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BASKET OPTION PRICING USING MELLIN TRANSFORMS

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Abstract. Analytical pricing formulas and Greeks are obtained for European and American basket put options using Mellin transforms. We assume assets are driven by geometric Brownian motion which exhibit correlation and pay a continuous dividend rate.

Keywords: American put option; Mellin transform; basket option.

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1. Introduction

An *option* is a financial contract that presents its holder with the right, but not the obligation, to buy (*call*) or sell (*put*) a given amount of asset at some future date. In practice, the underlying asset is often the price of a stock, commodity, foreign exchange rate, financial index or futures contract. While many *styles* of options exist, here we are concerned with the valuation of *European* and *American* varieties. American options may be exercised at any time $t < T$, while European options can only be exercised at time T. In both cases, their definitions can be extended to *basket* options, which differ by their dependence on $n \in \mathbb{N}$ underlying assets.

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Since the seminal paper of [1], much of the literature assumes assets are driven by geometric Brownian motion (GBM). Under this assumption, European option valuation relies on solving the Black-Scholes partial differential equation (PDE). With American options, the early-exercise condition gives rise to a *free boundary*, in which no closed-form solution exists. The corresponding PDE is given by the inhomogeneous Black-Scholes equation. However, using the Mellin transform to solve the PDE has only recently been considered. Using the Mellin transform to solve the PDE is distinguishable from convential methods in that; one, the technique requires no change of variables or reduction to a diffusion equation; and two, it enables option formulas to be expressed in terms of market asset prices, rather than logarithmic asset prices. For pricing financial derivatives, the Mellin technique was first introduced in [2], where the authors consider the European call option without dividends. Thereafter, the dividend-paying single-asset case is solved in [5] by applying the Mellin transform to the PDE, [4] *via* the discounted expectation formula for options, and [3, 10] by an application of Mellin convolution. For American options, the dividend-paying single-asset case is solved additionally in [5]. Mellin transforms have been used to price other styles of options, including perpetuals in [9]. The general multi-asset formulas for pricing European and American basket options on dividend-paying assets are derived herein.

In section 2, we extend the existing Mellin-type pricing formulas for European basket options on n assets with continuous dividend rates and correlation. The expressions for the analogous American basket option are derived in section 3. As a corollary, new expressions for the *Greeks* of multi-asset European and American options are provided in section 4.

2. European Options

In this section, Mellin transforms are used to derive the formula for the price of a European basket put option where assets have a continuous dividend rate and correlation. For an option issued on n assets, let $\boldsymbol{S} = (S_1, \ldots, S_n)'$, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)'$ and $\boldsymbol{q} = (q_1, \ldots, q_n)'$. The value $V = V(\boldsymbol{S}, t; K; T; \boldsymbol{\sigma}; r; \boldsymbol{q})$ is dependent on the underlying asset prices $0 \le S_i(t) < \infty$, the exercise price $K > 0$, the maturity time $0 \le t \le T$, the asset volatilities (or standard deviations) $\sigma_i \ge 0$, the risk-free interest rate $r \ge 0$, and continuous dividend rates $q_i \ge 0$, $\forall i$. The assets are assumed to be driven by geometric Brownian motion,

$$dS_i = \mu_i S_i dt + \sigma_i S_i dW_i, \tag{2.1}$$

where the Wiener processes satisfy $dW_i \sim Normal(0, dt)$ and $corr(dW_i, dW_j) = \rho_{ij}$ for $\rho_{ij} \in [-1, 1]$. The risk-neutral drift

$$\mu_i = r - q_i - \frac{\sigma_i^2}{2} \tag{2.2}$$

ensures the no-arbitrage condition holds. For multivariate Brownian motion with drift, say $\boldsymbol{X}_t$, the characteristic function $\Phi(\boldsymbol{u}; t) := \exp[-t\Psi(\boldsymbol{u})] = \mathbb{E}[\exp(i\boldsymbol{u}'\boldsymbol{X}_t)]$ is given by the exponent

$$\Psi(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}'\Sigma\boldsymbol{u} - i\boldsymbol{\mu}'\boldsymbol{u}. \tag{2.3}$$

It is known under these conditions that the corresponding PDE for the price of a European basket option is the generalized Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - q_i) S_i \frac{\partial V}{\partial S_i} - rV = 0. \tag{2.4}$$

