_2(y), f_3(z)) \leq D^*\ (f_1(x), f_2(y), f_3(z))$ for all $x,y,z \in X$

then f_1, f_2 and f_3 have a unique common fixed point

Proof. Since f_1, f_2, f_3 are orbitally contractive, there exists $x_0 \in X$, and a three positive real numbers $\lambda_1, \lambda_2$and $\lambda_3$such that $D^*(a,b,f_i(x)) \leq \lambda_i D^*(a,b,x)$, for all $x \in O(f_1, f_2, f_3, x_0\ )$ and $\{a,b,x\} \neq \{x_0\ \}$. Defined a sequence x_n in X as,

$$x_n = \left\{ \begin{array}{c} f_1(x_{n-1}), n = 1,4,\ldots .3k-2 \\ f_2(x_{n-1}), n = 2,5,\ldots .3k-1 \\ f_3(x_{n-1}), n = 3,6,\ldots .3k \end{array} \right\}$$

To prove that $\{x_n\}$ is Cauchy sequence suppose $m \geq n$, as in the prove of Theorem (5), we shall prove the case,

$x_n = f_1(x_{n-1})$,$x_m = f_2(x_{m-1})$. Now, $D^*(x_n,x_n,x_m) = D^*(f_1(x_{n-1}), f_1(x_{n-1}), f_2(x_{m-1})$

$$= D^*(f_1(x_{n-1}), f_2(x_{m-1}), f_3(x_{n-1}))$$

$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1}) + \alpha_2 [D^*(x_{n-1}, f_1(x_{n-1}), f_2(x_{m-1})) + D^*(x_{m-1}, f_2(x_{m-1}), f_3(x_{n-1}))]$$

$$+\alpha_3 [D^*(x_{n-1},x_{m-1}, f_2(x_{m-1})) + D^*(x_{m-1},x_{n-1}, f_3(x_{n-1}))]$$

$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1}) + \alpha_2 [\lambda_1 D^*(x_{n-1},x_{n-1}, f_2(x_{m-1})) + \lambda_2 D^*(x_{m-1},x_{m-1}, f_3(x_{n-1}))]$$

$$+\alpha_3 [\lambda_2 D^*(x_{n-1},x_{m-1},x_{m-1}) + \lambda_3 D^*(x_{m-1},x_{n-1},x_{n-1})\}]$$

$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1})$$

$$+\alpha_2 [\lambda_1 \lambda_2 D^*(x_{n-1},x_{n-1},x_{m-1}) + \lambda_2 \lambda_3 D^*(x_{m-1},x_{m-1},x_{n-1})]$$

$$+\alpha_3 [\lambda_2 D^*(x_{n-1},x_{m-1},x_{m-1}) + \lambda_3 D^*(x_{m-1},x_{n-1},x_{n-1})\}]$$

$$\leq (\alpha_1 + \alpha_2(\lambda_1\lambda_2 + \lambda_2\lambda_3) + \alpha_3(\lambda_2 + \lambda_3))D^*(x_{n-1},x_{m-1},x_{n-1})$$

$\leq \beta\ D^*(x_{n-1},x_{m-1},x_{n-1})$ where $\beta = \alpha_1 + 2\alpha_2 + 2\alpha_3$ and $0 \leq \beta < 1$.

Continuo this way we obtained,

$$D^*(x_n,x_n,x_m) \ \leq \beta\ D^*(x_{n-2},x_{m-2},x_{n-2}) \leq \ldots$$

$\leq \beta^n D^*(x_0,x_{m-n},x_0) = \beta^n (\max\{\lambda_1,\lambda_2,\lambda_3\})^{m-n-1} D^*(x_0,x_1,x_0) \to 0$ as $n,m \to \infty$.

So $\{x_n\}$ is Cauchy hence, converges to some point say $x \in X$.

