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SOME ALGEBRAIC PROPERTIES OF ORDER-PRESERVING FULL CONTRACTION TRANSFORMATION SEMIGROUP

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Abstract. Let X_n be a finite set, CT_n the semigroup of full contraction and OCT_n order-preserving full contraction transformation semigroup of a finite set. In this paper the Local and global U-depth as well as the status of OCT_n were investigated where U is the generating set. The local and global depth were found from the known generating Set of OCT_n and also, the status of the semigroup OCT_n was obtained from the product of global depth and the order of generating of OCT_n. For $\alpha \in OCT_n$, local depth of is equal to its defect, global depth and status of OCT_n are $n-1$ and $2(n-1)^2$ respectively. We also look at the structure of Greens relations of order-preserving full contraction transformation semigroup.

Keywords: order-preserving; local depth; global depth; status; Green's relation.

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1. Introduction

The algebraic theory of semigroup has been well studied during the second half of the twentieth century. Many ideas in study of semigroup were directly motivated by their analogues in

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the ring theory. However, certain important and interesting concepts such as the concepts of Green's relation, idempotent rank, nilpotent rank , product of idempotent and so on developed independent , just as the set of all permutation of a finite set proves to be an important source of examples in group theory, the set of all mappings of a finite set in to itself provide us with a corresponding objects in the semigroup theory. The importance of mappings in semigroup theory can be judged by the fact that every semigroup is isomorphic to a semigroup of mappings in this sense it turns out that the understanding of semigroup of mappings in semigroup theories is of paramount significance. Transformation semigroups are one of the most fundamental mathematical objects. They occur in theoretical computer science, where properties of language depend on algebraic properties of various transformation semigroups related to them. Algebraic and Combinatorial properties of transformation semigroup have been studied over a long period and interesting results have emerged for example Ganyushkin and Mazorchuk (2009); Umar (1996). One of the reasons is due to the fact that every finite inverse semigroup can be embedded in a symmetric inverse semigroup. On studying a new class of semigroup it is mostly of interest to know the type of semigroup in question for example , whether the semigroup is regular, inverse (see Clifford and Preston, 1967). Another interest is in the characterisation of the five classical Green's relations, mostly when the semigroup is regular. For non-regular semigroup, generalisation of Green's relations like starred analogues are investigated (Maclister 1976). Regular semigroups were introduced by Green (1951) and was the paper in which popular Green's relations were introduced. It was his study of regular semigroups which led Green to define relations. Recently, Garba et.al.(2017) characterised the Green's relations and starred Green's relations in CT_n, CI_n and also starred Green's relation in OCT_n.

2. Basic Definitions

Let $O_n = \{\alpha \in T_n \setminus S_n : (\forall x, y \in X_n) x \leq y \Rightarrow x\alpha \leq y\alpha\}$
$CT_n = \{\alpha \in T_n \setminus S_n : (\forall x, y \in X_n) |x\alpha - y\alpha| \leq |x - y|\}$
and $OCT_n = CT_n \cap O_n$ be the subsemigroups of $T_n \setminus S_n$ consisting of all order-preserving maps, all contraction maps and all order -preserving maps contraction maps respectively, $(S_n is symmetric group)$

Definition 2.1: A non-empty subset U of semigroup S is said to be a generating set for S, written as $\langle U\rangle = S$, if every elements of S is expressible as a product of elements in U. if S is finite, local U-depth of $s \in S$ is the smallest integer k such that $s = u_1u_2u_3\dots u_k$ for some $u_1u_2\dots u_k \in U$ and the global U-depth of S is the smallest integer in such that $U^{(m)} = S$ where $U^{(m)} = U \cup U^2 \cup U^3 \cup \dots \cup U^m$.

Definition 2.2:. On the set X_n, the equivalence $Ker(\alpha) = \{(x,y) : x\alpha = y\alpha\}$ partitions X_n into $Ker(\alpha) - classes$ called blocks of α. A block of α is called stationary if it contains the image of its members under α. Let A be a subset of X_n and let $\{A_1, A_2, \dots A_r\}$ be partition of X_n. the A is called convex if for all $x, y \in X_n$ $(x, y \in Aandx \le z \le y \Rightarrow z \in A)$.

Definition 2.3: The status of semigroup S is defined as the least value of the product of the order of generating set $|U|$ and the global U-depth $\Delta(U)$.