We note (2.4) must satisfy the boundary conditions

$$V(\boldsymbol{S}, T) = \theta(\boldsymbol{S}) = \Big(K - \sum_{i=1}^{n} S_i\Big)^{+} \quad \text{and} \quad V(\boldsymbol{S}, t) \to 0 \text{ as } \boldsymbol{S} \to \infty. \tag{2.5}$$

Let $\mathscr{M}\{f(\boldsymbol{x}); \boldsymbol{w}\}$ denote the multidimensional Mellin transform of a function $f(\boldsymbol{x}) \in \mathbb{R}^{n+}$ given by,

$$\hat{f}(\boldsymbol{w}) := \mathscr{M}\{f(\boldsymbol{x}); \boldsymbol{w}\} = \int_{\mathbb{R}^{n+}} f(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{w}-1} d\boldsymbol{x} \tag{2.6}$$

where complex variable $\boldsymbol{w} = (w_1, ..., w_n)'$ exists in an appropriate domain of convergence in $\mathbb{C}^n$. Conversely, the inverse multidimensional Mellin transform of a function $\hat{f}(\boldsymbol{w}) \in \mathbb{C}^n$ is defined by

$$f(\boldsymbol{x}) = \mathscr{M}^{-1}\{\hat{f}(\boldsymbol{w}); \boldsymbol{x}\} = (2\pi i)^{-n} \int_{\gamma} \hat{f}(\boldsymbol{w})\boldsymbol{x}^{-\boldsymbol{w}} d\boldsymbol{w}, \tag{2.7}$$

where $\gamma = \underset{j=1}{\overset{n}{\times}} \gamma_j$ are strips in $\mathbb{C}^n$ defined by $\gamma_j = \{a_j + ib_j : a_j \in \mathbb{R}, b_j = \pm\infty\}$ with $a_j \in \Re(w_j)$. Thus, to find the multidimensional Mellin transform of the generalized Black-Scholes equation apply (2.6) to (2.4):

$$\frac{\partial \hat{V}}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j w_i w_j\hat{V}+\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 w_i\hat{V}+(r-q_i)\sum_{i=1}^{n}w_i\hat{V}-r\hat{V}=0. \tag{2.8}$$

By use of (2.2) and (2.3) we may rearrange the expression to obtain the ordinary differential equation

$$\frac{d\hat{V}(\boldsymbol{w},t)}{dt}=(\Psi(\boldsymbol{w}i)+r)\hat{V}(\boldsymbol{w},t). \tag{2.9}$$

Solving *via* the final time condition (2.5) yields

$$\hat{V}(\boldsymbol{w},t)=\hat{\theta}(\boldsymbol{w})e^{-(\Psi(\boldsymbol{w}i)+r)(T-t)}. \tag{2.10}$$

Hence, by Mellin inversion we obtain our result.

Theorem 1. *The Mellin-type formula for a European basket put option on n assets is given by*

$$V_E^P(\boldsymbol{S},t)=\mathscr{M}^{-1}\left\{\hat{\theta}(\boldsymbol{w})\Phi(\boldsymbol{w}i,T-t)\right\}e^{-r(T-t)}, \tag{2.11}$$

where $\Phi(*)$ is the characteristic function of a multivariate Brownian motion with drift and the Mellin transform of the payoff function is given by

$$\hat{\theta}(\boldsymbol{w})=\frac{\beta_n(\boldsymbol{w})K^{1+\Sigma\boldsymbol{w}}}{(\sum\boldsymbol{w})(\sum\boldsymbol{w}+1)} \tag{2.12}$$

for multinomial beta function $\beta_n(\boldsymbol{w})=\prod_{j=1}^{n}\Gamma(w_j)/\Gamma(\sum_{i=1}^{n}w_i)$, $\boldsymbol{w}\in\mathbb{C}^n$, and $\Re(\boldsymbol{w})>0$.