Now we want to prove that x is fixed point for f_1,

$$D^*(f_1(x),x,x) = lim_{n\to\infty}\ D^*\ (f_1(x),x_{3n-1},x_{3n})$$

$$= lim_{n\to\infty}\ D^*\ (f_1(x), f_2(x_{3n-2}), f_3\ (x_{3n-1}))$$

$$\leq lim_{n\to\infty} \alpha_1 D^*(x,x_{3n-2},x_{3n-1})$$

$$+\alpha_2 [D^*(x, f_1(x), f_2(x_{3n-2})) + D^*(x_{3n-2}, f_2(x_{3n-2}), f_3(x_{3n-1}))]$$

$$+ \alpha_3 [D^*(x,x_{3n-2}, f_2(x_{3n-2})) + D^*(x_{3n-2},x_{3n-1}, f_3(x_{3n-1}))]$$

$$= lim_{n\to\infty}(\alpha_1 D^*(x,x_{3n-2},x_{3n-1})$$

$$+\alpha_2 \left[D^*(x, f_1(x), x_{3n-1})) + D^*(x_{3n-2}, x_{3n-1}, x_{3n})\right]$$
$$+\alpha_3 \left[D^*(x, x_{3n-2}, x_{3n-1}) + D^*(x_{3n-2}, x_{3n-1}, x_{3n})\right])$$
$$= \alpha_2 D^*(x, f_1(x), x).$$

Therefore $D^*(x, f_1(x), x) = 0$, So x fixed point for f_1.

Similarly for f_2, f_3

For the uniqueness, suppose there exist $y \neq x$, such that x, y are common fixed points for f_1, f_2, f_3, then

$$D^*(x,y,y) = D^*(\ f_1(x), f_2(y), f_3(y))$$
$$\leq \alpha_1 D^*(x,y,y) + \alpha_2\{D^*(x, f_1(x), f_2(y))$$
$$+D^*(y, f_2(y), f_3(y))\} + \alpha_3\{D^*(x,y, f_2(y)) + D^*(y,y, f_3(y))\}$$
$$= (\alpha_1+\alpha_2+\alpha_3) D^*(x,y,y) < D^*(x,y,y)$$

□

Theorem 7. *let X be a complete D^*-metric space and $f_1, f_2, f_3 : X \to X$ be any three maps satisfies the followings*

1) $D^*(f_1(x), f_2(y), f_3(z))$

$\leq \alpha \max\{D^*(x,y,z), D^*(x, f_1(x), f_2(y)), D^*(y, f_2(y), f_3(z)), D^*(x,y, f_2(y)), D^*(y,z, f_3(z))\}$

for all $x,y,z \in X$, $0 \leq \alpha < 1$

2) f_1, f_2, f_3 *are orbitally contractive orbitally.*

3) $D^*(f_1(x), f_1(x), f_2(y)) \leq D^*(f_1(x), f_2(y), f_3(z))$,

$D^*(f_1(x), f_1(x), f_3(z)) \leq D^*\ (f_1(x), f_2(y), f_3(z))$

and $D^*(f_2(y), f_2(y), f_3(z)) \leq D^*\ (f_1(x), f_2(y), f_3(z))$ *for all* $x,y,z \in X$

then f_1, f_2 *and* f_3 *have a unique common fixed point*

Proof. Since f_1, f_2, f_3 are orbitally contractive, there exists $x_0 \in X$, and a three positive real numbers $\lambda_1, \lambda_2$and $\lambda_3$such that $D^*(a,b, f_i(x)) \leq \lambda_i D^*(a,b,x)$, for all $x \in O(f_1, f_2, f_3, x_0\ )$ and $\{a,b,x\} \neq \{x_0\ \}$. Defined a sequence x_n in X as,

$$x_n = \left\{ \begin{array}{c} f_1(x_{n-1}), n = 1,4,\ldots.3k-2 \\ f_2(x_{n-1}), n = 2,5,\ldots.3k-1 \\ f_3(x_{n-1}), n = 3,6,\ldots.3k \end{array} \right\}$$

To prove that $\{x_n\}$ is Cauchy sequence suppose $m \geq n$,as in the prove of Theorem (5), we shall prove the case,