That is,$Stat(S) = min\{|U|\Delta(U) : \langle U\rangle = S\}$

Definition 2.4: The defect of α is define as non image point of α i.e $n - |Im(\alpha)|$.

3. Main Results

According to Garba et.al. (2017) the generating set of semigroup (OCT_n) is expressed as
$$U = \left\{ \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i & . & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & i\, i+1 & \cdots & n \\ 2 & 3 & \cdots & i+1 & \cdots & n \end{pmatrix} \right\}$$
$1 \le i \le n$.

Now, let $\beta = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i & \cdots & n-1 & \end{pmatrix} \equiv \beta_{i,i+1}$

and $\gamma = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 2 & 3 & \cdots & . & i+1 & \cdots & n \end{pmatrix} \equiv \gamma_{i,i+1}, \forall\, \beta, \gamma \in U.$

and from definition of generating set, any arbitrary $\alpha \in OCT_n$ is expressed as product of β and γ i.e. $\alpha = \beta\gamma \equiv \beta_{i,i+1} \cdot \gamma_{i,i+1}$

Results on Depth and Status of Order-preserving Full Contraction Transformation Semigroup

Theorem 4.1: Let $S = OCT_n$, for each $\alpha \in OCT_n$, the local U-depth of α is equal to $def(\alpha)$.

Proof: Let $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_i & \cdots & A_r \\ a_1 & a_2 & \cdots & a_{i+1} & \cdots & a_r \end{pmatrix}$ where $A_i < A_j < \dots < A_s$ are non singleton blocks in α and $a_i, a_j, \dots, a_s$ are the order of non single blocks respectively, that is $|A_i| = a_i, |A_j| = a_j \cdots, |A_s| = a_s$

Now, if $\alpha \in OCT_n$, there must exist a block A_i for which $|A_i| \geq 2$. Three cases were considered to prove this theorem.

Case 1: If $a_1 = 1$, then α is expressed as a product of β that is,

$$\alpha = \beta_{i,i+1}^{a_i-1}\,\beta_{j,j+1}^{a_j-1}\,\cdots\,\beta_{s,s+1}^{a_s-1}$$

Case 2: If $a_1 = 2$, in this case α can be expressed as a product of β and γ as follow

$$\alpha = \begin{cases} \beta_{i,i+1}\gamma_{i,i+1}\beta_{i+1,i+2}^{a_i-3}\beta_{j+1,j+2}^{a_j-1}\cdots\beta_{s+1,s+2}^{a_s-1}, & if|A_i| > 2 \\ \beta_{i,i+1}\gamma_{\min[A_j]-1,\min[A_j]}\beta_{j+1,j+2}^{a_j-2}\cdots\beta_{s+1,s+2}^{a_s-2}, & if|A_j|\&|A_s| > 2 \\ & \text{and } \ s+1, s+2 \in A_s \\ \beta_{i,i+1}\gamma_{\min[A_j]-1,\min[A_j]}\beta_{j+1,j+2}^{a_j-1}\cdots\beta_{s+1,s+2}^{a_s-1}, & if|A_s| \geq 2 \\ & \text{and if at least one of } \ s+1, s+2 \text{ is not in} A_s \end{cases}$$

Case 3: If $a_1 \geq 3$, in this case α can be expressed as product of β and γ i.e.

$$\alpha = \begin{cases} \gamma_{i,i+1}\gamma_{i+2,i+3}\beta_{j+2,j+3}^{a_j-1}\cdots\beta_{s+2,s+3}^{a_s-1}, & if|A_i| \leq 3 \\ \gamma_{i,i+1}\gamma_{i+1,i+2}\beta_{i+2,i+3}\beta_{j+2,j+3}^{a_j-1}\cdots\beta_{s+2,s+3}^{a_s-1}, & if|A_s| > 3 \end{cases}$$

Example Case 1: Let $\alpha \in OCT_8$

$$\alpha = \begin{pmatrix} 1 & \{2,3,4\} & 5 & 6 & \{7,8\} \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & \{2,3\} & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

Here, $def(\alpha) = 3$, the local U-depth of α is 3, these are $\beta_{23}\beta_{23}\beta_{56}$

Example Case 2: Let $\alpha \in OCT_n$

$$\alpha = \begin{pmatrix} 1 & 2 & \{3,4\} & 5 & \{6,7,8\} \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \{3,4\} & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & \{5,6\} & 7 & 8 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5\{6,7\} & 8 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$def(\alpha) = 3$, the local U-depth of α is 3, these are $\beta_{34}\gamma_{56}\beta_{67}$

