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CROSS-CONNECTION REPRESENTATION OF REGULAR SEMIGROUPS

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Abstract. Normal categories, their normal duals and the local isomorphisms that existed between these categories were introduced by K.S.S. Nambooripad in [4] in order to construct the cross-connection semigroup. In this paper we recall that the principal left and right ideals of a regular semigroups are normal categories and provides the construction of the cross-connection semigroup termed as the cross-connection representation of a regular semigroup.

Keywords: regular semigroup; normal category; cross-connection.

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1. Introduction

A category $\mathscr{C}$ having certain remarkable properties such as subobject relation, factorization of morphisms and pocessing sufficently many cones, termed as normal categories were introduced in [4]. For a regular semigroup S the categories of principal left [right] ideals $L(S)$ $[R(S)]$ are normal categories and conversely it is also seen that every normal category arises as ideal category of some regular semigroup. For each $a \in S$, ρ^a is a cone with vertex Sa in $L(S)$, and the set of all such cones under cone composition is the semigroup $TL(S)$. Similarly the category

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$[R(S)]$ and cones λ^a, $a \in S$ with vertex aS is the semigroup $TR(S)$. Further these cones determie certain set valued functors which provides the functor categories $N^*L(S)$ and $N^*R(S)$ called the normal duals of $L(S)$ and $R(S)$ and there esists a local isomorphisms $\Gamma S : \mathscr{R}(S) \to N^*L(S)$ called a connection. Clearly each $Se \in L(S)$ we see that $Se \in M\Gamma S(eS)$ and so the image of ΓS is total and so there is local isomorphism $\Gamma^* S : L(S) \to N^*R(S)$ called the dual connection. The connection and dual connection together provide the cross-connection $(R(S), L(S), \Gamma S)$.

The connection ΓS and dual connection $\Gamma^* S$ induces bi- functors $\Gamma(-,-) : R(S) \times L(S) \to \mathbf{Set}$ and $\Gamma^*(-,-) : L(S) \times R(S) \to \mathbf{Set}$ such that there is a natural bijection

$$\chi_{\Gamma S} : \Gamma S(-,-) \to \Gamma^* S(-,-)$$

whose component at $(Se, fS) \in \mathscr{L}(S) \times R(S)$ is

$$\chi_{\Gamma S}(Se, fS) : \Gamma S(Se, fS) \to \Gamma^* S(Se, fS)$$

between bi-functors $\Gamma S : R(S) \times L(S) \to \mathbf{Set}$ and $\Gamma^* S : L(S) \times R(S) \to \mathbf{Set}$ such that this bijection yeilds a pairs of cones (ρ^a, λ^a) and the collection of these cones together with the binary composition defined by

$$(\rho^a, \lambda^a) \circ (\rho^b, \lambda^b) = (\rho^a \rho^b, \lambda^b \lambda^a)$$

is the semigroup $\tilde{S}\Gamma S$ called the corss-connection representation of S.

2. Preliminaries

In the following we recall some basic notions and results concerning semigroups . A set S together with an associative binary operation is called a semigroup. An element $x \in S$ is regular if $xyx = x$ for some $y \in S$ and a semigroup S is called regular if all elements of S are regular. An element $x \in S$ is called an idempotent if $x^2 = x$, the collection of all idempotents in S will be denoted by $E(S)$. The principal left ideal genegared by $a \in S$ is the set $Sa = \{sa | s \in S\}$. Two elements of a semigroup S are said to be $\mathscr{L}, \mathscr{R}, \mathscr{J}$-equivalent if they generate the same principal left, right, two sided ideals respectively and these are equivalence relations. The join of the relations $\mathscr{L}$ and $\mathscr{R}$ is denoted by $\mathscr{D}$ and their intersection by $\mathscr{H}$. These equivalence relations are introduced by J.A.Green and are known as Green's relations and are of fundamental importance in the study of the structure of semigroups(cf. [1])