$x_n = f_1(x_{n-1})$, $x_m = f_2(x_{m-1})$. Now,

with no loss of generality suppose $m \geq n$

$$D^*(x_n,x_n,x_m) = D^*(f_1(x_{n-1}), f_1(x_{n-1}), f_2(x_{m-1})$$
$$\leq D^*(f_1(x_{n-1}), f_2(x_{m-1}), f_3(x_{n-1}))$$
$$\leq \alpha \max\{D^*(x_{n-1},x_{m-1},x_{n-1}), D^*(x_{n-1}, f_1(x_{n-1}), f_2(x_{m-1})),$$
$$D^*(x_{m-1}, f_2(x_{m-1}), f_3(x_{n-1})), D^*(x_{n-1},x_{m-1}, f_2(x_{m-1})), D^*(x_{m-1},x_{n-1}, f_3(x_{n-1}))\}$$
$$\leq \alpha \max\{D^*(x_{n-1},x_{m-1},x_{n-1}), \lambda_1 D^*(x_{n-1},x_{n-1}, f_2(x_{m-1})), \lambda_2 D^*(x_{m-1},x_{m-1}, f_3(x_{n-1})),$$
$$D^*(x_{n-1},x_{m-1}, f_2(x_{m-1})), D^*(x_{m-1},x_{n-1}, f_3(x_{n-1}))\}$$
$$\leq \alpha \max\{D^*(x_{n-1},x_{m-1},x_{n-1}), \lambda_1\lambda_2 D^*(x_{n-1},x_{n-1},x_{m-1}), \lambda_2\lambda_3 D^*(x_{m-1},x_{m-1},x_{n-1}),$$
$$\lambda_2 D^*(x_{n-1},x_{m-1},x_{m-1}), \lambda_3 D^*(x_{m-1},x_{n-1},x_{n-1})\}$$
$$\leq \alpha D^*(x_{n-1},x_{m-1},x_{n-1})$$

continuo this way we obtained,

$$D^*(x_n,x_n,x_m) \leq \alpha^2 D^*(x_{n-2},x_{m-2},x_{n-2}) \leq \ldots.$$
$$\leq \alpha^n D^*(x_0,x_{m-n},x_0)$$
$$= \alpha^n (\max\{\lambda_1,\lambda_2,\lambda_3\})^{m-n-1} D^*(x_0,x_1,x_0) \to 0 \text{ as } n,m \to \infty.$$

So $\{x_n\}$ is Cauchy hence, converges to some point say $x \in X$.

Now we want to prove that x is fixed point for f_1,

Suppose that $f_1(x) \neq x$

$$D^*(f_1(x),x,x) = lim_{n\to\infty} D^* \ (f_1(x),x_{3n-1},x_{3n})$$
$$= lim_{n\to\infty} D^* \ (f_1(x), f_2(x_{3n-2}), f_3 \ (x_{3n-1}))$$
$$\leq lim_{n\to\infty} \alpha \max\{D^*(x,x_{3n-2},x_{3n-1}), D^*(x, f_1(x), f_2(x_{3n-2})), D^*(x_{3n-2}, f_2(x_{3n-2}), f_3(x_{3n-1})),$$
$$D^*(x_{n-1},x_{m-1}, f_2(x_{3n-2})), D^*(x_{3n-1},x_{3n-1}, f_3(x_{3n-1}))\}$$
$$\leq \max lim_{n\to\infty} \alpha\{D^*(x,x_{3n-2},x_{3n-1}), D^*(x, f_1(x), f_2(x_{3n-2})),$$
$$D^*(x_{3n-2}, f_2(x_{3n-2}), f_3(x_{3n-1})), D^*(x_{n-1},x_{m-1}, f_2(x_{3n-2})), D^*(x_{3n-1},x_{3n-1}, f_3(x_{3n-1}))\}$$
$$= \max lim_{n\to\infty} \alpha\{D^*(x,x_{3n-2},x_{3n-1}), D^*(x, f_1(x),x_{3n-1}), D^*(x_{3n-2},x_{3n-1},x_{3n}),$$
$$D^*(x_{n-1},x_{3n-2},x_{3n-1}), D^*(x_{3n-1},x_{3n-1},x_{3n-})\}$$
$$= \max \alpha\{D^*(x,x,x), D^*(x, f_1(x),x), D^*(x,x,x), D^*(x,x,x, D^*(x,x,x)\}$$
$$= \alpha \max\{D^*(x, f_1(x),x), 0\}$$

$= \alpha D^*(x, f_1(x), x)$

$< D^*(x, f_1(x), x)$

Contradiction

Therefore $D^*(x, f_1(x), x) = 0$, So x fixed point for f_1.