Example Case 3: Let $\alpha \in OCT_n$ Let

$$\alpha = \begin{pmatrix} 1 & \{2,3,4\} & 5 \\ 3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \{2,3\} & 4 & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & \{3,4\} & 5 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

The $def(\alpha) = 2$, the local U-depth of α is 2, these are $\gamma_{23}\gamma_{34}$

Example Case 3 Let

$$\alpha = \begin{pmatrix} 1 & \{2,3,4\} & 5 & 6 & \{7,8\} \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$def(\alpha) = 3$ and the local U-depth of α is 3, these are $\gamma_{23}\gamma_{34}\beta_{78}$

Example Case 3: Let $\alpha \in OCT_n$ Let

$$\alpha = \begin{pmatrix} \{1,2,3\} & 4 & 5 & \{6,7\} & \{8,9\} & 10 \\ 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}$$

$def(\alpha) = 4$ and the local U-depth of α is 4, these are $\gamma_{23}\gamma_{34}\gamma_{67}\beta_{89}$

Remark

We observe that as the value of a_1 increases, the more the factor γ appears.

Lemma 4.1: $max\{def(\alpha) : \alpha \in OCT_n\} = n-1$

Proof: For each map $\alpha \in OCT_n$, since the local U-depth of α is equal to its defect, that is the smallest positive integer k for which $\alpha = u_1, u_2, \cdots, u_k$ where $u_1, u_2, \cdots, u_k \in U$. Maximizing

$def(\alpha)$ by making $h(\alpha)$ as large as possible and since $def(\alpha) = n - |h(\alpha)|$, clearly , the maximum value of $h(\alpha)$ is $n-1$. Thus, $max\{def(\alpha) : \alpha \in OCT_n\} = n-1$. Hence, for all $n \geq 3$, the global U-depth of OCT_n deduced easily.

Theorem 4.2: For each $n \geq 3$, the global U-depth of OCT_n is $n-1$

Proof: The proof follows from lemma 3.1.

Theorem 4.3: Let $S = OCT_n$, $n \geq 3$ Then $Stat(OCT_n) \leq 2(n-1)^2$.

Proof: Let U be the generating set, $|U|$ be the order of U and $\Delta(U)$ be the global U-depth of OCT_n. Then, $|U|$ is equal to $2(n-1)$ by corollary 2.1 and from theorem 3.2. above $\Delta(U)$ equal to $n-1$. Since the status of semigroup is the least value of the product of order of generating set and global U-depth, that is $Stat\,(S) = min\{|U|\Delta(U) : \langle U\rangle = S\}$ Hence, $Stat(OCT_n) \leq 2(n-1)^2$.

Results on Green's Relations of order-preserving full contraction transformation semigroup

Theorem 5.1 Let $S = OCT_n$ and $\alpha, \beta \in S$, then

a. $(\alpha,\beta) \in \mathscr{L}\,(OCT_n)$ if and only if either $\alpha = \beta$ or $im\,(\alpha) = im\,(\beta)$ and $ker\,(\alpha)$,$ker\,(\beta)$ are convex partitions of X_n;
b. $(\alpha,\beta) \in \mathscr{R}\,(OCT_n)$ if and only if $ker\,(\alpha) = ker\,(\beta)$;
c. $(\alpha,\beta) \in \mathscr{D}\,(OCT_n)$ if and only if either $ker\,(\alpha) = ker\,(\beta)$ or $|im\,(\alpha)| = |im\,(\beta)|$ and both $ker\,(\alpha)$,$ker\,(\beta)$ are convex partitions of X_n; and
d. $(\alpha,\beta) \in \mathscr{H}\,(OCT_n)$ if and only if $\alpha = \beta$.

Proof

(a) Suppose that $(\alpha,\beta) \in \mathscr{L}\,(OCT_n)$, then $\delta\beta = \alpha$ and $\gamma\alpha = \beta$ for some $\delta, \gamma \in OCT_n$. This clearly implies that $im\,(\alpha) = im\,(\beta)$. Therefore, $im\,(\gamma)$ and $im\,(\delta)$ must be transversal of $ker\,(\alpha)$ and $ker\,(\beta)$ respectively. But since $\delta, \gamma \in OCT_n$. It follow from [Garba et.al. (2017)], that $im\,(\gamma)$ and $im\,(\delta)$ are convex subsets of X_n. Thus, $ker\,(\alpha)$ and $ker\,(\beta)$ are convex partitions of X_n.