2.1. Categories, preorders and nornal categories. A category $\mathscr{C}$ consists of a class called the class of vertices or objects $v\mathscr{C}$ and a class of disjoint sets $\mathscr{C}(a,b)$ one for each pair $(a,b) \in v\mathscr{C} \times v\mathscr{C}$. An element $f \in \mathscr{C}$ is called a morphism from a to b, written $f : a \to b$; $a = dom\ f$ called the domain of f and $b = cod\ f$ called the codomain of f. For $a,b,c, \in v\mathscr{C}$, a map $\circ : \mathscr{C}(a,b) \times \mathscr{C}(b,c) \to \mathscr{C}(a,c)$ such that $(f,g) \to g \circ f$ called the *composition* of morphisms in $\mathscr{C}$. and for each $a \in v\mathscr{C}$, a unique $1_a \in \mathscr{C}(a,a)$ is called the identity morphism on a. Further these must satisfy the following axioms :

- The composition is associative : for $f \in \mathscr{C}(a,b), g \in \mathscr{C}(b,c)$ *and* $h \in \mathscr{C}(c,d)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- for each $a,b \in v\mathscr{C}, f \in \mathscr{C}\ (a,b)$

$$f \circ 1_a = f = 1_b \circ f$$

The following are some examples of categories.

- **Set**: the category in which objects are sets and morphisms are functions between sets.
- **Grp**: Category with groups as objects and homomorphisms as morphisms.

A **functor** $F : \mathscr{C} \to \mathscr{D}$ from a category $\mathscr{C}$ to a category $\mathscr{D}$ consists of a vertex map $vF : v\mathscr{C} \to v\mathscr{D}$ which assigns to each $a \in vC$ a vertex $F(a) \in D$ and a morphism map $F : C \to D$ which assigns to each morphism $f : a \to b$, a morphism

$$F(f) : F(a) \to F(b) \in \mathscr{D}$$

such that $F(1_a) = 1_{F(a)}$ for all $a \in v\mathscr{C}$; and $F(f)F(g) = F(fg)$ for all morphisms $f,g \in \mathscr{C}$ for which the composition fg exists.

Example 1. *The power set functor $\mathscr{P}$: **Set** $\to$ **Set**. Its object function assigns each object X in **Set** the usual power set $\mathscr{P}X$ and its arrow function assigns to each $f : X \to Y$ the map $\mathscr{P}f : \mathscr{P}X \to \mathscr{P}Y$ which send each $S \subset X$ to its image $fS \subset Y$.*

Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A natural transformation $\eta : F \to G$ is a family $\{\eta_a : F(a) \to G(a) | a \in v\mathscr{C}\}$ of maps in $\mathscr{D}$ such that for every map $f : a \to b$ in $\mathscr{C}$, the following diagram commutes

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_c} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta_{c'}]{} & G(b) \end{array}$$

The map η_a are called the components of η. If each component of η is an isomorphism then η is called a natural isomorphism. A category whose objects are functors between categories and morphisms are natural transformations between such functors with composition of morphisms, the composition of natural transformations is a category and is termed as the *functor category*.

A *preorder* $\mathscr{P}$ is a category such that for any $p, p' \in v\mathscr{P}$, the hom-set $\mathscr{P}(p, p')$ contains at most one morphism. In this case, the relation $\subseteq$ on the class $v\mathscr{P}$ of objects of $\mathscr{P}$ is defined by

$$p \subseteq p' \text{ if } \mathscr{P}(p, p') \neq \emptyset$$

is a quasi- order. A preorder $\mathscr{P}$ is said to be a *strict* if $\subseteq$ is a partial order.

Definition 2. *(Category with subobjects) Let $\mathscr{C}$ be a small category and $\mathscr{P}$ be a subcategory of $\mathscr{C}$ such that $\mathscr{P}$ is a strict preorder with $v\mathscr{P} = v\mathscr{C}$. Then $(\mathscr{C}, \mathscr{P})$ is a category with $subobjects$ if*

(1) every $f \in \mathscr{P}$ is a monomorphism in $\mathscr{C}$

(2) if $f = hg$ for $f, g \in \mathscr{P}$, then $h \in \mathscr{P}$.

Example 3. *In categories $Set, Grp, Vect_K, Mod_R$ the relation on objects induced by the usual set inclusion is a subobject relation.*

In a category $(\mathscr{C}, \mathscr{P})$ with subobjects, morphisms in $\mathscr{P}$ are called inclusions. If $c' \to c$ is an inclusion, we write $c' \subseteq c$ and denotes this inclusion by $j^c_{c'}$. An inclusion $j^c_{c'}$ *splits* if there exists $q : c \to c' \in \mathscr{C}$ such that $j^c_{c'} q = 1_{c'}$ and the morphism q is called a retraction.