Similarly for f_2, f_3

For the uniqueness, suppose there exist $y \neq x$, such that x, y are common fixed points for f_1, f_2, f_3, then

then $D^*(x,y,y) = D^*(f_1(x), f_2(y), f_3(y))$

$\leq \alpha \max\{D^*(x,y,y), D^*(x, f_1(x), f_2(y)), D^*(y, f_2(y), f_3(y)), D^*(x,y,f_2(y)), D^*(y,y,f_3(y))\}$

$\leq \alpha \max\{D^*(x,y,y), D^*(x,x,y), D^*(y,y,y), D^*(x,y,y), D^*(y,y,y)\}$

$= \alpha D^*(x,y,y) < D^*(x,y,y)$

This is a contradiction

Hence f_1, f_2 and f_3 have a unique common fixed point. □

Theorem 8. *let X be a complete D^*-metric space and $f_1, f_2, f_3 : X \to X$ be any three maps satisfies the followings*

1)$D^*(f_1(x), f_2(y), f_3(z)) \leq \alpha_1 D^*(x,y,z) + \alpha_2 \max\{D^*(x, f_1(x), f_2(y)), D^*(y, f_2(y), f_3(z))\}$

for all $x, y, z \in X$, $0 \leq \alpha_1 + 2\alpha_2 < 1$

2) f_1, f_2, f_3 are orbitally contractive.

3)$D^*(f_1(x), f_1(x), f_2(y)) \leq D^*(f_1(x), f_2(y), f_3(z))$,

$D^*(f_1(x), f_1(x), f_3(z)) \leq D^*\ (f_1(x), f_2(y), f_3(z))$

and $D^*(f_2(y), f_2(y), f_3(z)) \leq D^*\ (f_1(x), f_2(y), f_3(z))$ for all $x, y, z \in X$.

then f_1, f_2 ,$f_3$have a unique common fixed point.

Proof. Since f_1, f_2, f_3 are orbitally contractive, there exists $x_0 \in X$, and a three positive real numbers $\lambda_1, \lambda_2$and $\lambda_3$such that $D^*(a,b,f_i(x)) \leq \lambda_i D^*(a,b,x)$, for all $x \in O(f_1, f_2, f_3, x_0)$ and $\{a,b,x\} \neq \{x_0\}$. Defined a sequence x_n in X as,

$$x_n = \left\{ \begin{array}{c} f_1(x_{n-1}), n = 1, 4, \ldots .3k-2 \\ f_2(x_{n-1}), n = 2, 5, \ldots .3k-1 \\ f_3(x_{n-1}), n = 3, 6, \ldots .3k \end{array} \right\}$$

To prove that $\{x_n\}$ is Cauchy sequence suppose $m \geq n$, as in the prove of Theorem (5), we shall prove the case,

with no loss of generality suppose $m \geq n$

$$D^*(x_n,x_n,x_m) = D^*(f_1(x_{n-1}),f_1(x_{n-1}),f_2(x_{m-1})$$
$$\leq D^*(f_1(x_{n-1}),f_2(x_{m-1}),f_3(x_{n-1}))$$
$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1})$$
$$+\alpha_2 \max\{D^*(x_{n-1},f_1(x_{n-1}),f_2(x_{m-1})),D^*(x_{m-1},f_2(x_{m-1}),f_3(x_{n-1}))\}$$
$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1})$$
$$+\alpha_2 \max\{\lambda_1 D^*(x_{n-1},x_{n-1},f_2(x_{m-1})),\lambda_2 D^*(x_{m-1},x_{m-1},f_3(x_{n-1}))\}$$
$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1})$$
$$+\alpha_2 \max\{\lambda_1\lambda_2 D^*(x_{n-1},x_{n-1},x_{m-1}),\lambda_2\lambda_3 D^*(x_{m-1},x_{m-1},x_{n-1}))\}$$
$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1})$$
$$+\alpha_2\{\lambda_1\lambda_2 D^*(x_{n-1},x_{n-1},x_{m-1})+\lambda_2\lambda_3 D^*(x_{m-1},x_{m-1},x_{n-1}))\}$$
$$\leq \alpha_1 D^*(x_{n-1},x_{m-1},x_{n-1})+2\alpha_2 D^*(x_{n-1},x_{n-1},x_{m-1})$$
$$= (\alpha_1+2\alpha_2)\ D^*(x_{n-1},x_{m-1},x_{n-1})$$