The derivation of (2.12) proceeds as follows. Consider the following expression for the J-dimensional Mellin transform of the put payoff function on J assets:

$$\int_{\mathbb{R}^{J+}}(K-\sum_{j=1}^{J}S_i)^+\prod_{j=1}^{J}S_j^{w_j-1}dS_j=\frac{\prod_{j=1}^{J}\Gamma(w_j)}{\Gamma(2+\sum_{j=1}^{J}w_j)}K^{1+\Sigma_{j=1}^{J}w_j}. \tag{2.13}$$

When $J=1$ the expression holds. Assume $J=n$, then for $J=n+1$

$$\begin{aligned} LHS &= \int_{\mathbb{R}^{(n+1)+}}(K-\sum_{j=1}^{n+1}S_i)^+\prod_{j=1}^{n+1}S_j^{w_j-1}dS_j \\ &= \frac{\prod_{j=1}^{n}\Gamma(w_j)}{\Gamma(2+\sum_{j=1}^{n}w_j)}\int_0^K(K-S_{n+1})^{1+\Sigma_{j=1}^{n}w_j}S_{n+1}^{w_{n+1}-1}dS_{n+1} \\ &= \frac{\prod_{j=1}^{n+1}\Gamma(w_j)}{\Gamma(2+\sum_{j=1}^{n+1}w_j)}K^{1+\Sigma_{j=1}^{n+1}} \end{aligned}$$

from Fubini's theorem and (3.191.1) in [6]. The result follows from the definition of the multinomial beta function and properties of gamma functions.

Remark 1. *An application of generalized put-call parity computes the price of a European call from a put (see* [7]*).*

3. American Options

In this section, Mellin transforms are used to derive the formula for the price of an American basket put option where assets have a continuous dividend rate and correlation. For multiple assets, the continuation region exists for $\sum_{i=1}^n S_i > S^*$, while the exercise region exists for $\sum_{i=1}^n S_i < S^*$. The value $V = V(S,t;K;T;\sigma;r;q)$ of an American option on one asset is known to satisfy the inhomogeneous generalized Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-q_i)S_i\frac{\partial V}{\partial S_i} - rV = f, \tag{3.1}$$

where the *early exercise function* is

$$f(\boldsymbol{S},t) = \begin{cases} -rK + \sum_{i=1}^n q_i S_i, & 0 < \sum_{i=1}^n S_i \le S^*(t), \\ 0, & S^*(t) < \sum_{i=1}^n S_i < \infty. \end{cases} \tag{3.2}$$

Similar to the European case, the boundary conditions imposed on (3.1) are

$$V(\boldsymbol{S},T) = \theta(\boldsymbol{S}) = \Big(K - \sum_{i=1}^{n} S_i\Big)^+ \quad \text{and} \quad V(\boldsymbol{S},t) \to 0 \text{ as } \boldsymbol{S} \to \infty. \tag{3.3}$$

The smooth pasting conditions along the boundary are

$$\left.\frac{\partial V(\boldsymbol{S},t)}{\partial S_i}\right|_{\sum_{i=1}^n S_i = S^*} = -1 \quad \text{and} \quad \theta(\boldsymbol{S}) = K - S^*. \tag{3.4}$$

The multidimensional Mellin transform of (3.1) is given by the expression

$$\frac{\partial \hat{V}}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j w_i w_j \hat{V} + \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 w_i \hat{V} + (r-q_i)\sum_{i=1}^{n} w_i\hat{V} - r\hat{V} = \hat{f}. \tag{3.5}$$

By use of (2.2) and (2.3) we may rearrange (3.5) to obtain the ordinary differential equation

$$\frac{d\hat{V}(\boldsymbol{w},t)}{dt} - (\Psi(\boldsymbol{w}i) + r)\hat{V}(\boldsymbol{w},t) = \hat{f}(\boldsymbol{w},t). \tag{3.6}$$

Solving *via* the final time condition (3.4) and applying Duhamel's principle yields

$$\hat{V}(\boldsymbol{w},t) = \hat{\theta}(\boldsymbol{w})e^{-(\Psi(\boldsymbol{w}i)+r)(T-t)} - \int\limits_t^T \hat{f}(\boldsymbol{w},s)e^{-(\Psi(\boldsymbol{w}i)+r)(s-t)}ds. \tag{3.7}$$

Hence, by Mellin inversion we obtain our result.