Definition 4. *A morphism f in a category $\mathscr{C}$ with subobjects is said to have factorization if f can be expressed as $f = pm$ where p is an epimorphism and m is an embedding.*

A normal factorization of $f \in \mathscr{C}(c, d)$ is a factorization of the form $f = quj$ where $q : c \to c'$ is a retraction, $u : c' \to d'$ is an isomorphism and $j = j^d_{d'}$ is an inclusion where $c', d' \in v\mathscr{C}$ with

$c' \subseteq c$, $d' \subseteq d$. The morphism qu is known as the epimorphic component of f and is denoted by f°.

Definition 5. *Let $\mathscr{C}$ be a category with subobjects and $d \in v\mathscr{C}$. A map $\gamma : v\mathscr{C} \to \mathscr{C}$ is called a cone from the base $v\mathscr{C}$ to the vertex d if*

(1) $\gamma(c) \in \mathscr{C}(c,d)$ for all $c \in v\mathscr{C}$
(2) if $c \subseteq c'$ then $j_c^{c'}\gamma(c') = \gamma(c)$

For a cone γ denote by c_γ the vertex of γ and for $c \in v\mathscr{C}$, the morphism $\gamma(c) : c \to c_\gamma$ is called the component of γ at c. A cone γ is said to be normal if there exists $c \in v\mathscr{C}$ such that $\gamma(c) : c \to c_\gamma$ is an isomorphism. We denote by $T\mathscr{C}$ the set of all normal cones in $\mathscr{C}$.

Definition 6. *A category $\mathscr{C}$ with subobjects is called a normal category if any morphism in $\mathscr{C}$ has a normal factorization, every inclusion in $\mathscr{C}$ splits and for each $c \in v\mathscr{C}$ there is a normal cone γ with vertex c and $\gamma(c) = 1_{c_\gamma}$.*

Observe that given a normal cone γ and an epimorphism $f : c_\gamma \to d$ the map $\gamma * f : a \to \gamma(a)f$ from $v\mathscr{C}$ to $\mathscr{C}$ is a normal cone with vertex d.

Remark 7. *The set of all normal cones $T\mathscr{C}$ in $\mathscr{C}$ with the cone composition*

$$\gamma^1 \cdot \gamma^2 = \gamma^1 * (\gamma^2_{c_{\gamma^1}})^\circ$$

is a regular semigroup.

For a cone $\gamma \in T\mathscr{C}$, the set

$$M\gamma = \{c \in v\mathscr{C} : \gamma(c) \,\text{is an isomorphism}\}$$

is the M-set of the normal cone γ. A normal cones $\gamma \in T\mathscr{C}$ define set valued functors $H(\gamma,-) : \mathscr{C} \to \mathbf{Set}$ and the category whose objects set $vN^*\mathscr{C} = \{H(\gamma,-) : \gamma \in T\mathscr{C}\}$ and morphisms $\sigma : H(\gamma,-) \to H(\gamma',-)$ given by

$$\begin{array}{ccc} H(\gamma,-) & \xrightarrow{\eta_c} & \mathscr{C}(c_\gamma,-) \\ \sigma\big\downarrow & & \big\downarrow \mathscr{C}(\bar{\sigma},-) \\ H(\gamma',-) & \xrightarrow[\eta_{c'}]{} & \mathscr{C}(c_{\gamma'},-) \end{array}$$

is the functor category $N^*\mathscr{C}$ called the normal dual of $\mathscr{C}$.

2.2. Cross-connections of normal categories. Given normal categories $\mathscr{C}$, $\mathscr{D}$ and $\Gamma : \mathscr{D} \to N^*\mathscr{C}$ a local isomorphism, a connection of normal categories is defined as follows (see cf. [4] for a detailed discussion).

Definition 8. *Let $\mathscr{C}$ and $\mathscr{D}$ be normal categories, the local isomorphism $\Gamma : \mathscr{D} \to N^*\mathscr{C}$ such that for every $c \in v\mathscr{C}$ there is some $d \in v\mathscr{D}$ with $c \in M\Gamma(d)$ is called a connection and is denoted as the triple $(\mathscr{D}, \mathscr{C}, \Gamma)$.*

When the image of the local isomorphism $\Gamma : \mathscr{D} \to N^*\mathscr{C}$ is total in $N^*\mathscr{C}$ we have the dual connection $\Gamma^* : \mathscr{C} \to N^*\mathscr{D}$ as well and they together termed as a cross-connection denoted as the triple $(\mathscr{C}, \mathscr{D}, \Gamma)$ and when there is no ambigity regarding the categories we simply say Γ is a cross-connection.