Conversely, suppose that $im\,(\alpha) = im\,(\beta) = \{c_1, c_2, \ldots, c_r\}$ and that $ker\,(\beta)$ are convex partitions of X_n.Let $\{a_1, a_2, \ldots, a_r\}$ and $\{b_1, b_2, \ldots, b_r\}$ be convex tranversal of $ker\,(\alpha)$ and $ker\,(\beta)$ respectively,arranged in away that $a_i \in c_i\alpha^{-1}$ and $b_i \in c_i\beta^{-1}$ for each $1 \leq$

$i \leq r$. define maps δ and γ by $ker(\delta) = ker(\alpha)$,$ker(\gamma) = ker(\beta)$,$\left(c_i\alpha^{-1}\right)\delta = b_i$ and $\left(c_i\beta^{-1}\right)\gamma = a_i$ for each $1 \leq i \leq r$. Then, $\delta, \gamma \in OCT_n$ and $\delta\beta = \alpha$ and $\gamma\alpha = \beta$. So that $(\alpha, \beta) \in \mathscr{L}(OCT_n)$

(b.) Suppose that $(\alpha, \beta) \in \mathscr{R}(OCT_n)$, then $\beta\delta = \alpha$ and $\alpha\gamma = \beta$ for some $\delta, \gamma \in OCT_n$. From this it follows that $ker(\alpha) = ker(\beta)$. Conversely, suppose that $ker(\alpha) = ker(\beta) = \{c_1, c_2, \ldots, c_r\}$. Then, since $\alpha, \beta \in OCT_n$, without loss of generality we write
$\alpha = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ i & i+1 & \ldots & i+r-1 \end{pmatrix}$ and $\beta = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ j & j+1 & \ldots & j+r-1 \end{pmatrix}$ for some $i, j \in X_n$, then the maps

$$\delta = \begin{pmatrix} \{1,2,\ldots,j\} & j+1 & \ldots & j+r-2 & \{j+r-1, j+r, \ldots, n\} \\ i & i+1 & \ldots & i+r-2 & i+r-1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} \{1,2,\ldots,i\} & i+1 & \ldots & i+r-2 & \{i+r-1, i+r, \ldots, n\} \\ j & j+1 & \ldots & j+r-2 & j+r-1 \end{pmatrix}$$

are in OCT_n and satisfy $\beta\delta = \alpha$ and $\alpha\gamma = \beta$, So that $(\alpha, \beta) \in \mathscr{R}(OCT_n)$

(c.) Suppose that $\alpha \neq \beta$ and $(\alpha, \beta) \in \mathscr{D}(OCT_n)$, then $(\alpha\gamma) \in \mathscr{L}(OCT_n)$ and $(\gamma, \beta) \in \mathscr{R}(OCT_n)$, for some $\gamma \in OCT_n$ from the theorem $im(\alpha) = im(\gamma)$, $ker(\gamma) = ker(\beta)$ and that $ker(\gamma)$, $ker(\alpha)$ are convex partitions of X_n. This implies that $|im(\alpha)| = |im(\beta)|$ and $ker(\alpha)$, $ker(\beta)$ are convex partitions of X_n. Conversely, suppose that $|im(\alpha)| = |im(\beta)|$, and both $ker(\alpha)$, $ker(\beta)$ are convex partitions of X_n. Then we can choose $\gamma \in OCT_n$ such that $ker(\alpha) = ker(\beta)$ and $im(\gamma) = im(\alpha)$. Hence $(\alpha\gamma) \in \mathscr{L}(OCT_n)$ and $(\gamma, \beta) \in \mathscr{R}(OCT_n)$.

(d.) This follows from (Garba et.al. 2017,) in the characterisation of starred Green's relations of order-preserving partial one-one contraction transformation semigroup (OCI_n).

4. Conclusion

In conclusion, we have shown that in the semigroup OCT_n, the minimum length factorisation of each $\alpha \in OCT_n$ is equal to the defect of α and the Status of semigroup OCT_n were obtained using the global $U-depth$ to be Stat$(OCT_n) \leq 2(n-1)^2$ with relevant examples. Also, the structure of Green's relations on semigroup OCT_n were characterized.

Conflict of Interests

The authors declare that there is no conflict of interests.

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