Let $\beta = (\alpha_1+2\alpha_2)$ then $\beta \leq 1$

$$D^*(x_n,x_m,x_n) \leq \beta\ D^*(x_{n-1},x_{m-1},x_{n-1})$$

continuo this way we obtained

$$\leq \beta^2\ D^*(x_{n-2},x_{m-2},x_{n-2})$$
$$.$$
$$.$$
$$\leq \beta^n((\max\{\lambda_1,\lambda_2,\lambda_3\})^{m-n-1} D^*(x_0,x_1,x_0) \to 0\ as\ n,m \to \infty.$$

Now we want to prove that x is fixed point for f_1

Suppose that $f_1(x) \neq x$

$$D^*(f_1(x),x,x) = lim_{n\to\infty} D^*\ (f_1(x),x_{3n-1},x_{3n})$$
$$= lim_{n\to\infty} D^*\ (f_1(x),f_2(x_{3n-2}),f_3\ (x_{3n-1}))$$
$$\leq lim_{n\to\infty}\{\alpha_1 D^*(x,x_{3n-2},x_{3n-1})$$
$$+\alpha_2 \max\{D^*(x,f_1(x),f_2(x_{3n-2})),D^*(x_{3n-2},f_2(x_{3n-2}),f_3(x_{3n-1}))\}\}$$
$$\leq lim_{n\to\infty}\{\alpha_1 D^*(x,x_{3n-2},x_{3n-1})$$

$$+\alpha_2 \max\{D^*(x, f_1(x), x_{3n-1}), D^*(x_{3n-2}, x_{3n-1}, x_{3n})\}\}$$
$$\leq lim_{n\to\infty}\alpha_1 D^*(x, x_{3n-2}, x_{3n-1})$$
$$+\alpha_2 \max\{\lim_{n\to\infty} D^*(x, f_1(x), x_{3n-1}), \lim_{n\to\infty} D^*(x_{3n-2}, x_{3n-1}, x_{3n})\}$$
$$= \alpha_2 D^*(x, f_1(x), x)$$
$$< D^*(x, f_1(x), x)$$

Contradiction

Therefore $D^*(x, f_1(x), x) = 0$, So x fixed point for f_1.

Similarly for f_2, f_3

For the uniqueness, suppose there exist $y \neq x$, such that x, y are common fixed points for f_1, f_2, f_3, then

then $D^*(x,y,y) = D^*(f_1(x), f_2(y), f_3(y))$

$$\leq \alpha_1 D^*(x,y,y) + \alpha_2 \max\{D^*(x, f_1(x), f_2(y)), D^*(y, f_2(y), f_3(y))\}$$
$$= \alpha_1 D^*(x,y,y) + \alpha_2 D^*(x,y,y),$$
$$= (\alpha_1 + \alpha_2) D^*(x,y,y)$$
$$< D^*(x,y,y).$$

This is a contradiction

Hence f_1, f_2 and f_3 have a unique common fixed point. □

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] A.H.Siddiqi Functional Analysis with applications, Tata McGraw, 1987.

[2] Alhihi, S. A. and Al-Fayyad, M. A. Topological properties of semi-linear uniform spaces. Glob. J. Pure Appl. Math. 12 (6) 2016, 5329–5341.

[3] Alhihi, S. A., More properties of semi linear uniform spaces. Appl. Math. 6 (2015), 1069-1075.

[4] C. T. Aage and J. N. Salunke, A note on fixed point and common fixed point theorems in 2-metric spaces, Int. J. Math. Anal. 2 (2008), 1381-1392.

[5] C. T. Aage and J.N.Salunke, Some fixed points theorems in generalized D^*-metric space, Appl. Sci. 12 (2010), 1-13.

[6] Dhage, B. C. Generalized metric spaces and mapping with fixed point, Bull. Calcutta Math. Soc. 84 (4), 329-338.

[7] Engelking, R. General Topology, North-Holand, Amsterdam,1968.

[8] Gahler, S. 2-metriscue Raume undihve topologische strulcture. Math. Nachv. 26 (1963), 115-148.

[9] Graves, L. M. On the completing of a Housdroff space, Ann. Math. 38 (1937), 61-64.

[10] G. V. R. Babu and K. N. V. V. Prasad, Common fixed point theorems of different compatible mapping using Ciric's contraction type condition, Math. Commun. 11(2006), 87-102.

[11] Harish Chandra,Common fixed point of weakly compatible maps in symmetric spaces, Appl. Math. Sci. 6 (2012), 2703-2710.

[12] Huaug, L. -G. and Zhang, X. Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[13] James, I.M. Topological and Uniform Spaces. Undergraduate Texts in Mathematics. Springer-Verlag 1987.