Note that the functor Γ induces a bi- functor $\Gamma(-,-) : \mathscr{D} \times \mathscr{C} \to \mathbf{Set}$ such that for $(c,d) \in v\mathscr{C} \times v\mathscr{D}$ the set

$$\Gamma(c,d) = \{\gamma * f^\circ, \text{where } f : c_\gamma \to c\}$$

and for $g : c \to c'$, $h : d \to d'$ then $(g,h) \in \mathscr{C} \times \mathscr{D}$

$$\Gamma(f,g) = \Gamma(c,d) \to \Gamma(c',d').$$

In a similar way there is bifunctor $\Gamma^*(-,-) : \mathscr{C} \times \mathscr{D} \to \mathbf{Set}$ and a natural bijection

$$\chi_{\Gamma(c,d)} : \Gamma(c,d) \to \Gamma^*(c,d).$$

For a cross-connection $\Gamma : \mathscr{D} \to N^*\mathscr{C}$, we have the set

$$E_\Gamma = \{(c,d) : c \in v\mathscr{C}, d \in v\mathscr{D} \,\text{and}\, c \in M\Gamma(d)\}$$

and it is easily seen that $(c,d) \in E_\Gamma$ if and only if $d \in M\Gamma^*(c)$ and for each $(c,d) \in E_\Gamma$ there is a unique cone $\gamma(c,d) \in \mathscr{C}$ such that

$$c_{\gamma(c,d)} = c, \quad \text{and} \quad \Gamma(d) = H(\gamma(c,d), -).$$

Similarly a unique cone $\gamma^*(c,d) \in \mathscr{D}$ such that

$$c_{\gamma^*(c,d)} = d, \quad \text{and} \quad \Gamma^*(d) = H(\gamma^*(c,d), -).$$

Define

$$U\Gamma = \bigcup \{\Gamma(c,d) : (c,d) \in v\mathscr{C} \times v\mathscr{D}\}$$
$$U\Gamma^* = \bigcup \{\Gamma^*(c,d) : (c,d) \in v\mathscr{C} \times v\mathscr{D}\}$$

by Proposition below $U\Gamma$ and $U\Gamma^*$ are regular subsemigroups of the semugroup of normal cones $T\mathscr{C}$, (see [4] for a detailed discussion).

Proposition 9. *A normal cone $\gamma \in U\Gamma$ if and only if $\gamma = \gamma(c,d) * f$ for some $(c,d) \in E_\Gamma$ and some isomorphism $f : c \to c'$ and $U\Gamma$ is a regular subsemigroup of $T\mathscr{C}$ such that*

$$E(U\Gamma) = \{\gamma(c,d) : (c,d) \in E_\Gamma\}.$$

Moreover $\mathscr{C}$ is isomorphic to $L(U\Gamma)$.

Given a cross-connection $\Gamma : \mathscr{D} \to N^*\mathscr{C}$, we shall say $\gamma \in U\Gamma$ is linked to $\gamma^* \in U\Gamma^*$ if there exists $(c,d) \in \mathscr{C} \times \mathscr{D}$ such that

$$\gamma \in \Gamma(c,d) \quad \text{and} \quad \gamma^* = \chi_{\Gamma(c,d)}(\gamma).$$

All linked pairs

$$\hat{S}\Gamma = \{(\gamma, \gamma^*) \in U\Gamma \times U\Gamma^*\}$$

together with the binary composition defined by

$$(\gamma, \gamma^*) \circ (\delta, \delta^*) = (\gamma\delta, \delta^*\gamma^*)$$

is a regular semigroup and is called the corss-connection semigroup.