Theorem 2. *The Mellin-type formula for an American basket put option on n assets is given by*

(3.8)

$$V_A^P(\boldsymbol{S},t) = e^{-r(T-t)}\mathscr{M}^{-1}\Big\{\hat{\theta}(\boldsymbol{w})\Phi(\boldsymbol{w}i,T-t)\Big\} - \mathscr{M}^{-1}\Big\{\int_t^T \hat{f}(\boldsymbol{w},s)\Phi(\boldsymbol{w}i,s-t)e^{-r(s-t)}ds\Big\},$$

where $\Phi(*)$ is the characteristic function of a multivariate Brownian motion with drift, $\hat{\theta}(*)$ is the Mellin transform of the payoff function given by (2.12)*, and the Mellin transform of the early exercise function is given by*

$$\hat{f}(\boldsymbol{w},t) = \frac{\beta_n(\boldsymbol{w})(S^*)^{\sum \boldsymbol{w}}}{\sum \boldsymbol{w}}\left[\frac{\boldsymbol{q}'\boldsymbol{w}S^*}{\sum \boldsymbol{w}+1} - rK\right] \tag{3.9}$$

for free boundary $S^*(t)$, multinomial beta function $\beta_n(\boldsymbol{w}) = \prod_{j=1}^n \Gamma(w_j)/\Gamma(\sum_{i=1}^n w_i)$, $\boldsymbol{w}\in\mathbb{C}^n$, and $\Re(\boldsymbol{w})>0$.

The derivation for (3.9) proceeds as follows. Consider the following expression for the J-dimensional Mellin transform of the early exercise function on J assets:

$$\int_{\mathbb{R}^{J+}}\left(-rK+\sum_{i=1}^J q_iS_i\right)\prod_{j=1}^J S_j^{w_j-1}dS_j = \frac{\prod_{j=1}^J\Gamma(w_j)(S^*)^{\sum_{j=1}^J w_j}}{\Gamma(1+\sum_{j=1}^J w_j)}\left[\frac{S^*\sum_{j=1}^J q_jw_j}{\sum_{j=1}^J w_j+1} - rK\right].$$

When $J=1$ the expression holds. Assume $J=n$, then for $J=n+1$

$$\begin{aligned}
LHS &= \int_{\mathbb{R}^{(n+1)+}}\left(-rK+\sum_{i=1}^{n+1} q_iS_i\right)\prod_{j=1}^{n+1} S_j^{w_j-1}dS_j\\
&= \frac{-rK\prod_{j=1}^n\Gamma(w_j)}{\Gamma(1+\sum_{j=1}^n w_j)}\int_0^{S^*}(S^*-S_{n+1})^{\sum_{j=1}^n w_j}S_{n+1}^{w_{n+1}-1}dS_{n+1}\\
&+ \frac{\sum_{j=1}^{n+1} q_jw_j\prod_{j=1}^n\Gamma(w_j)}{\Gamma(2+\sum_{j=1}^n w_j)}\int_0^{S^*}(S^*-S_{n+1})^{1+\sum_{j=1}^n w_j}S_{n+1}^{w_{n+1}-1}dS_{n+1}\\
&= \frac{-rK\prod_{j=1}^{n+1}\Gamma(w_j)}{\Gamma(1+\sum_{j=1}^{n+1} w_j)}(S^*)^{\sum_{j=1}^{n+1}} + \frac{\sum_{j=1}^{n+1} q_jw_j\prod_{j=1}^{n+1}\Gamma(w_j)}{\Gamma(2+\sum_{j=1}^{n+1} w_j)}(S^*)^{1+\sum_{j=1}^{n+1}}
\end{aligned}$$

from Fubini's theorem and equation (3.191.1) in [6]. The result follows from the definition of the multinomial beta function and properties of gamma functions.

Remark 2. *An application of generalized put-call symmetry gives the price of an American call option from a put (see* [8]*).*

Note that the early exercise premium only contributes to the price of the option when $\sum_{i=1}^{n} S_i(s) \leq S^*(s)$. Otherwise the second term of (3.8) is zero. By imposing the smooth pasting conditions (3.4) on (3.8), we obtain an implicit equation describing the free boundary.