[14] Jay G. Mehta and M. L. Joshi, On Common fixed point theorems in complete metric spaces, Gen. Math. Notes, 2 (1) (2011), 55-63.

[15] M. Aamri and D.EL moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002),181-188.

[16] Menger, K. Untersuchungen ūber allegemeine Metrik, Math. Ann. 100 (1928), 75-163.

[17] M. L. Joshi and Jay G. Mehta, Common fixed point for weakly compatible maps in complete metric spaces, Int. J. Comput. Appl. II (4) (2010), 32-34.

[18] Mustafa, Z. and Sims, B. Some remavles concerning D-metric spaces in proceeding of the international Conference on Fixed point theory and Applications, pp. 189-198, Vallencia, Spainsnly 2004.

[19] Nguyen Van Dung, Nguyen Trung Hieu, Nguyen Thi Thanh Ly and Vo Duc Thinh, Remarks on fixed point problem of 2- metric spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 167.

[20] P. K. Jain, Khalil Ahmad, Metric spaces, Alpha science international, 2004.

[21] Rahul Tiwari, D. P. Shukla, Kashi Ram Ahiwar and Chetan, Fixed point theorems of self-mapping in a complete 2-metric spaces, Int. J. Contemp. Math. Sci. 7 (2012), 1501-1507.

[22] R. K. Namdeo and Sweetee Dubey, Coincidence and fixed point of non-expansive type mapping on 2-metric spces, Int. Math. Forum 2 (2007), 803-811.

[23] Rawashdeh, A. and Tallafha, A. Fixed Point in Modified Semi-linear Uniform Spaces. Glob. J. Pure Appl. Math. 13 (2017), 4461-4476.

[24] Rawashdeh, A. and Tallafha, A. Fixed point in semi-linear uniform spaces and convex metric spaces. Submitted.

[25] S. Czerwile, Nonlinear Set-valued contraction mappings in b-metric spaces Attisemin, Mat. Fiz. Univ. Modern 49 (1998), 263-279.

[26] Shaban Sedghi, Nabi Shobe and Shahram Sedghi, Common fixed point theorems for two mapping in D^*-metric space, J. Prime Res. Math. 4(2008), 132-142 .

[27] Shaban Sedghi, Nabi Shobe and Haiyun Zhou, A common fixed point theorem in D^*-metric spaces, J. Prime Res. Math. 3 (2007), Article ID 27906, 13 pages.

[28] Sunny Chauhan, Mohammad Imdad, and Calogerovetro, Unified metrical common fixed point theorems in 2-metric spaces via implicit relation, J. Operators, 2013(2013), Article ID 186910, 11 pages.

[29] Tallafha, A. and Khalil, R., Best Approximation in Uniformity type spaces. Eur. J. Pure Appl. Math. 2 (2) (2009), 231-238.

[30] Tallafha, A. Some properties of semi-linear uniform spaces. Boletin da Sociedade Paranaense Dematematica, 29 (2) (2011), 9-14.

[31] Tallafha, A. Open Problems in Semi-Linear Uniform Spaces. J. Appl. Funct. Anal. 8 (2013), 223-228.

[32] Tallafha, A. Fixed point in Semi-Linear Uniform Spaces. Eur. J. Math. Sci. 12 (3) (2013), 334-340.

[33] Tallafha, A. Alhihi, S. Fixed point in a non-metrizable spaces. Accepted for publication as a part of the proceedings volume ”Computational Analysis: Contributions from AMAT2015” in Springer- New York.

[34] Tallafha, A. Alhihi, S. Contraction on Semi-linear Uniform Space. Submitted.

[35] T. Veerapandi and Aji. M. Pillai, Some common fixed point theorems in D^*-metric space, Int. J. Math. Sci. Appl. 1 (3) (2011), 1049-1063 .

[36] V. Popa, A general fixed point theorem for weakly compatible mappings satisfying an implicit relation, Filomat 19(2005), 45- 55.

[37] V. Popa, A general fixed point theorem for weakly compatible mappings in compact metric spaces , Turk. J. Math. 25(2001),456-474.

[38] Vishal Gupta, S. B. Singh and RavinderKumar, Some fixed point theorems in 2-metric spaces, Adv. Appl. Sci. Res. 3(2012), 2807-2814.