Theorem 10. *Given cross-connection $\Gamma : \mathscr{D} \to N^*\mathscr{C}$ and the cross-connection semigroup $\hat{S}\Gamma$ the projections $\pi : (\gamma, \gamma^*) \mapsto \gamma$ is homomorphosm of $\hat{S}\Gamma$ onto $U\Gamma$ and $\pi^* : (\gamma, \gamma^*) \mapsto \gamma^*$ is an anti-homomorphosm of $\hat{S}\Gamma$ onto $U\Gamma^*$. Consequently $\hat{S}\Gamma$ is a subdirect product of $U\Gamma$ and $(U\Gamma^*)^{op}$.*

3. Cross-Connections of Regular Semigroups

Recall that for any set X, the full transformation semigroup $\mathscr{T}(X)$ consisting of mappings from X into X with composition of maps is a regular semigroup. A semigroup S is, if for some X a subsemigroup of $\mathscr{T}(X)$ is called a semigroup of mappings and a right regular representation of semigroup S a homomorphism $\rho : a \mapsto \rho_a$ of S into the full trasnformation semigroup $\mathscr{T}S$ and S_ρ denote the image of ρ. Clearly $\rho : S \to S_\rho$ is surjective and S is said to be reductive if ρ is injective.

Let S be a regular semigroup with $E(S)$ denotes the set of its idempotents. For each $a \in S$ the map $\rho_a : x \mapsto xa$, $[\lambda_a : x \mapsto ax]$, for any $x \in S$ is called right [left] translation determined by a. Now for the category $L(S)$ whose object sets $\nu L(S) = \{Se : e \in E(S)\}$ the principal left ideals of S is generated by an idempotent with morphisms partial right translations $\rho : Se \to Sf$ where $\rho = \rho_u|Se$ for some $u \in S$.

Proposition 11. *Let S be a regular semigroup. $L(S)$ the category of principal left ideals of S is generated by an idempotent with morphisms*

$$L(S)(Se, Sf) = \{\rho : Se \to Sf : (st)\rho = s(t\rho) \quad \forall s,t \in Se\}$$

then $\rho \in L(S)$ is $\rho = \rho(e,u,f) = \rho_u|Se$ where $u \in eSf$. Then $L(S)$ is with subobjects and further

1. *$\rho(e,u,f) = \rho(e',v,f')$ if and only if $e\mathscr{L}e', f\mathscr{L}f', u \in eSf, v \in e'Sf'$ and $v = e'u$.*
2. *For any $g \in \mathscr{R}_u \cap \omega(e)$ and $h \in E(\mathscr{L}_u), \rho = \rho(e,g,g)\rho(g,u,h)\rho(h,h,f)$ is a normal factorization of ρ, where $\omega(e) = \{f : ef = fe = e\}$.*

3.1. Semigroup of normal cones.

Let S be a regular semigroup $a \in S$ and $f \in E(\mathscr{L}_a)$. Then ρ^a is a normal cone in $L(S)$ with vertex Sf called the principal cone generated by a. The component of ρ^a at Se is

$$\rho^a(Se) = \rho(e, ea, f)$$

a cone is said to be normal if there is at least one component which is an isomorphism. The M-set of a cone ρ^a is given by $M\rho^a = \{Se : e \in E(\mathscr{R}_a)\}$. Now it is easily seen that for a regular semigroup S, the category $L(S)$ is a normal category. Further, the set of all normal cones in $L(S)$, $[R(S)]$ with composition of cones is a semigroup and is written as $\mathscr{T}L(S)$, $[\mathscr{T}R(S)]$. A cone ρ^a is an idempotent in $\mathscr{T}L(S)$ if and only if $a \in E(S)$.

Proposition 12. *Let S be a regular semigroup, $\mathscr{T}L(S)$ the semigroup of normal cones in S Then the map $a \mapsto \rho^a$ is a homomorphism from S to $\mathscr{T}\mathscr{L}(S)$.*

Dually we have the category $R(S)$ and for $a \in S$ and $f \in E(\mathscr{R}_a)$, $\lambda^a \in \mathscr{T}R(S)$ is a normal cone with vertex fS called the principal cone generated by a, the component of λ^a at eS is $\lambda^a(eS) = \lambda(e, ae, f)$. The M set $M\lambda^a$ is $\{eS : e \in E(\mathscr{L}_a)\}$ and the map $a \mapsto \lambda^a$ is an anti-homomorphism from S to $\mathscr{T}R(S)$.