Corollary 1. *The free boundary $S^*(t)$ is given by the solution of the expression*

$$K - S^*(t) = \frac{e^{-r(T-t)}}{2\pi i} \int_\gamma \hat{\theta}(\boldsymbol{w}) \Phi(\boldsymbol{w}i, T-t) \boldsymbol{S}^*(t)^{-\boldsymbol{w}} d\boldsymbol{w} - \int_\gamma \int_t^T \hat{f}(\boldsymbol{w}, s) \Phi(\boldsymbol{w}i, s-t) e^{-r(s-t)} \boldsymbol{S}^*(t)^{-\boldsymbol{w}} ds d\boldsymbol{w}. \tag{3.10}$$

The free boundary can be obtained by solving for $S^*(t)$ where $\boldsymbol{S}^*(t) = (S_1^*, \ldots S_n^*)$ over the space of possible prices in $\mathbb{R}^{n+}$ such that $S^* = \sum_{i=1}^{n} S_i^*$. By setting the free boundary equal to zero, (3.8) reduces to (2.11).

4. Option Sensitivities

Option *sensitivities* or *Greeks* describe the relationship between the value of an option and changes in one of its underlying parameters. They play a vital role for risk management and portfolio optimization, since they have the ability to describe how vulnerable an option is to a particular risk factor. They are easily obtained for European and American options by passing the appropriate derivative operator under the complex integral in (3.8). For succinctness, the variable change $\tau = T - t$ is used in some of the following expressions. The first partial derivative with respect to a given asset, Delta, is given by

$$\Delta_1 := \frac{\partial V}{\partial S_i} = -e^{-r\tau} \mathscr{M}^{-1} \left\{ \frac{w_i}{S_i} \hat{\theta}(\boldsymbol{w}) \Phi(\boldsymbol{w}i, \tau) \right\} + \mathscr{M}^{-1} \left\{ \frac{w_i}{S_i} \int_0^\tau \hat{f}(\boldsymbol{w}, \tau - s) \Phi(\boldsymbol{w}i, s) e^{-rs} ds \right\}. \tag{4.1}$$

Theta, the first partial derivative with respect to time is

$$\Theta := -\frac{\partial V}{\partial t} = -e^{-r(T-t)} \mathscr{M}^{-1} \left\{ (\Psi(\boldsymbol{w}i) + r) \hat{\theta}(\boldsymbol{w}) \Phi(\boldsymbol{w}i, T-t) \right\} + \mathscr{M}^{-1} \left\{ \int_t^T (\Psi(\boldsymbol{w}i) + r - 1) \hat{f}(\boldsymbol{w}, s) \Phi(\boldsymbol{w}i, s-t) e^{-r(s-t)} ds \right\}. \tag{4.2}$$

Rho, the first partial derivative with respect to the risk-free rate of return is given by

$$\rho := \frac{\partial V}{\partial r} = -\tau e^{-r\tau}\mathscr{M}^{-1}\Big\{(\sum_{j=1}^{n} w_i - 1)(T-t)\hat{\theta}(\boldsymbol{w})\Phi(\boldsymbol{w}i,\tau)\Big\} - \mathscr{M}^{-1}\Big\{\int_t^T (\sum_{j=1}^{n} w_i - 1)(s-t)\hat{f}(\boldsymbol{w},s)\Phi(\boldsymbol{w}i,s-t)e^{-r(s-t)}ds\Big\}. \tag{4.3}$$

By eliminating the second term for each Greek we obtain the corresponding European option sensitivities. Since most payoff functions are independent of the derivative operator, these expressions also hold for many path-independent multi-asset options. The American case differs because the exercise region varies with time and depends on the payoff function. Even in the simplest case of the basket option, the Mellin transform of the early exercise function is dependent on the derivative operator and must be considered to obtain expressions for other multi-asset Greeks.

Remark 3. *By direct substitution of the Greeks, we may prove that* (i) *formula* (2.11) *is a classical solution to the European pricing problem* (2.4)-(2.5) *and* (ii) *formula* (3.8) *is a classical solution to the American pricing problem* (3.1)-(3.4).

5. Conclusion

In the context of Mellin transforms, we obtain analytic solutions for the fair value of basket put options and Greeks on n assets with continuous dividend rates and correlation. Solutions are obtained for both European and American option styles. The decomposition of the solution enables the direct computation of either European or American basket option prices. By solving for the Mellin transform of alternate payoff functions, the results presented here may be used to price more complicated multi-asset options.

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Conflict of Interests

The authors declare that there are no conflict of interests.

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