The normal dulals $N^*L(S)$ and $N^*R(S)$ are functor categories and the local isomorphism $\Gamma S : R(S) \to N^*L(S)$ is the composite $\Gamma S := \bar{G} \cdot FS_\rho$ where $FS_\rho : R(S) \to R(\mathscr{T}L(S))$ is

$$FS_\rho(eS) = \rho^e(\mathscr{T}L(S)) \text{ and } FS_\rho(\lambda(e,u,f)) = \rho(\rho^e, \rho^u, \rho^f) \tag{1}$$

and $\bar{G} : R(\mathscr{T}L(S)) \to N^*L(S)$ by $\bar{G}(\rho(\rho^e, \rho^u, \rho^f)) = H(\rho^e, -)$

The explicit relation $\Gamma S : R(S) \to N^*L(S)$ is furnished below.

Theorem 13. *The functor $\Gamma S : R(S) \to N^*L(S)$ defined by*

$$v\Gamma S(eS) = H(\rho^e, -) \textit{ and } \Gamma S(\lambda(e,u,f)) = \eta_{\rho^e}\mathscr{L}(S)(\rho(f,u,e), -)\eta^{-1}_{\rho^f} \tag{2}$$

is a local isomorphisms and is termed as a connection.

Definition 14. *S be a regular semigroup and $\Gamma S : R(S) \to N^*L(S)$ is the connection $v\Gamma S(eS) = H(\rho^e, -)$, then*

$$M\Gamma S(eS) = MH(\rho^e, -) = M\rho^e = \{Se, : \rho^e(Se') \textit{ is isomorphism}\}.$$

Proposition 15. *S be a regular semigroup and $\Gamma S : R(S) \to N^*L(S)$ is the connection, then for each $eS \in R(S)$, there is an $Se \in M\Gamma S(eS)$ and so the image of ΓS is total in $N^*L(S)$.*

From the Proposition above we obtain the following Theorem

Theorem 16. *For the connection $\Gamma S : R(S) \to N^*L(S)$. There exists a connection $\Gamma^* S : L(S) \to N^*R(S)$ such that for each $Se \in L(S)$, there is $eS \in M\Gamma^* S(Se)$ and the connection $\Gamma^* S$ is termed as the dual connection to ΓS.*

Note that the connection $\Gamma S : R(S) \to N^*L(S)$ and the dual connection $\Gamma^* S : L(S) \to N^*R(S)$ together constitute the cross-connection $(L(S), R(S), \Gamma S)$.

3.2. Bifunctors and duality.

Consider the local isomorphism $\Gamma S : R(S) \to N^*L(S)$, note that this local isomorphism determines unique bifunctor

$$\Gamma S(-,-) : L(S) \times R(S) \to \mathbf{Set}$$

given by

$$\Gamma S(Se, fS) = \Gamma S(fS)(Se) = H(\rho^f, Se) = \{\rho^f * \rho(f,u,e)^\circ : u \in fSe\} \tag{3}$$

$$\Gamma S(\rho, \lambda) = \Gamma S(fS)(\rho)\Gamma S(\lambda)(Se') = \Gamma S(\lambda)(Se)\Gamma S(f'S)(\rho) \tag{4}$$

for all (Se, fS) and $(\rho, \lambda) : (Se, fS) \to (Se', f'S)$.

Definition 17. *For each $(Se, fS) \in L(S) \times R(S)$ the bifunctor $\Gamma S(-,-) : L(S) \times R(S) \to$ **Set** determins a set $\Gamma S(Se, fS)$. Then $U\Gamma S$ defined by*

$$U\Gamma S = \bigcup \{\Gamma S(Se, fS) : (Se, fS) \in L(S) \times R(S)\}$$

is a semigroup.

Proposition 18. *A cone $\rho^a \in U\Gamma S$ is a normal cone if and only if $\rho^a = \rho^f * \rho$ where ρ is an isomorphism $Sf \to Sa$.*

Proof. Let $\rho^a = \rho^f * \rho$ where ρ is an isomorphism $Sf \to Sa$, then we have $\rho^a \in H(\rho^f, Se)$ such that $e\mathscr{L}a$ which implies $\rho^a \in \Gamma S(Se, fS)$ and so $\rho^a \in U\Gamma S$. Conversely let $\rho^a \in U\Gamma S$, then $\rho^a \in \Gamma S(Se, fS)$ and so $f\mathscr{R}a\mathscr{L}e$ thus $\rho^a = \rho^f * \rho(f,a,e)$ where $\rho(f,a,e)$ is an isomorphism. $\square$

Remark 19. *$E(U\Gamma S) = \{\rho^e : e \in E(S)\}$ and for any $a \in S$, $\rho^a = \rho^e * \rho(e,a,f)$ where $e\mathscr{R}a$ and $f \in E(\mathscr{L}_a)$ and so $\rho^a \in U\Gamma$.*

Also it is easy to observe that there is a bifunctor $\Gamma^*S(-,-) : L(S) \times R(S) \to \mathbf{Set}$ which is the dual of $\Gamma S(-,-)$ and a semigroup

$$U\Gamma^*S = \bigcup\{\Gamma^*S(Se, fS) : (Se, fS) \in L(S) \times R(S)\}$$

a cone $\lambda^a \in U\Gamma^*S$ is a normal cone if and only if $\lambda^a = \lambda^e * \lambda(e, af)$ where $\lambda(e, af)$ is an isomorphism $eS \to aS$.

Theorem 20. *Let S be a regular semigroup. $\Gamma S(-,-)$ and $\Gamma^*S(-,-)$ are the bifunctors determined by ΓS and Γ^*S respectively. Then there is a natural isomorphism $\chi_{\Gamma S} : \Gamma(-,-)S \to \Gamma^*S(-,-)$ whose components are defined by*

$$\chi_{\Gamma S}(Se, fS) : \rho^f * \rho(f, u, e)^\circ \mapsto \lambda^e * \lambda(e, u, f)^\circ$$

for each $(Se, fS) \in v(L(S) \times R(S))$.

Let $(L(S), R(S), \Gamma S)$ be a cross-connection. Then the bifunctors $\Gamma S(-,-)$ and $\Gamma^*S(-,-)$ together with natural isomorphism $\chi_{\Gamma S}$ determines a pairs cones (ρ^a, λ^a) where $\chi_{\Gamma S}(Se, fS)(\rho^a) = \lambda^a$, $a \in S$ called the linked pair. The linked pair of cones $(\rho^a, \lambda^a) : a \in S$ together with the composition defined by

$$(\rho^a, \lambda^a) \circ (\rho^a, \lambda^a) = (\rho^a \rho^b, \lambda^b \lambda^a)$$

the semigroup $\hat{S}\Gamma S$ called the cross-connection semigroup and the map $\varphi(S) : S \to \hat{S}\Gamma S$ defined by

$$\varphi(S)(a) = (\rho^a, \lambda^a)$$

is an isomorphism of S onto $\hat{S}\Gamma S$.

Theorem 21. *S be a regular semigroup, $(L(S), R(S), \Gamma S)$ a cross-connection and $\hat{S}\Gamma S = (\rho^a, \lambda^a)$ its cross-connection semigroup. Then the projections $\pi : (\rho^a, \lambda^a) \mapsto \rho^a$ is homomorphosm of $\hat{S}\Gamma S$ onto $U\Gamma S$ and $\pi^* : (\rho^a, \lambda^a) \mapsto \lambda^a$ is an anti-homomorphosm of $\hat{S}\Gamma S$ onto $U\Gamma^*S$. Consequently $\hat{S}\Gamma S$ is a subdirect product of $U\Gamma S$ and $U\Gamma^*S^{op}$.*

4. Conclusion

In [4], it is shown that given two normal categories $\mathscr{C}$, $\mathscr{D}$ and a local isomorphism $\Gamma : \mathscr{D} \to N^*\mathscr{C}$ whose image is total, there is a cross-connetion $(\mathscr{C}, \mathscr{D}, \Gamma)$ such that $\hat{S}\Gamma$ is a semigroup and is called the cross-connection semigroup . In this paper we demonstate that for a regular semigroup S the principal left/right ideal categories $\mathscr{L}(S)$ and $\mathscr{R}(S)$ of S are normal categories and there exists local isomorhisms $\Gamma S : \mathscr{R}(S) \to N^*\mathscr{L}(S)$, $\Gamma^* S : \mathscr{L}(S) \to N^*\mathscr{R}(S)$ and cross-connection $(\mathscr{L}(S), \mathscr{R}(S), \Gamma S)$ such that the cross-connection semigroup $\hat{S}\Gamma S$ is the cross-connection representation of S.

Conflict of Interests

The author declares that there is no conflict of interests.

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